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Computer Algebra in Scientific Computing

CASC 2001

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Preface

CASC 2001 continues a tradition — started in 1998 — of international conferences on the latest advances in the application of computer algebra systems (CASs) to the solution of various problems in scientific computing. The three earlier conferences in this sequence, CASC'98, CASC'99, and CASC 2000, were held, respectively, in St. Petersburg, Russia, in Munich, Germany, and in Samarkand, Uzbekistan, and proved to be very successful.

We have to thank the program committee, listed overleaf, for a tremendous job in soliciting and providing reviews for the submitted papers. There were more than three reviews per submission on average. The result of this job is reflected in the present volume, which contains revised versions of the accepted papers. The collection of papers included in the proceedings covers various topics of computer algebra methods, algorithms and software applied to scientific computing.

In particular, five papers are devoted to the implementation of the analysis of involutive systems with the aid of CASs. The specific examples include new efficient algorithms for the computation of Janet bases for monomial ideals, involutive division, involutive reduction method, etc.

A number of papers deal with application of CASs for obtaining and validating new exact solutions to initial and boundary value problems for partial differential equations in mathematical physics. Several papers show how CASs can be used to obtain analytic solutions of initial and boundary value problems for ordinary differential equations and for studying their properties.

Several papers present the application of CASs to the solution of differential geometry tasks. A number of papers deals with group and Lie symmetry analysis as applied, in particular, to equations governing plane motions of a viscous heat-conducting gas.

There are also papers devoted to problems that are typical CA applications: polynomial ideals, polynomial algebra, and quantifier elimination.

A number of papers is aimed in a new interesting direction for the development of applied CAS packages: the integration of object-oriented programming in CAS environments with the capabilities of Java.

A novel feature of this conference is an enhanced emphasis on engineering applications of computer algebra. In particular, such applied problems are considered as the modelling of shape memory metal alloys, the stability of a satellite with a solar sail, reliability problems in aerospace systems, automatic motion planning of an automobile or aircraft within moving traffic, detection of all singular positions of planar mechanisms, etc.

The invited lecture by R. Maeder shows in detail how MATHEMATICA routines can be moved to and efficiently be implemented in a parallel environment.

The CASC 2001 workshop was supported financially by a generous grant from the Deutsche Forschungsgemeinschaft (DFG) and Visual Analysis AG. We are grateful to W. Meixner for his technical help in the preparation of the camera ready manuscript for this volume. We also thank the publisher, Springer Verlag, for their support in preparing these proceedings.

Our particular thanks are due to the members of the local organizing committee in Konstanz who have ably handled local arrangements in this particularly pleasant location in the very South of Germany.

Munich, July 2001

V.G. Ganzha
E.W. Mayr
E.V. Vorozhtsov

Workshop Organization

CASC 2001 was organized jointly by the Technische Universität München, the FH Weingarten-Ravensburg, and the FH Konstanz.

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Jets. A MAPLE-Package for Formal Differential Geometry

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Abstract. The MAPLE-package `jets` was first designed to be an extension of the package `desolv`. In the current stage it became an independent package going beyond symmetries to handle different aspects of formal differential geometry, including some important parts of the variational bicomplex. We demonstrate this by computing the set of all Hamiltonian structures of a order at most 3, which are compatible with D_x. This set includes among others the famous KdV-operator $D_{xxx} + \frac{2}{3}uD_x + \frac{1}{3}u_x$.

1 Introduction

The MAPLE-package `jets`, originally an extension of the package `desolv`[1] adding to it the facility of computing generalized symmetries of differential equations, is at the current stage an independent package going beyond symmetries to handle different aspects of what I. M. Gel'fand, in his 1970 address to the International Congress in Nice, called "formal differential geometry". Important parts of the variational bicomplex, as playing a crucial role in the formal theory, are implemented in `jets`. Most of the implementation of the variational aspects in `jets`, such as variational symmetries, higher Euler operators, homotopy operators and conservation laws, was done by GEHRT HARTJEN as part of his diploma thesis [Har]. As dual to functional forms and the vertical derivative also functional multi-vectors and the Nijenhuis-Schouten bracket are also implemented in `jets`, enabling one to handle Hamiltonian systems of evolution equations and non-linear integrable systems. The package adds to MAPLE the important feature of dealing with jet calculus, a thing which is still missing in modern computer algebra systems. Almost every formula appearing in [Olv] can now be computed using `jets`.

2 Hamiltonian Structures and the Nijenhuis-Schouten Bracket

As mentioned in the abstract, the aim of this paper is to demonstrate a non-trivial application of the package `jets` by computing the set of all Hamiltonian

[1] `desolv` was written by Khai Vu and Colin McIntosh. `jets` still uses `desolv` to solve linear PDE systems.

structures of a order at most 3, which are compatible with D_x. This is done in section 3. To this end we define the notion of functional multi-vectors, Hamiltonian structures and the Nijenhuis-Schouten bracket. The notions used in sequel are standard and can be found in [Olv]. Further details are found in [Bar].

Let $E \to M$ be a fibred manifold in p independent variables $(x^i) = (x^1, \ldots, x^p)$ and q dependent variables $(u^\alpha) = (u^1, \ldots, u^q)$. By $J_\infty(E) \to M$ we denote the infinite jet bundle having the jet variables (x^i, u^α_J) as coordinates, where J is an arbitrary multi-index. By $\mathcal{A}$ we denote the space of differential expressions over E, i.e. smooth real-valued functions of finitely many arbitrary jet variables. By $\mathcal{V}^1$ we denote the space of evolutionary vector fields, or equivalently the space of characteristics over a jet bundle. This space can be identified with the Cartesian power $\mathcal{A}^q$. Further we define locally $\mathcal{F}^0 := \mathcal{A}/\mathsf{Div}(\mathcal{A}^p)$ and call it the space of functionals[2]. By $\mathcal{F}^1$ we denote the $\mathcal{F}^0$-dual space of $\mathcal{V}^1$. We can also identify it with $\mathcal{A}^q$. Further let $\mathcal{F}^n$ (resp. $\mathcal{V}^n$) denote the space of functional n-forms (resp. n-vectors).

We first note the following two basic formulas. The first one relates the prolongation of an evolutionary vector field and the Fréchet derivative

$$\mathsf{pr}\,\mathbf{v}_Q(L) = \mathsf{D}_L Q, \tag{1}$$

where $Q = (Q^1, \ldots, Q^q)^{\mathrm{tr}}$ is a characteristic, $\mathbf{v}_Q = Q^\alpha \frac{\partial}{\partial u^\alpha}$ and evolutionary vector field, $\mathsf{pr}\,\mathbf{v}_Q = D_J Q^\alpha \frac{\partial}{\partial u^\alpha_J}$ (prolongation formula) and $\mathsf{D}_L = (\frac{\partial L}{\partial u^1_J}\mathsf{D}_J, \ldots, \frac{\partial L}{\partial u^q_J}\mathsf{D}_J)$ (Fréchet derivative). The proof follows immediately from the prolongation formula and the definition of the Fréchet derivative. The second formula is the standard LEIBNIZ rule

$$\mathsf{pr}\,\mathbf{v}(L \cdot P) = \mathsf{pr}\,\mathbf{v}L \cdot P + L \cdot \mathsf{pr}\,\mathbf{v}P \tag{2}$$

where $\mathbf{v}$ is a generalized vector field and L, P are arbitrary differential expression.

We still need the following lemma.

Lemma 1. *For a differential operator $\mathcal{D} = P^J D_J$ ($P^J \in \mathcal{A}$) and differential function $T \in \mathcal{A}$, we have the following Leibniz rule:*

$$\mathsf{pr}\,\mathbf{v}_Q(\mathcal{D}T) = \mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})T + \mathcal{D}\mathsf{pr}\,\mathbf{v}_Q(T), \tag{3}$$

or equivalently by (1)

$$\mathsf{D}_{\mathcal{D}T}(Q) = \mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})T + \mathcal{D}\mathsf{D}_T Q. \tag{4}$$

PROOF. [Olv], Formula (5.38). □

Definition 1 (Adjoint operator). *The formal adjoint operator of a matrix differential operator $\mathcal{D} = (P^J_{\alpha\beta} D_J)$ is defined by*

$$\mathcal{D}^* = ((-1)^{|J|} D_J P^J_{\beta\alpha}).$$

[2] $\mathsf{Div}P = D_i P^i$, where $P = (P^1, \ldots, P^p)$ and $D_i = D_{x^i}$

Definition 2 (Euler operator). *For $L \in \mathcal{A}$ the operator*

$$\mathsf{E}(L) := \mathsf{D}_L^*(1) \tag{5}$$

is called the EULER *operator.*

Lemma 2 ([Olv], Formula (4.15)). *A Lagrangian $L \in \mathcal{A}$ transforms infinitesimally according to the rule*

$$\mathcal{L}_{\mathbf{v}} L = \mathsf{pr}\,\mathbf{v} L + L \mathsf{Div}(\xi), \tag{6}$$

where $\mathbf{v} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}$ is a generalized vector field[3]*.*

PROOF. [Olv], Theorem 4.12. □

Corollary 1 (Lie derivative of functionals). *For a Lagrangian L viewed as an element of $\mathcal{F}^0$, i.e. as a functional 0-form, the Lie derivative $\mathcal{L}_{\mathbf{v}}$ satisfies*

$$\mathcal{L}_{\mathbf{v}} L = \mathsf{pr}\,\mathbf{v}_Q L = \mathsf{E}(L) \cdot Q. \tag{7}$$

PROOF. The following are identities between functionals. For a generalized vector field $\mathbf{v}$ with characteristic Q

$$\begin{aligned}
\mathcal{L}_{\mathbf{v}} L &\overset{(6)}{=} \mathsf{pr}\,\mathbf{v} L + L \mathsf{Div}(\xi) \\
&= \mathsf{pr}\,\mathbf{v}_Q L + \xi^i D_i L + L D_i \xi^i \\
&= \mathsf{pr}\,\mathbf{v}_Q L + \mathsf{Div}(L\xi) \\
&= \mathsf{pr}\,\mathbf{v}_Q L \\
&\overset{(1)}{=} 1 \cdot \mathsf{D}_L(Q) \\
&= \mathsf{D}_L^*(1) \cdot Q \\
&\overset{(5)}{=} \mathsf{E}(L) \cdot Q
\end{aligned}$$

□

Definition 3 (Lie derivative of vector fields). *Let $\mathbf{v}$ be a generalized vector field and R a characteristic, i.e. $R \in \mathcal{V}^1$. Define the Lie derivative of R with respect to $\mathbf{v}$ by*

$$\begin{aligned}
\mathcal{L}_{\mathbf{v}}(R) &= \mathsf{pr}\,\mathbf{v}_Q R - \mathsf{pr}\,\mathbf{v}_R Q \\
&\overset{(1)}{=} \mathsf{pr}\,\mathbf{v}_Q R - \mathsf{D}_Q R,
\end{aligned} \tag{8}$$

where Q is the characteristic of $\mathbf{v}$.

Proposition 1 (Lie derivative of functional 1-forms). *For the Lie derivative of a source form $\Delta \in \mathcal{F}^1$ the following two statements are equivalent:*

[3] [Olv] proves this for point vector fields only. The above Lie derivative coincides with the notion of projected Lie derivative $\mathcal{L}_{\mathbf{v}}^\sharp$ introduced in [And], Chapter 3.

(i) Δ *transforms infinitesimally according to*

$$\mathcal{L}_{\mathbf{v}_Q}\Delta = \mathsf{pr}\,\mathbf{v}_Q\Delta + \mathsf{D}_Q^*\Delta. \tag{9}$$

(ii) $\mathcal{L}_{\mathbf{v}_Q}$ *satisfies the following* Leibniz rule *for an arbitrary characteristic* R

$$\mathcal{L}_{\mathbf{v}_Q}(\Delta \cdot R) = \mathcal{L}_{\mathbf{v}_Q}\Delta \cdot R + \Delta \cdot \mathcal{L}_{\mathbf{v}_Q}R. \tag{10}$$

This is an identity of functionals, i.e. the left and right hand sides are equal up to local divergence.

PROOF. Both directions follow from the following equalities:

$$\begin{aligned}
&\mathsf{E}(\mathcal{L}_{\mathbf{v}_Q}(\Delta \cdot R)) - \mathsf{E}(\Delta \cdot \mathcal{L}_{\mathbf{v}_Q}R)\\
&\overset{(7)}{=} \mathsf{E}(\mathsf{pr}\,\mathbf{v}_Q(\Delta \cdot R)) - \mathsf{E}(\Delta \cdot \mathcal{L}_{\mathbf{v}_Q}R)\\
&\overset{(2),(8)}{=} \mathsf{E}(\mathsf{pr}\,\mathbf{v}_Q\Delta \cdot R + \Delta \cdot \mathsf{pr}\,\mathbf{v}_Q R) - \mathsf{E}(\Delta \cdot (\mathsf{pr}\,\mathbf{v}_Q R - \mathsf{D}_Q R))\\
&= \mathsf{E}(\mathsf{pr}\,\mathbf{v}_Q\Delta \cdot R + \Delta \cdot \mathsf{D}_Q R)\\
&= \mathsf{E}((\mathsf{pr}\,\mathbf{v}_Q\Delta + \mathsf{D}_Q^*\Delta) \cdot R).
\end{aligned}$$

□

Remark 1. The identity of functionals

$$\mathcal{L}_{\mathbf{v}_Q}\Delta \cdot R = \mathsf{pr}\,\mathbf{v}_Q\Delta \cdot R + \Delta \cdot \mathsf{pr}\,\mathbf{v}_R Q, \tag{11}$$

which is part of the proof, appears as formula (4.2) in [GDo2].

Lemma 3. *The following identity holds for a general* $\mathcal{K} : \mathcal{F}^1 \to \mathcal{V}^1$

$$(\mathsf{pr}\,\mathbf{v}_{.}(\mathcal{K})\Delta)^*\Sigma = (\mathsf{pr}\,\mathbf{v}_{.}(\mathcal{K}^*)\Sigma)^*\Delta \tag{12}$$

PROOF. For an arbitrary characteristic S

$$\begin{aligned}
&\mathsf{E}(S \cdot ((\mathsf{pr}\,\mathbf{v}_{.}(\mathcal{K})\Delta)^*\Sigma - (\mathsf{pr}\,\mathbf{v}_{.}(\mathcal{K}^*)\Sigma)^*\Delta))\\
&= \mathsf{E}(\mathsf{pr}\,\mathbf{v}_S(\mathcal{K})\Delta \cdot \Sigma - \mathsf{pr}\,\mathbf{v}_S(\mathcal{K}^*)\Sigma \cdot \Delta)\\
&= \mathsf{E}(\mathsf{pr}\,\mathbf{v}_S(\mathcal{K})\Delta \cdot \Sigma - \Sigma \cdot \mathsf{pr}\,\mathbf{v}_S(\mathcal{K})\Delta)\\
&= 0.
\end{aligned}$$

□

Definition 4 (Nijenhuis-Schouten bracket). *For* $\mathcal{D}, \mathcal{E} \in \mathcal{V}^2$ *the Nijenhuis-Schouten bracket* $[\mathcal{D}, \mathcal{E}] : \mathcal{F}^1 \times \mathcal{F}^1 \times \mathcal{F}^1 \to \mathcal{F}^0$ *is defined as follows:*

$$[\mathcal{D}, \mathcal{E}](\Delta_1, \Delta_2, \Delta_3) := \mathcal{L}_{\mathcal{D}\Delta_1}\Delta_2 \cdot \mathcal{E}\Delta_3 + \mathcal{L}_{\mathcal{E}\Delta_1}\Delta_2 \cdot \mathcal{D}\Delta_3 + \text{(cycle)}, \tag{13}$$

where the word (cycle) means summation over all cyclic permutations of the indices 1, 2, 3. $\mathcal{D}$ *and* $\mathcal{E}$ *are viewed as differential operators from* $\mathcal{F}^1$ *into* $\mathcal{V}^1$.

This definition is a generalisation of the classical Nijenhuis-Schouten bracket from differential geometry, which is one of its advantages. It appears in [GDo2], Formula (3.3). Nevertheless there are two major *drawbacks* of this definition. The first one is that the right hand side is a functional, so it has no normal form. This means that checking the vanishing of the bracket or extracting conditions for its vanishing is not a direct procedure. The second one is that one needs more than total differentials of the Δ_i's, meaning that we cannot compute with general Δ_i's, complicating the check of vanishing of the bracket. Besides, from this definition we do not see that the bracket of two 2-vectors is a $(3,0)$-tensor, *even* a 3-vector. In the following we want to make use of the freedom of adding divergences to circumvent these drawbacks. The following formula cures both drawbacks.

Proposition 2 (Nijenhuis-Schouten bracket). *For $\mathcal{D},\mathcal{E} \in \mathcal{V}^2$ the following formula is an equivalent definition of the Nijenhuis-Schouten bracket $[\mathcal{D},\mathcal{E}]$*

$$\begin{aligned}[\mathcal{D},\mathcal{E}](\Delta) = {} & \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta}(\mathcal{E}) - \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta + (\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta)^* + \\ & \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta}(\mathcal{D}) - \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta + (\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta)^*. \end{aligned} \tag{14}$$

PROOF.

$$\begin{aligned}
&\mathsf{E}([\mathcal{D},\mathcal{E}](\Delta_1,\Delta_2,\Delta_3)) \\
&= \mathsf{E}(\mathcal{L}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3) + \mathsf{E}(\mathcal{L}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3) + (\text{cycle}) \\
&\overset{(11)}{=} \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3) + \mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D}\Delta_1)) + \\
&\quad \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3) + \mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E}\Delta_1)) + \\
&\quad (\text{cycle}) \\
&\overset{(3)}{=} \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3) + \mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1) + \mathsf{E}(\Delta_2\cdot\mathcal{D}\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}\Delta_1) + \\
&\quad \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3) + \mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1) + \mathsf{E}(\Delta_2\cdot\mathcal{E}\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}\Delta_1) + \\
&\quad (\text{cycle}) \\
&= \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3) + \mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1) - \mathsf{E}(\mathcal{D}\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}\Delta_1) + \\
&\quad \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3) + \mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1) - \mathsf{E}(\mathcal{E}\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}\Delta_1) + \\
&\quad (\text{cycle}) \\
&= \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3) + \mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1) - \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}\Delta_1\cdot\mathcal{D}\Delta_2) + \\
&\quad \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3) + \mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1) - \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}\Delta_1\cdot\mathcal{E}\Delta_2) + \\
&\quad (\text{cycle}) \\
&\overset{(\text{cycle})}{=} \mathsf{E}(\Delta_3\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}(\mathcal{E})\Delta_2 + \Delta_1\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_2}(\mathcal{E})\Delta_3 + \Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1 \\
&\quad +\Delta_3\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}(\mathcal{D})\Delta_2 + \Delta_1\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_2}(\mathcal{D})\Delta_3 + \Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1) \\
&= \mathsf{E}(\Delta_3\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}(\mathcal{E})\Delta_2 - \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_2}(\mathcal{E})\Delta_1\cdot\Delta_3 + (\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta_1)^*\Delta_2\cdot\Delta_3 \\
&\quad +\Delta_3\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}(\mathcal{D})\Delta_2 - \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_2}(\mathcal{D})\Delta_1\cdot\Delta_3 + (\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta_1)^*\Delta_2\cdot\Delta_3) \\
&= \mathsf{E}(\Delta_3\cdot(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}(\mathcal{E}) - \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta_1 + (\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta_1)^* \\
&\quad +\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}(\mathcal{D}) - \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta_1 + (\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta_1)^*)\Delta_2).
\end{aligned}$$

□

Remark 2. The right hand side of the formula

$$\begin{aligned}[\mathcal{D},\mathcal{E}](\Delta_1,\Delta_2,\Delta_3) \\ = \Delta_3 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}(\mathcal{E})\Delta_2 + \Delta_1 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_2}(\mathcal{E})\Delta_3 + \Delta_2 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1 \\ +\Delta_3 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}(\mathcal{D})\Delta_2 + \Delta_1 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_2}(\mathcal{D})\Delta_3 + \Delta_2 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1,\end{aligned} \tag{15}$$

which is part of the proof, appears as formula (7.30) in [Olv]. This formula is an identity of functionals. This definition still has the first drawback, that trivial functionals do not in general vanish identically, but only up to local divergence. The second drawback is eliminated and one can see the $(3,0)$-tensoriality of the expression. But due to the first drawback it still *not* completely easy to see that this expression is in fact a 3-vector. If we instead use Proposition 2 to define the bracket, these properties follow immediately:

Lemma 4. *The Nijenhuis-Schouten bracket satisfies the following properties:*

(i) $[\mathcal{D},\mathcal{E}]$ *is a 3-vector, i.e. is totally skew-adjoint:*
  - *(a)* $[\mathcal{D},\mathcal{E}](\Delta)$ *is a total differential operator in the source form* Δ.
  - *(b)* $[\mathcal{D},\mathcal{E}](\Delta)$ *is skew-adjoint.*
  - *(c)* $[\mathcal{D},\mathcal{E}](\Delta)\Sigma = -[\mathcal{D},\mathcal{E}](\Sigma)\Delta$ *is skew-adjoint.*

(ii) $[\mathcal{D},\mathcal{E}] = [\mathcal{E},\mathcal{D}]$.

Proof. (i.b) follows immediately from the skew-adjointness of $\mathcal{D},\mathcal{E}$ and for (i.c) we further need to notice that $\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta = (\mathsf{pr}\,\mathbf{v}_{\cdot}(\mathcal{E})\Delta)\mathcal{D}$ and (12) for functional bi-vectors i.e. skew-adjoint operators $\mathcal{K}: \mathcal{F}^1 \to \mathcal{V}^1$. □

Definition 5 (Poisson bracket). *Let* $\mathcal{D}: \mathcal{F}^1 \to \mathcal{V}^1$ *be a differential operator. The Poisson bracket of two functionals* L, P *is defined by*

$$\{L,P\} = \mathsf{E}(L)\cdot\mathcal{D}\mathsf{E}(P), \tag{16}$$

which is again a functional.

Definition 6 (Hamiltonian structure). *A differential operator* $\mathcal{D}: \mathcal{F}^1 \to \mathcal{V}^1$ *is called Hamiltonian if its Poisson bracket (16) is skew-symmetric*

$$\{L,P\} = -\{P,L\}, \tag{17}$$

and satisfies the Jacobi identity

$$\{\{L,P\},R\} + \{\{P,R\},L\} + \{\{R,P\},L\} = 0, \tag{18}$$

for all functionals L, P, R. *These are identities between functionals.*

Proposition 3. *A differential operator* $\mathcal{D}$ *is Hamiltonian, if and only if* $\mathcal{D}$ *is a 2-vector satisfying* $[\mathcal{D},\mathcal{D}] = 0$.