Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

781

Harmonic Analysis Iraklion 1978

Proceedings

Edited by
N. Petridis, S. K. Pichorides, and N. Varopoulos

Springer-Verlag
Berlin Heidelberg New York

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

781

Harmonic Analysis Iraklion 1978

Proceedings of a Conference Held at the University of Crete, Iraklion, Greece, July 1978

Edited by
N. Petridis, S. K. Pichorides, and N. Varopoulos

Springer-Verlag
Berlin Heidelberg New York 1980

Editors

Nicholas Petridis
Eastern Illinois University
Department of Mathematics
Charleston, IL 61920
USA

Stylianos K. Pichorides
N. R. C. Demokritos
Aghia Paraskevi
Attikis
Greece

Nicolas Varopoulos
Department of Mathematics
University of Paris XI
Orsay 91
France

AMS Subject Classifications (1980): 42-XX, 43-XX

ISBN 3-540-09756-2 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-09756-2 Springer-Verlag New York Heidelberg Berlin

Library of Congress Cataloging in Publication Data. Symposium on Harmonic Analysis, University of Crete, 1978. Harmonic analysis, Iraklion 1978. (Lecture notes in mathematics; 781) Bibliography: p. Includes index. 1. Harmonic analysis--Congresses. I. Petridis, N. II. Pichorides, S. K., 1940- III. Varopoulos, N., 1940- IV. University of Crete. V. Title. VI. Series: Lecture notes in mathematics (Berlin); 781. QA3.L28 no. 781 [QA403] 510s [515'.2433] 80-10989
ISBN 0-387-09756-2

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2141/3140-543210

FOREWORD

This volume represents the talks delivered at the Symposium on Harmonic Analysis, held at the University of Crete, in Iraklion, Greece, the first week of July 1978.

The conference was organized by the newly founded University of Crete on the occasion of its first anniversary.

The manuscripts of the lectures are published here, as supplied to us by the speakers, except for retyping to make them uniform in appearance.

The common feature of these lectures is that either they strictly belong to Harmonic Analysis (classical and abstract) or they use methods belonging to it.

We believe that we express the feelings of all participants if we extend our thanks not only to our host, the University of Crete, but also to a number of local communities (Iraklion, Aghios Nikolaos, Acharnes, Anogia, etc.) which transformed their love for and their faith in the new University to an unforgettable hospitality for its guests.

We also wish to thank

- The Ministry of Science and Culture,
- The Mayor and the Town Council of Iraklion,
- The National Tourist Organization of Greece,

for financial support.

The Editorial Committee

N. Petridis, S. Pichorides, N. Varopoulos

CONTENTS

+ To appear elsewhere.

CRITERIA FOR ABSOLUTE CONVERGENCE OF FOURIER SERIES

Nicolas K. Artémiades

To give, even a partly expository, talk to an audience containing such a number of experts does not seem to be an easy task. I am afraid certain people will hear the speaker explaining their theorems to them. But that will just have to be.

1. <u>INTRODUCTION</u>

One of the primary objectives in the theory of Fourier series is the study of the class $\mathcal{A}$ of all Lebesgue integrable complex-valued functions on the circle T (the additive group of the reals modulo 2π) whose Fourier series are absolutely convergent. Let A be the set of all continuous functions which belong to $\mathcal{A}$. It is well known that A is a Banach space under pointwise linear operations and the norm $||f||_A = \sum_{n\varepsilon Z} |\hat{f}(n) < +\infty$, where $\hat{f}(n)$ is the n^{th} Fourier coefficient of the function $f \varepsilon A$. Also, A is an algebra under pointwise multiplication and $||fg||_A \leq ||f||_A \, ||g||_A$. This means that A is a commutative Banach algebra with the constant function 1 as its identity element.

The Banach Algebra structure of A (due to N. Wiener) suggests a great number of problems which constitute the so called "modern approach" to the study of A. For example, one of these problems on which attention has been concentrated is to find "under what conditions upon the function F, defined on some subset D of the complex plane, is it true that $F \circ f \varepsilon A$ whenever $f \varepsilon A$ and $f(\mathbb{R})$ D ?" Another problem is to "determine the closed ideals of A". A particular case of this last problem is the so called "Problem of Spectral Synthesis" which can be formulated as follows: "Is every closed ideal of A of the form I_E, where I_E is the closed ideal of A formed by all functions in A which vanish on the closed set E ?" A negative answer to this question was given in 1959 by P. Malliavin.

But I will not continue further towards the direction suggested by the Banach Algebra structure of A.

The classical approach to the study of A has in the main concentrated attention on seeking conditions on an individual function f, which are sufficient and/or necessary to ensure that $f \in A$. Wiener proved that the property of a continuous function on T to belong to A is a local one. This simply means that if f is continuous on T and if for each $a \in T$ there exists a function $g_a \in A$ which is equal to f in some neighborhood of a, then $f \in A$. In the classical approach, emphasis is given to comparing this local property to other properties, as for example is the modulus of continuity. Into this direction of research fall developments concerning the restrictions of the class A, noted A(E), to closed subsets E of the circle. There are closed subsets of the circle (called Helson sets) such that every continuous function on E belongs to A(E). In general, it is true that more E is "fat" more severe is the condition to be imposed on the modulus of continuity of a function f in order that $f \in A(E)$.

In many instances, the study of a problem in A is facilitated if it is transferred to an A(E) for a certain E.

To finish with this very brief expository part of the article I would like to mention two well known criteria for a function f to be in .

Criterion of M. Riesz. $f \in A$ iff it can be expressed in the form $f = u*\upsilon$ with $u,\upsilon \in L^2(T)$.

Unfortunately, this criterion is very difficult to apply in any specific case that is not already decidable on more evident grounds.

Steckin's Criterion. For every $f \in L^2(T)$ and every integer $n \geq 0$ set

$$e_n(f) = \inf ||f-P||_{L^2(T)}$$

where the infimum is taken over all trigonometric polynomials with at most n coefficients different from zero ($P(t) = \sum_{m=1}^{n} \gamma_m e^{i\lambda_m t}$). We have $\sum_{n \in Z} |\hat{f}(n)| < +\infty$ iff $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} e_n(f) < +\infty$.

The main drawback with this criterion is the extreme difficulty encountered in estimating the numbers $e_n(f)$ for a given function.

2. Some other criteria

It is well known (Kahane [2], p. 9) that "every continuous function on T with non negative Fourier coefficients belongs to A".

This result can easily be generalized as follows:

Theorem 1. Let $f: T \to \mathbb{C}$ be continuous with the property that there is a $a \in \mathbb{R}$ such that $a \leq \arg \hat{f}(n) \leq a + \frac{\pi}{2}$ ($n \in Z$). Then $f \in A$.

Also every $f \in A$ is a linear combination of continuous functions on T with the above property.

Proof. The second part of the theorem is obvious. To prove the first part let us assume, without loss of generality, that $a = 0$. For if $a \neq 0$ we may consider $g(x) = f(x) \cdot e^{-ia}$ instead of f, since $0 \leq \arg \hat{g}(n) \leq \frac{\pi}{2}$. Next, set

$$F(x) = \frac{f(x) + \overline{f(-x)}}{2}, \quad G(x) = \frac{f(x) - \overline{f(-x)}}{2i}.$$

Clearly, both F and G are continuous and $\hat{F}(n) = \mathrm{Re}\hat{f}(n) \geq 0$, $\hat{G}(n) = \mathcal{J}m\hat{f}(n) \geq 0$, so that by the previous result F and G belong to A which means $f \in A$ since A is a linear space.

Theorems 2 and 3 below provide criteria for f to be in A.

Theorem 2. Let $f \in L^1(T)$. Then $f \in A$ iff the following condition is satisfied.

"There is a Lebesgue point, a, for f such that the sequences

(*) $\langle(\mathrm{Re}\,\hat{f}_a(n)^-\rangle_{n\in Z}$, $\langle(\ m\,\hat{f}_a(n))^-\rangle_{n\in Z}$

belong to ℓ'.

<u>Proof</u>. Suppose (*) is satisfied. We first consider the case $a=0$. For N a positive integer set $\sigma_N(t) = \sum_{n=-N}^{N} (1-\frac{n}{N})\hat{f}(n)e^{int}$.

By a theorem of Lebesgue we have that $\lim_{N\to\infty}\sigma_N(t) = f(t)$, if f is a Lebesgue point for f. Hence

$$(1)\qquad \lim_{N\to\infty}\sigma_N(0) = \lim \sum_{n=-N}^{N} (1-\frac{|n|}{N})\hat{f}(n) = \hat{f}(0) = \text{finite.}$$

Also

$$(2)\qquad {}_N(0) = \sum_{n=-N}^{N} (1-\frac{|n|}{N})\hat{f}(n) =$$

$$= \sum_{n=-N}^{N} (1-\frac{|n|}{N})(\mathrm{Re}\hat{f}(n))^+ + i \sum_{n=-N}^{N} (1-\frac{n}{N})\ (\mathcal{J}\,m\hat{f}(n))^+$$

$$- \sum_{n=-N}^{N} (1-\frac{|n|}{N})(\mathrm{Re}\hat{f}(n))^- - i \sum_{n=-N}^{N} (1-\frac{|n|}{N})(\mathcal{J}\,m\hat{f}(n))^- .$$

If we let $N\to\infty$ then the $\sigma_N(0)$ are uniformly bounded because of (1), while the last two sums of the right-hand side of (2) are bounded (more precisely they converge because of the hypothesis (*)). Therefore

$$\lim_N \sum_{n=-N}^{N} (1-\frac{n}{N})(\mathrm{Re}\hat{f}\ (n))^+ < +\infty$$

$$\lim_N \sum_{n=-N}^{N} (1-\frac{n}{N})(\mathcal{J}\,m f(n))^+ < +\infty$$

Since the Cesãro summability of a series with non-negative terms implies the convergence of the series, it follows that $\sum_{n\in Z}|\hat{f}(n)| < +\infty$, i.e. $f\in A$.

Next assume $a\neq 0$. Then 0 is a Lebesgue point for f_a so that, by the last result, we have $\sum_{n\in Z}|\hat{f}_a(n)| < +\infty$. But $\hat{f}_a(n) = e^{ian}\hat{f}(n)$ so that $f \in A$.

Now, if $f \in A$ and a is any Lebesgue point for f we have $\sum_{n \in Z} |\hat{f}(n)| = \sum_{n \in Z} |e^{ian} \hat{f}(n)| = \sum_{n \in Z} |\hat{f}_a(n)| < +\infty$ and condition (*) is clearly satisfied.

Corollary 1. Let $f \in L^1(T)$. Then f is equal a.e to a linear combination of positive definite functions iff condition (*) is satisfied.

Proof. It follows from Theorem 2 and Herglotz's characterization of continuous functions with non-negative coefficients as positive definite.

Theorem 3. Let $f \in L^1(T)$. Then $f \in A$ iff the following condition is satisfied:

"f is ess. bounded in a neighborhood of some real number a, and both sequences

(**) $\langle (\mathrm{Re}\, f_a(n))^- \rangle_{n \in Z}$, $(\mathcal{J}m\, f_a(n))^- \rangle_{n \in Z}$ belong to ℓ^1."

Proof. Suppose (**) is satisfied and let $a = 0$. Using the notation of Theorem 2 we have:

$$\sigma_N(t) = \frac{1}{2} \int_T f(y) K_N(t-y)\, dy$$

where $K_N(y) = \sum_{n=-N}^{N} (1 - \frac{|n|}{N}) e^{iny} = \frac{\sin^2(N|2)y}{N \sin^2(y|2)}$.

Next assume $|f(y)| \leq M$ a.e for $y \in (-h,h)$ $(h > 0)$.

We have

$$\sigma_N(0) = \frac{1}{2\pi} \int_{-h}^{h} f(y) K_n(y)\, dy + \frac{1}{2\pi} \int_{-h}^{h} (\ ) + \frac{1}{2\pi} \int (\ ).$$

Observe that the first of the last three integrals is bounded by M, and the other two converge to zero as $N \to \infty$ by the Lebesgue dominated convergence theorem. Therefore the $\sigma_N(0)$ are uniformly bounded.

From this point on the proof proceeds exactly the same way as in Theorem 2., that is by letting $N \to \infty$ in (2).

Corollary 2. Let $f \in L^1(T)$. Then f is equal a.e. to a linear combination of positive definite functions iff (**) is satisfied.

Using a technique similar to the one used above one proves the following analogues of theorems 2 and 3.

Theorem 2'. Let $f \in L^1(\mathbb{R})$. Then $\hat{f} \in L^1(\mathbb{R})$ (where $\hat{f}$ is the Fourier transform of f) iff there is a Lebesgue point a for f such that $(\mathrm{Re}\hat{f}_a)^-$ $(\mathcal{J}m\,\mathrm{Re}\hat{f}_a)^-$ belong to $L^1(\mathbb{R})$.

Theorem 3'. Let $f \in L^1(\mathbb{R})$. Then $\hat{f} \in L^1(\mathbb{R})$ iff f is essentially bounded in a neighborhood of some real number a and $(\mathrm{Re}\hat{f}_a)^-$, $(\mathcal{J}m\hat{f}_a)^-$ belong to $L^1(\mathbb{R})$.

One might find theorems 2 and 3 interesting also because of the following remark.

Remark

Call a numerical series $\Sigma a_n + ib_n$, $(a_n, b_n \in \mathbb{R})$ "one sidedly absolutely convergent" (O.A.C.), iff: (at least one of Σa_n^+, Σa_n^-) and (at least one of Σb_n^+, Σb_n^-) is finite.

Now, it is possible that a series $\Sigma(a_n + ib_n)$ is not OAC while the series $\Sigma(a_n + ib_n)e^{i\lambda}$ is OAC. In other words a non OAC series can, in some cases, be converted to an OAC series by just multiplying each term by a factor of the form $e^{i\lambda}$ (λ = some constant) or perhaps in some other way.

Example: Let $c_n = a_n + ib_n$ where $c_{2n} = 1+i$, $c_{2n+1} = 1-i$, $n = 0,1,2,\ldots$ and $\lambda = \frac{\pi}{4}$. Then it is easily seen that Σc_n is not OAC while $\Sigma c_n e^{i\lambda}$ is.

Theorems 2 and 3 essentially say that the Fourier series of f converges absolutely iff $\Sigma\hat{f}_a(n)$ is OAC.

REFERENCES

[1] Artémiades, N. "Criteria for absolute convergence of Fourier series", Proc. Amer. Math. Soc. 50 (1975) 179-183.

[2] Kahane, J.-P., "Séries de Fourier absolument convergentes", Ergebnisse der Math. 50, Springer-Verlag (1970).

[3] Zygmund, A., Trigonometric Series (2nd edition) 2 vols., Cambridge, England, 1959.

FRACTIONAL CARTESIAN PRODUCTS IN HARMONIC ANALYSIS

by

Ron C. Blei(*)

The Hebrew University and The University of Connecticut

Our purpose here is to explain the fractional cartesian products of [1] which naturally filled gaps between ordinary cartesian products of sets in a framework of harmonic analysis. The idea of fractional powers of sets appears fairly general and we would first like to describe briefly--taking a somewhat metamathematical point of view--the philosophy behind these products. Let E be a given set. Let L be a positive integer, Y be a fixed indexing space and $\{f_i\}_{i=1}^{L}$ be a collection of functions from Y onto E. Consider now the following subset of the usual L-fold product of E:

$$E_{(f_i)} = \{(f_i(y))_{i=1}^{L} : y \in Y\} \subset E^L$$

If the f_i's are 'independent' (for any $x_1,\ldots,x_L \in E$, the system of equations $f_i(y) = x_i$, $i = 1,\ldots,L$, has a solution in Y) then $E_{(f_i)} = E^L$. On the other extreme, if the f_i's are mutually 'dependent'

$$(f_i(y_1) = f_i(y_2) \Rightarrow f_j(y_1) = f_j(y_2) \text{ for any } f_i, f_j \text{ and } y_1, y_2 \in Y)$$

then $E_{(f_i)}$ can be canonically identified with E. If, however, the type of interdependencies between the f_i's falls somewhere between independence and mutual dependence, $E_{(f_i)}$ is then a set that falls somewhere between E and E^L.

To see how to formulate an intermediate type of interdependencies we observe that independence and dependence can be measured in

---

(*) Author was supported partially by NSF Grant MCS 76-07135.

the following way. First, by replacing Y with an appropriate quotient of Y, we can assume without loss of generality that $y \to (f_i(y))_{i=1}^{L}$ is an injection. Let s be a positive integer, and $A_1, \ldots, A_L \subset E$ be arbitrary where $|A_1| = \cdots = |A_L| = s$ ($|\cdot|$ denotes cardinality). Write:

$$\phi_{(A_i)}(s) = |\{y \varepsilon Y : f_1(y) \varepsilon A_1 \text{ and } \ldots \text{ and } f_L(y) \varepsilon A_L\}|.$$

Note that if the f_i's are independent then

$$\phi_{(A_i)}(s) = s^L;$$

on the other hand, if the f_i's are mutually dependent then

$$\phi_{(A_i)}(s) \leq s .$$

An intermediate interdependency for $\{f_i\}_{i=1}^{L}$ that corresponds to $1 < r < L$ can be described by the relation (asymptotic in s).

$$(1) \quad \psi(E_{(f_i)};s) = \sup\{\phi_{(A_i)}(s) : A_1, \ldots, A_L \subset E,\ |A_1|, \ldots, |A_L| \leq s\} \sim s^r.$$

This is the basic idea underlying the fractional products of [1] where prescribed interdependencies between concrete f_i's simulated the desired fractional power of a set.

We now move to a harmonic analytic context, where we start with $E = \{\gamma_i\}_{i=1}^{\infty}$, an infinite independent set in some discrete abelian group Γ; that is, for any L, L' > 0 the relation

$$\prod_{j=1}^{L} \gamma_j^{\lambda_j} = \prod_{j=1}^{L'} \gamma_j^{\nu_j}$$

where the λ_j's and ν_j's are arbitrary integers, implies that $L = L'$ and $\lambda_j = \nu_j$ for all j. For example, E could be the canonical basis in $\bigoplus Z$ (the infinite direct sum of Z) whose compact dual group is $\bigotimes T$ (the infinite direct product of T). We proceed to construct a fractional cartesian product of E. Let $J \geq K > 0$ be given integers, and let

$$\mathcal{J} = \{1,\ldots,J\}.$$

For the sake of typographical convenience, we write $N = \binom{J}{K}$. Let

$$\{S_1,\ldots,S_N\}$$

be the collection of all K-subsets of $\mathcal{J}$ (sets containing K elements of $\mathcal{J}$), where each $S_\alpha \subset \mathcal{J}$ is enumerated as

$$S_\alpha = (\alpha_1,\ldots,\alpha_K).$$

Let $P_1,\ldots,P_N$ be the projections from $(Z^+)^J$ onto $(Z^+)^K$ defined as follows: For $1 \leq \alpha \leq N$ and $j = (j_1,\ldots,j_J) \in (Z^+)^J$,

$$P_\alpha(j) = (j_{\alpha_1},\ldots,j_{\alpha_K}).$$

Next, let f be any one-one function from $(Z^+)^J$ onto E, and

$$f_\alpha = f \circ P_\alpha : (Z^+)^J \to E;$$

write

$$E_{(f_\alpha)} = E_{J,K} = \{(f_1(j),\ldots,f_N(j)) : j \in (Z^+)^J\} \subset E^N \subset \Gamma^N.$$

The outstanding feature of $E_{J,K}$ is that

$$\psi(E_{J,K};s) \sim s^{J/K}$$

(see (1) for the definition of ψ), which is, in fact, an analogue of a basic harmonic analytic (or probabilistic) property of $E_{J,K}$ that will now be discussed. First, we recall that a spectral set $F \subset \Gamma$ is a $\Lambda(p)$ set, $2 < p < \infty$, if there is a constant $A > 0$ so that for all functions $h \in L^2(G)$ whose spectrum is in F ($G = \Gamma\hat{}$), we have

$$(2) \qquad A\|h\|_2 \geq \|h\|_p .$$

The 'smallest' A for which (2) holds is the $\Lambda(p)$ constant of F and is denoted by $A(p,F)$.

Definition. Let $\beta \in [1,\infty)$ $F \subset \Gamma$ is a Λ^β set if $A(p,F)$ is $O(p^{\beta/2})$. F is said to be exactly-Λ^β when F is Λ^a if and only if $a \in [\beta,\infty)$, and exactly non-Λ^β when F is Λ^a if and only if $a \in (\beta,\infty)$.

J-fold cartesian products of independent sets are the prototypical examples that are exactly Λ^J (see [2]). The gaps that were kept open between J and J + 1 are neatly filled, as we are about to see, by the fractional products that have just been defined.

Theorem. Let $E \subset \Gamma$ be an independent set. Then, $E_{J,K} \subset \Gamma^N$ is exactly $\Lambda^{J/K}$, and, moreover, there is $\eta_{J,K} > 0$ so that for all $q > 2$

$$(*) \qquad \eta_{J,K}\, q^{J/2K} \leq A(q,E_{J,K}) \leq q^{J/2K} .$$

To avoid a fog of indices, we sketch the proof of the theorem in the case $J = 3$ and $K = 2$ -- the general case follows a similar line.

The right hand inequality in (*) is based on a simple combinational criterion that is a link between the algebraic structure of a spectral set and its harmonic analytic features. Let F be a subset of Γ, s be a positive integer and $\gamma \in \Gamma$. Let $r_s(F,\gamma)$ denote the number of ways to write γ in the form of

(3) $$\gamma = \gamma_1 \cdots \gamma_s ,$$

where $\gamma_1, \ldots, \gamma_s$ are (not necessarily distinct) elements in F, and where different permutations on the right hand side of (3) are counted as different representations. An application of Plancherel formula and the Schwartz inequality yields

(4) $$A(2s,F) \leq \sup \{ [r_s(F,\gamma)]^{1/2s} : \gamma \in \Gamma \}$$

(see Théorème 3 in [2]).

We now present $E_{3,2} \subset \Gamma^3$ as

$$E_{3,2} = \{[\gamma_{ij}, \gamma_{jk}, \gamma_{ik}) : i,j,k = 1, \ldots\}$$

where $\{\gamma_{ij}\}_{i,j=1}^{\infty}$ is some fixed enumeration of our independent set E, and proceed to estimate $r_s(E_{3,2},\delta)$ for any given $(\delta_1,\delta_2,\delta_3) = \delta \in \Gamma^3$. Suppose that

(5) $$(\delta_1,\delta_2,\delta_3) = (\prod_{n=1}^{s} \gamma_{i_n j_n}, \prod_{n=1}^{s} \gamma_{j_n k_n}, \prod_{n=1}^{s} \gamma_{i_n k_n}) .$$

The independence of E implies that the only way that δ can be obtained as a product of s elements from $E_{3,2}$ is for these elements to have in their first, second and third coordinates the members of E that appear in the first, second and third coordinates of (5), respectively. Let

$$A_1 = \{(i_1 j_1), \ldots, (i_s j_s)\} ,$$

$$A_2 = \{(j_1 k_1), \ldots, (j_s k_s)\} ,$$

$$A_3 = \{(i_1 k_1), \ldots, (i_s k_s)\} ,$$