Supersymmetry and Supergravity

by

Julius Wess

and

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PREFACE

The strong interest with which these lectures on supersymmetry and supergravity were received at Princeton University encouraged me to make their contents accessible to a larger audience. They are not a systematic review of the subject. Instead, they offer an introduction to the approach followed by Bruno Zumino and myself in our attempt to develop and understand the structure of supersymmetry and supergravity.

This book consists of two parts. The first develops a formalism which allows us to construct supersymmetric gauge theories. The second part extends this formalism to local supersymmetry transformations.

At the end of each chapter, two papers are cited which I recommend to the reader. I am aware that this selection does not do justice to many authors who have contributed to the subject. However, I would like to draw attention to the more complete lists of references found in P. Fayet and S. Ferrara, *Supersymmetry*, Physics Reports 32C, No. 5, 1977, and P. Van Nieuwenhuizen, *Supergravity*, Physics Reports 68C, No. 4, 1981.

Throughout the text, important equations are numbered in boldface. They are collected at the end of each chapter. Exercises are also included along with each chapter; many of them contain information essential to a deeper understanding of the subject.

This book was prepared in collaboration with Jonathan Bagger, without whom it would never have been written. Both Jon and I would like to thank Winnie Waring for her devoted assistance in the preparation of the manuscript. As a tribute to her high standards, we have tried our best to avoid errors in factors and signs. Many people have helped eliminate these errors. In particular, we would like to thank Martin Müller for his assistance with the second half of the book.

I wish to express my gratitude to the Federal Republic of Germany for the grant which made possible my stay at The Institute for Advanced Study as an Albert Einstein Visiting Professor, and Jon would like to express his appreciation to the U.S. National Science Foundation for his Graduate Fellowship at Princeton University.

In conclusion, I would like to thank Stephen Adler and the Members of the Institute for Advanced Study, as well as David Gross and the Department of Physics at Princeton University, for their most encouraging and critical interest in these lectures.

JULIUS WESS
UNIVERSITY OF KARLSRUHE
MAY, 1982

CONTENTS

Supersymmetry and Supergravity

I. WHY SUPERSYMMETRY?

Supersymmetry is a subject of considerable interest among physicists and mathematicians. Not only is it fascinating in its own right, but there is also a growing belief that it may play a fundamental role in particle physics. This belief is based on an important result of Haag, Sohnius, and Lopuszanski. They proved that the supersymmetry algebra is the only graded Lie algebra of symmetries of the S-matrix consistent with relativistic quantum field theory. In this chapter, we shall discuss their theorem and its proof. (Readers specifically interested in supersymmetric theories might prefer to start directly with Chapter II or III.)

Before we begin, however, we first present the supersymmetry algebra:

$$\begin{aligned} \{Q_\alpha{}^A, \bar{Q}_{\dot\beta B}\}_+ &= 2\sigma_{\alpha\dot\beta}{}^m P_m\, \delta^A{}_B \\ \{Q_\alpha{}^A, Q_\beta{}^B\}_+ &= \{\bar{Q}_{\dot\alpha A}, \bar{Q}_{\dot\beta B}\}_+ = 0 \\ [P_m, Q_\alpha{}^A]_- &= [P_m, \bar{Q}_{\dot\alpha A}]_- = 0 \\ [P_m, P_n]_- &= 0. \end{aligned} \qquad \textbf{(I)}$$

The Greek indices ($\alpha, \beta, \ldots, \dot\alpha, \dot\beta, \ldots$) run from one to two and denote two-component Weyl spinors. The Latin indices ($m, n, \ldots$) run from one to four and identify Lorentz four-vectors. The capital indices ($A, B, \ldots$) refer to an internal space; they run from 1 to some number $N \geq 1$. The algebra with $N = 1$ is called the supersymmetry algebra, while those with $N > 1$ are called extended supersymmetry algebras. All the notation and conventions used throughout this book are summarized in Appendix A.

We are now ready to consider the theorem. Of all the graded Lie algebras, only the supersymmetry algebras (together with their extensions to include central charges, which we shall discuss at the end of the chapter) generate symmetries of the S-matrix consistent with relativistic quantum field theory. The proof of this statement is based on the Coleman-Mandula theorem, the most precise and powerful in a series of no-go theorems about the possible symmetries of the S-matrix.

The Coleman-Mandula theorem starts from the following assumptions:

(1) the S-matrix is based on a local, relativistic quantum field theory in four-dimensional spacetime;

(2) there are only a finite number of different particles associated with one-particle states of a given mass; and

(3) there is an energy gap between the vacuum and the one particle states.

The theorem concludes that the most general Lie algebra of symmetries of the S-matrix contains the energy-momentum operator P_m, the Lorentz rotation generator M_{mn}, and a finite number of Lorentz scalar operators B_ℓ. The theorem further asserts that the B_ℓ must belong to the Lie algebra of a compact Lie group.

Supersymmetries avoid the restrictions of the Coleman-Mandula theorem by relaxing one condition. They generalize the notion of a Lie algebra to include algebraic systems whose defining relations involve anticommutators as well as commutators. These new algebras are called superalgebras or graded Lie algebras. Schematically, they take the following form:

$$\{Q,Q'\}_+ = X \qquad [X,X']_- = X'' \qquad [Q,X]_- = Q''. \tag{1.2}$$

Here Q, Q', and Q'' represent the odd (anticommuting) part of the algebra, and X, X', and X'' the even (commuting) part.

The operators X are determined by the Coleman-Mandula theorem. They are either elements of the Poincaré algebra $\mathscr{P} = \{P_m, M_{mn}\}$ or elements of a Lorentz-invariant compact Lie algebra $\mathscr{A}$. The algebra $\mathscr{A}$ is a direct sum of a semisimple algebra $\mathscr{A}_1$ and an Abelian algebra $\mathscr{A}_2$, $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2$.

The generators Q may be decomposed into a sum of representations irreducible under the homogeneous Lorentz group $\mathscr{L}$:

$$Q = \sum Q_{\underline{\alpha_1 \cdots \alpha_a}, \underline{\dot\alpha_1 \cdots \dot\alpha_b}}. \tag{1.3}$$

The $Q_{\underline{\alpha_1 \cdots \alpha_a}, \underline{\dot\alpha_1 \cdots \dot\alpha_b}}$ are symmetric with respect to the underlined indices $\alpha_1 \cdots \alpha_a$ and $\dot\alpha_1 \cdots \dot\alpha_b$. They belong to irreducible spin-$\frac{1}{2}(a + b)$ representations of $\mathscr{L}$. Since the Q's anticommute, the connection between spin and statistics tells us that $a + b$ must be odd.

We shall now invoke two additional assumptions to prove that $a + b = 1$. These assumptions are:

(1) the operators Q act in a Hilbert space with positive definite metric; and

(2) both Q and its hermitian conjugate $\bar{Q}$ belong to the algebra.

We start by considering the anticommutator

$$\{Q_{\underline{\alpha_1\cdots\alpha_a},\underline{\dot{\alpha}_1\cdots\dot{\alpha}_b}},\bar{Q}_{\underline{\dot{\beta}_1\cdots\dot{\beta}_a},\underline{\beta_1\cdots\beta_b}}\}, \tag{1.4}$$

where all the indices are assigned the value 1. The product

$$Q_{\underbrace{1\cdots1}_{a},\underbrace{\dot{1}\cdots\dot{1}}_{b}}\;\bar{Q}_{\underbrace{\dot{1}\cdots\dot{1}}_{a},\underbrace{1\cdots1}_{b}} \tag{1.5}$$

belongs to a spin-$(a + b)$ representation of $\mathscr{L}$, so

$$\left\{Q_{\underbrace{1\cdots1}_{a},\underbrace{\dot{1}\cdots\dot{1}}_{b}},\bar{Q}_{\underbrace{\dot{1}\cdots\dot{1}}_{a},\underbrace{1\cdots1}_{b}}\right\} \tag{1.6}$$

must close into an even element of the algebra with spin $(a + b)$. From the Coleman-Mandula theorem, we know that this element is either zero or a component of P_m. For $a + b > 1$, it must be zero.

The anticommutator (1.6) is a positive definite operator in a Hilbert space with a positive definite metric. This tells us that $Q_{\underbrace{1\cdots1}_{a},\underbrace{\dot{1}\cdots\dot{1}}_{b}} = 0$ for $a + b > 1$. Since the $Q_{\underline{\alpha_1\cdots\alpha_a},\underline{\dot{\alpha}_1\cdots\dot{\alpha}_b}}$ are irreducible under $\mathscr{L}$, they all must vanish for $a + b > 1$. From this we conclude that the odd part of the supersymmetry algebra is composed entirely of the spin-$\frac{1}{2}$ operators $Q_\alpha{}^L$ and $\bar{Q}_{\dot{\alpha}M}$.

The anticommutator of $Q_\alpha{}^L$ and $\bar{Q}_{\dot{\alpha}M}$ closes into $P_{\alpha\dot{\alpha}}$,

$$\{Q_\alpha{}^L,\bar{Q}_{\dot{\alpha}M}\} = P_{\alpha\dot{\alpha}}C^L{}_M, \tag{1.7}$$

where $P_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}{}^m P_m$. In Exercise 1 we show that the finite-dimensional matrix $C^L{}_M$ is hermitian. It may therefore be diagonalized by a unitary transformation. Since $\{Q_1{}^L,\bar{Q}_{\dot{1}L}\}$ is positive definite, the matrix $C^L{}_M$ has positive definite eigenvalues. This lets us choose a basis in the odd part of the algebra such that

$$\{Q_\alpha{}^L,\bar{Q}_{\dot{\alpha}M}\} = 2P_{\alpha\dot{\alpha}}\delta^L{}_M. \tag{1.8}$$

We now turn our attention to the anticommutator of two odd elements, both with undotted indices. The right-hand side of this expression may be decomposed into symmetric and antisymmetric parts. The symmetric part has spin 1. From the Coleman-Mandula theorem, the only possible candidate is the Lorentz generator $M_{\underline{\alpha\beta}}$:

$$\{Q_\alpha{}^L,Q_\beta{}^M\} = \varepsilon_{\alpha\beta}X^{\widetilde{LM}} + M_{\underline{\alpha\beta}}Y^{\underline{LM}}. \tag{1.9}$$

From the fact that P_m commutes with $Q_\alpha{}^L$ (see Exercise 2), we find that the $Y^{\underset{\smile}{LM}}$ must vanish. This lets us write the commutator (1.9) as follows:

$$\{Q_\alpha{}^L, Q_\beta{}^M\} = \varepsilon_{\alpha\beta} a^{\ell,\underset{\smile}{LM}} B_\ell . \tag{1.10}$$

Here B_ℓ is a hermitian element of $\mathscr{A}_1 \oplus \mathscr{A}_2$ and $a^{\ell,\underset{\smile}{LM}}$ is antisymmetric in L and M. With this result, the supersymmetry algebra takes the following form:

$$\begin{aligned}
\{Q_\alpha{}^L, \bar{Q}_{\dot\beta M}\} &= 2\sigma_{\alpha\dot\beta}{}^m P_m \delta^L{}_M \\
[P_m, Q_\alpha{}^L] &= [P_m, \bar{Q}_{\dot\beta M}] = 0 \\
\{Q_\alpha{}^L, Q_\beta{}^M\} &= \varepsilon_{\alpha\beta} a^{\ell,\underset{\smile}{LM}} B_\ell = \varepsilon_{\alpha\beta} X^{\underset{\smile}{LM}} \\
\{\bar{Q}_{\dot\alpha L}, \bar{Q}_{\dot\beta M}\} &= \varepsilon_{\dot\alpha\dot\beta} a^*{}_{\ell,\underset{\smile}{LM}} B^\ell = \varepsilon_{\dot\alpha\dot\beta} X^+{}_{\underset{\smile}{LM}} \\
[Q_\alpha{}^L, B_\ell] &= S_\ell{}^L{}_M Q_\alpha{}^M \\
[B^\ell, \bar{Q}_{\dot\alpha L}] &= S^{*\ell}{}_L{}^M \bar{Q}_{\dot\alpha M} \\
[B_\ell, B_m] &= i c_{\ell m}{}^k B_k .
\end{aligned} \tag{1.11}$$

We shall now use the Jacobi identities to further restrict the coefficients $a^{\ell,\underset{\smile}{LM}}$ and $S_\ell{}^L{}_M$ in (1.11). The ordinary Jacobi identity may be easily extended to include anticommutators, as is done in Exercise 3:

$$\{A, \{B,C]] \pm \{B, \{C,A]] \pm \{C, \{A,B]] = 0. \tag{1.12}$$

The bracket structure { ,] signifies either commutator or anticommutator, according to the even or odd character of A, B, and C. The signs are determined by the odd elements. If the odd elements are in a cyclic permutation of the first term, the sign is positive; if not, it is negative. By exploring the Jacobi identities in a certain order, we shall arrive at our results as quickly as possible.

We first consider the identity

$$[B_\ell, \{Q_\alpha{}^L, \bar{Q}_{\dot\beta M}\}] + \{Q_\alpha{}^L, [\bar{Q}_{\dot\beta M}, B_\ell]\} - \{\bar{Q}_{\dot\beta M}, [B_\ell, Q_\alpha{}^L]\} = 0. \tag{1.13}$$

The first term vanishes because B_ℓ and P_m commute. The second and third terms give

$$-\{Q_\alpha{}^L, \bar{Q}_{\dot\beta K}\} S^{*\ell}{}_M{}^K + \{\bar{Q}_{\dot\beta M}, Q_\alpha{}^K\} S_\ell{}^L{}_K = 0, \tag{1.14}$$

or

$$2P_{\alpha\dot\beta}[S^{*\ell}{}_M{}^L - S_\ell{}^L{}_M] = 0. \tag{1.15}$$

Equation (1.15) is true only if

$$S^{*\ell}{}_M{}^L = S_\ell{}^L{}_M, \qquad \textbf{(1.16)}$$

so $S_\ell{}^L{}_M$ is hermitian.

Next we use the identity

$$[B_\ell, \{Q_\alpha{}^L, Q_\beta{}^M\}] + \{Q_\alpha{}^L, [Q_\beta{}^A, B_\ell]\} - \{Q_\beta{}^M[B_\ell, Q_\alpha{}^L]\} = 0 \qquad (1.17)$$

to prove that the generators $X^{\underset{\smile}{LM}} = a^{\ell,\underset{\smile}{LM}} B_\ell$ form an invariant subalgebra of $\mathscr{A}_1 \oplus \mathscr{A}_2$. Evaluating (1.17) with the help of (1.11), we find

$$\varepsilon_{\alpha\beta}\{[B_\ell, X^{\underset{\smile}{LM}}] + S_\ell{}^M{}_K X^{\underset{\smile}{LK}} - S_\ell{}^L{}_K X^{\underset{\smile}{MK}}\} = 0. \qquad (1.18)$$

This shows that the commutator of B_ℓ with $X^{\underset{\smile}{LM}}$ closes into the set of generators $X^{\underset{\smile}{LM}}$. The $X^{\underset{\smile}{LM}}$ are linear combinations of the B_ℓ, so we conclude that the $X^{\underset{\smile}{LM}}$ form an invariant subalgebra of $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2$.

We now use the identity

$$[Q_\alpha{}^L, \{Q_\beta{}^M, \bar{Q}_{\dot{\gamma}K}\}] + [Q_\beta{}^M, \{\bar{Q}_{\dot{\gamma}K}, Q_\alpha{}^L\}] + [\bar{Q}_{\dot{\gamma}K}, \{Q_\alpha{}^L, Q_\beta{}^M\}] = 0 \qquad (1.19)$$

to show that the generators $X^{\underset{\smile}{LM}}$ commute with all the generators of $\mathscr{A}$. Combining (1.19) with (1.11), we find

$$\varepsilon_{\alpha\beta}[\bar{Q}_{\dot{\gamma}K}, X^{\underset{\smile}{LM}}] = 0, \qquad (1.20)$$

so

$$[X^{\underset{\smile}{KN}}, X^{\underset{\smile}{LM}}] = \frac{1}{2}\varepsilon^{\beta\alpha}[\{Q_\alpha{}^K, Q_\beta{}^N\}, X^{\underset{\smile}{LM}}] = 0. \qquad (1.21)$$

This implies that the $X^{\underset{\smile}{LM}}$ form an Abelian (invariant) subalgebra of $\mathscr{A}$. Since $\mathscr{A}_1$ is semisimple, the $X^{\underset{\smile}{LM}}$ are elements of $\mathscr{A}_2$ and commute with all the generators of $\mathscr{A}$:

$$[X^{\underset{\smile}{LM}}, B_\ell] = 0. \qquad (1.22)$$

For this reason, they are called central charges. Inserting (1.22) into (1.18),

$$S_\ell{}^M{}_K X^{\underset{\smile}{LK}} - S_\ell{}^L{}_K X^{\underset{\smile}{MK}} = 0, \qquad (1.23)$$

and substituting $X^{\underset{\smile}{MK}} = a^{\ell,\underset{\smile}{MK}} B_\ell$, we find

$$S_\ell{}^M{}_K a^{k,\underset{\smile}{LK}} - S_\ell{}^L{}_K a^{k,\underset{\smile}{MK}} = 0. \qquad (1.24)$$

From the fact that $S_\ell{}^M{}_K$ is hermitian and $a_k^{\underset{\smile}{MK}}$ antisymmetric, we conclude

$$S_\ell{}^M{}_K a^{k,\underset{\smile}{KL}} = -a^{k,\underset{\smile}{MK}} S^{*\ell}{}_K{}^L. \quad \textbf{(1.25)}$$

In Exercise 4 we show that the $S_\ell{}^M{}_K$ form a representation of $A_1 \oplus A_2$. Equation (1.25) tells us that the matrices a_k intertwine the representation S_ℓ with its complex conjugate $-S_\ell{}^*$. Central charges exist only if the algebra $A_1 \oplus A_2$ permits such intertwiners. A trivial example is given by $S_\ell^{\underset{\smile}{MK}} = 0$. Another is provided by orthogonal groups, where $S_\ell = -S_\ell{}^*$. A third example is given in Exercise 5.

No further restrictions follow from the other Jacobi identities, as may be proven by checking them all. We have therefore found the most general supersymmetry algebra:

$$\begin{aligned}
[P_m, P_n] &= 0 \\
[P_m, Q_\alpha{}^L] &= [P_m, \bar{Q}_{\dot\alpha L}] = 0 \\
[P_m, B_\ell] &= [P_m, X^{\underset{\smile}{LM}}] = 0 \\
\{Q_\alpha{}^L, \bar{Q}_{\dot\alpha M}\} &= 2\sigma_{\alpha\dot\alpha}{}^m P_m \delta^L{}_M \\
\{Q_\alpha{}^L, Q_\beta{}^M\} &= \varepsilon_{\alpha\beta} X^{\underset{\smile}{LM}} \\
\{\bar{Q}_{\dot\alpha L}, \bar{Q}_{\dot\beta M}\} &= \varepsilon_{\dot\alpha\dot\beta} X^+{}_{\underset{\smile}{LM}} \\
[X^{\underset{\smile}{LM}}, \bar{Q}_{\dot\alpha K}] &= [X^{\underset{\smile}{LM}}, Q_\alpha{}^K] = 0 \\
[X^{\underset{\smile}{LM}}, X^{\underset{\smile}{KN}}] &= [X^{\underset{\smile}{LM}}, B_\ell] = 0 \\
[B_\ell, B_m] &= i c_{\ell m}{}^k B_k \\
[Q_\alpha{}^L, B_\ell] &= S_\ell{}^L{}_M Q_\alpha{}^M \\
[\bar{Q}_{\dot\alpha L}, B^\ell] &= -S^{*\ell}{}_L{}^M \bar{Q}_{\dot\alpha M} \\
X^{\underset{\smile}{LM}} &= a^{\ell,\underset{\smile}{LM}} B_\ell .
\end{aligned} \quad \textbf{(1.26)}$$

This is the most general graded Lie algebra of symmetries of the S-matrix consistent with relativistic quantum field theory. If central charges exist, they must be of the form $X^{\underset{\smile}{LM}} = a^{\ell,\underset{\smile}{LM}} B_\ell$, where a^ℓ intertwines the representations S_ℓ and $-S^{*\ell}$.

References

S. Coleman and J. Mandula, *Phys. Rev. 159*, 1251 (1967).
R. Haag, J. Lopuszanski, and M. Sohnius, *Nucl. Phys. B88*, 257 (1975).

EQUATIONS

$$\begin{aligned} \{Q_\alpha{}^A, \bar{Q}_{\dot\beta B}\}_+ &= 2\sigma_{\alpha\dot\beta}{}^m P_m \delta^A{}_B \\ \{Q_\alpha{}^A, Q_\beta{}^B\}_+ &= \{\bar{Q}_{\dot\alpha A}, \bar{Q}_{\dot\beta B}\}_+ = 0 \\ [P_m, Q_\alpha{}^A]_- &= [P_m, \bar{Q}_{\dot\alpha A}]_- = 0 \\ [P_m, P_n]_- &= 0. \end{aligned} \tag{I}$$

$$\{A, \{B, C]] \pm \{B, \{C, A]] \pm \{C, \{A, B]] = 0. \tag{1.12}$$

$$S^{*\ell}{}_M{}^L = S_\ell{}^L{}_M. \tag{1.16}$$

$$S_\ell{}^M{}_K a^{k,\underset{\smile}{KL}} = -a^{k,\underset{\smile}{MK}} S^{*\ell}{}_K{}^L. \tag{1.25}$$

$$\begin{aligned} [P_m, P_n] &= 0 \\ [P_m, Q_\alpha{}^L] &= [P_m, \bar{Q}_{\dot\alpha L}] = 0 \\ [P_m, B_\ell] &= [P_m, X^{\underset{\smile}{LM}}] = 0 \\ \{Q_\alpha{}^L, \bar{Q}_{\dot\alpha M}\} &= 2\sigma_{\alpha\dot\alpha}{}^m P_m \delta^L{}_M \\ \{Q_\alpha{}^L, Q_\beta{}^M\} &= \varepsilon_{\alpha\beta} X^{\underset{\smile}{LM}} \\ \{\bar{Q}_{\dot\alpha L}, \bar{Q}_{\dot\beta M}\} &= \varepsilon_{\dot\alpha\dot\beta} X^+{}_{\underset{\smile}{LM}} \\ [X^{\underset{\smile}{LM}}, \bar{Q}_{\dot\alpha K}] &= [X^{\underset{\smile}{LM}}, Q_\alpha{}^K] = 0 \\ [X^{\underset{\smile}{LM}}, X^{\underset{\smile}{KN}}] &= [X^{\underset{\smile}{LM}}, B_\ell] = 0 \\ [B_\ell, B_m] &= i c_{\ell m}{}^k B_k \\ [Q_\alpha{}^L, B_\ell] &= S_\ell{}^L{}_M Q_\alpha{}^M \\ [\bar{Q}_{\dot\alpha L}, B^\ell] &= -S^{*\ell}{}_L{}^M \bar{Q}_{\dot\alpha M} \\ X^{\underset{\smile}{LM}} &= a^{\ell,\underset{\smile}{LM}} B_\ell. \end{aligned} \tag{1.26}$$

EXERCISES

(1) Prove that $C^L{}_M$ in (1.7) is hermitian by comparing the anticommutator (1.7) with its hermitian conjugate.

(2) Show that $[Q_\alpha, P_m] = 0$. Start from the fact that there are no spin -$\frac{3}{2}$ generators. Deduce that $[P_{\alpha\dot\alpha}, Q_\gamma{}^L] = Z^L{}_M \varepsilon_{\alpha\gamma} \bar{Q}_{\dot\alpha}{}^M$, where the $Z^L{}_M$ are some set of numbers. Use the Jacobi identity for $[P_{\beta\dot\beta}, [P_{\alpha\dot\alpha}, Q_\gamma{}^L]]$ to prove that all the $Z^L{}_M$ vanish. This shows that the $Q_\alpha{}^L$ are translationally invariant.

(3) Prove the Jacobi identity (1.12). In particular, verify

$$[B_1, [B_2,B_3]] + [B_2, [B_3,B_1]] + [B_3, [B_1,B_2]] = 0$$
$$[Q_1, [B_2,B_3]] + [B_2, [B_3,Q_1]] + [B_3, [Q_1,B_2]] = 0$$
$$[B_1, \{Q_2,Q_3\}] + \{Q_2, [Q_3,B_1]\} - \{Q_3, [B_1,Q_2]\} = 0$$
$$[Q_1, \{Q_2,Q_3\}] + [Q_2, \{Q_3,Q_1\}] + [Q_3, \{Q_1,Q_2\}] = 0.$$

(4) Use the identity

$$[B_\ell, [B_m,Q_\alpha{}^L]] + [B_m, [Q_\alpha{}^L,B_\ell]] + [Q_\alpha{}^L, [B_\ell,B_m]] = 0$$

to prove

$$[S_m,S_\ell] = ic_{m\ell}{}^k S_k .$$

(The matrix S_ℓ has elements $S_\ell{}^K{}_M$.) Show that $-S^*_\ell$ satisfies the same commutation relations.

(5) The Pauli matrices $\boldsymbol{\sigma}$ and their conjugates $-\boldsymbol{\sigma}^*$ both form representations of SU(2). Show that ε is an intertwiner between these representations. Verify that the commutator

$$\{Q_\alpha{}^L,Q_\beta{}^M\} = \varepsilon_{\alpha\beta}\varepsilon^{LM}(c_1 Z_1 + ic_2 Z_2)$$

is consistent with the Jacobi identities if Z_1 and Z_2 are central charges.

II. REPRESENTATIONS OF THE SUPERSYMMETRY ALGEBRA

An exciting feature of the supersymmetry algebra is that there exist quantum field theories in which the supersymmetry generators Q_α may be represented in terms of conserved currents $J_\alpha{}^m$:

$$Q_\alpha = \int d^3\mathbf{x} J_\alpha{}^0$$
$$\frac{\partial}{\partial x^m} J_\alpha{}^m = 0. \tag{2.1}$$

The currents $J_\alpha{}^m$ are local expressions of the field operators. The algebra (I) is satisfied because of the canonical equal-time commutation relations, and the Hilbert space spans a representation space of the supersymmetry algebra. In this chapter we shall study the supersymmetry representations of one-particle states.

The energy-momentum four-vector P_m commutes with the supersymmetry generators Q_α and $\bar{Q}_{\dot{\alpha}}$. The mass operator P^2 is a Casimir operator, so irreducible representations of the supersymmetry algebra are of equal mass. We shall construct these irreducible representations by the method of induced representations, considering fixed time-like ($P^2 < 0$) and light-like ($P^2 = 0$) momenta.

Before we do this, however, we shall first prove that every representation of the supersymmetry algebra contains an equal number of bosonic and fermionic states. We begin by introducing a fermion number operator N_F, such that $(-)^{N_F}$ has eigenvalue $+1$ on bosonic states that -1 on fermionic states. It follows immediately that

$$(-)^{N_F} Q_\alpha = -Q_\alpha (-)^{N_F}. \tag{2.2}$$

For any finite-dimensional representation of the algebra (such that the trace is well-defined), we find

$$\begin{aligned} \mathrm{Tr}[(-)^{N_F}\{Q_\alpha{}^A, \bar{Q}_{\dot{\beta}B}\}] &= \mathrm{Tr}[(-)^{N_F}(Q_\alpha{}^A \bar{Q}_{\dot{\beta}B} + \bar{Q}_{\dot{\beta}B} Q_\alpha{}^A)] \\ &= \mathrm{Tr}[-Q_\alpha{}^A(-)^{N_F}\bar{Q}_{\dot{\beta}B} + Q_\alpha{}^A(-)^{N_F}\bar{Q}_{\dot{\beta}B}] \\ &= 0. \end{aligned} \tag{2.3}$$