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Global Differentiable Dynamics

Proceedings of the Conference
held at Case Western Reserve University,
Cleveland, Ohio, June 2–6, 1969

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FOREWORD

Early in 1969, the Mathematics Department of Case Western Reserve University was given funds by the National Science Foundation to conduct one of the first of the regional conferences devoted to a special topic in mathematics. Because of the presence in the department of a strong group in dynamical systems, with Professor Otomar Hájek at the fore, it was decided that the regional conference would be devoted to global differentiable dynamics.

Since one of the objectives of the regional conference program of the National Science Foundation is to enrich the research and educational capabilities of mathematicians of a specific region, it is clear that most of the participants would be drawn from the particular region in which the National Science Foundation invests its funds. However, it is clear that, without some catalyst from outside the particular region, such a regional conference could only bring together mathematicians who would discuss among themselves the same problems on which they had been working. Without the introduction of the outside lecturer to draw together the results of a large discipline and to outline the problems of the next few years, the regional conference would do nothing more than draw together the same people who meet regularly at the regional meetings of the American Mathematical Society.

The Regional Conference on Global Differentiable Dynamics, held at Case Western Reserve University in Cleveland, Ohio, 2-6 June 1969 was fortunate to have Professor Lawrence Markus of the University of Minnesota as its principal speaker; Professor Markus gave ten lectures in five days on global differentiable dynamics. Realizing the importance of new ideas and different points of view, the

Mathematics Department of Case Western Reserve University supplemented the grant of funds by the National Science Foundation, and invited other leading mathematicians from outside the region to give hour addresses complementing the lectures of Professor Markus. Professor Joseph Auslander of the University of Maryland spoke on the structure and homomorphisms of minimal sets, Professor M. L. Cartwright of Cambridge University and Case Western Reserve University spoke on the basic frequencies of almost periodic flows, Professor Walter Gottschalk of Wesleyan University spoke on ambits, Professor G. S. Jones of the University of Maryland spoke on periodic and near-periodic flows, and Professor Emilio Roxin of the University of Rhode Island spoke on differential games of pursuit.

The present volume comprises the invited addresses, together with many of the papers given by the participants in the conference. The major exception is that of the lectures of Professor Markus which, at the request of the National Science Foundation, were to be published elsewhere in the form of a monograph (cf. Lawrence Markus, Lectures in Differentiable Dynamics, Regional Conf. Series in Math. No. 3, Amer. Math. Soc., Providence, 1971).

Both Professor Hájek, who was the principal organizer and director of the affairs of the conference, and I believe that the conference was far more successful than we had hoped for, our criteria being the scientific interaction of the participants, the informality and depth of the discussions among the participants, and, most significant of all, the initiation of mathematical collaboration among several of the participants. The lectures of Professor Markus and the other invited speakers mentioned above were of the highest caliber and served to stimulate the interest of all the participants. Special thanks are due to the secretarial staff of the Mathematics

Department, as well as to many members of the academic staff, who helped in the arrangements for the conference.

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FLOWS OF CHARACTERISTIC 0^+

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1. INTRODUCTION

The purpose of this paper is to classify a certain class of dynamical systems on the plane; namely, those in which all closed positively invariant sets are positively D-stable, i.e. stable in Ura's sense (see [11]). Such flows are called flows of characteristic 0^+. In Section 2 we give some of the basic definitions and notations that are used throughout the paper. In Section 3 we prove some results of a more general nature which are later applied to flows of characteristic 0^+ on the plane. It is proved that if the phase space X of a flow is normal and connected and the set of critical points S is globally + asymptotically stable, then S is connected. Further, if the phase space X of a flow of characteristic 0^+ is connected and locally compact, then a compact subset M of X is a positive attractor implies that M is globally + asymptotically stable.

In Section 4 we discuss flows of characteristic 0^+ on the plane. Three mutually exclusive and exhaustive cases are considered. It is shown that if the set of critical points S of such a flow is empty, then the flow is parallelizable. If S is compact, then it either consists of a single point which is a Poincaré center, or it is globally + asymptotically stable. If S is not compact, then either $R^2 = S$, or S is + asymptotically stable; S and the region of positive attraction $A^+(S)$ of S, each has a countable number of components. Further, each component of $A^+(S)$ is homeomorphic to R^2. At the end of this section, we summarize all the results of this section in the form of a complete classification of such flows.

In Section 5 we discuss flows of characteristic $0^{\pm}$ on the plane, i.e., those in which every closed invariant set is positively and negatively stable in Ura's sense. We prove that such a flow is either parallelizable, or it has a single critical point which is a global Poincaré center, or all points are critical points.

2. NOTATIONS AND DEFINITIONS

Let R, R^+, and R^- denote the sets of real numbers, non-negative, and non-positive real numbers, respectively. Given a topological space X and a mapping π of the product space $X \times R$ into X, we say (X,π) defines a dynamical system or flow on the phase space X if the following conditions are satisfied.

1. Identity axiom: $\pi(x,0) = x$.
2. Homomorphism axiom: $\pi(\pi(x,t),s) = \pi(x,s+t)$.
3. Continuity axiom: π is continuous on $X \times R$.

For brevity, we denote $\pi(x,t)$ by xt. For each $x \in X$, we let $C(x)$ denote the trajectory or orbit through x, i.e. $C(x) = xR$. Similarly, the positive and negative semi-trajectories through x are represented by $C^+(x)$ and $C^-(x)$, respectively, i.e. $C^+(x) = xR^+$ and $C^-(x) = xR^-$. We let $L^+(x)$ denote the positive (or ω -) limit set of x, i.e. $L^+(x) = \cap\{\overline{C^+(xt)}: t \in R\}$. Similarly, $L^-(x)$ denotes the negative (or α -) limit set of x. A point x

is called a critical or rest point if $xR = x$. A subset M of X is said to be invariant if $C(M) = M$, and positively (negatively) invariant if $C^+(M) = M$ ($C^-(M) = M$). A closed invariant set M is minimal if it has no proper subset which is closed and invariant.

Throughout this paper, we use ∂M and $\overline{M}$ to represent the boundary and closure of M. Given a Jordan curve C on the plane R^2, we let $\text{int}(C)$ denote the bounded component of $R^2 - C$. Let $(R^2)^* = R^2 \cup \{\omega\}$ be the one point compactification of the plane.

A closed invariant set M is said to be positively Liapunov stable, or more simply, positively stable, if for every neighborhood U of M, there exists a neighborhood V of M such that $C^+(V) \subset U$. M is said to be a positive attractor if there exists a neighborhood U of M such that $L^+(U) \subset M$. The largest such neighborhood U is called the region of positive attraction of M and will be denoted by $A^+(M)$. M is said to be + asymptotically stable if it is both positively stable and a positive attractor. It is said to be globally + asymptotically stable if it is + asymptotically stable and $A^+(M) = X$.

For each $x \in X$, the (first) positive (negative) prolongation $D^+(x)$ ($D^-(x)$) of x is given by

$$D^+(x) = \bigcap_{N \in \eta(x)} \overline{\{C^+(N)\}} \qquad (D^-(x) = \bigcap_{N \in \eta(x)} \overline{\{C^-(N)\}}),$$

where $\eta(x)$ is the neighborhood filter of x.

The (first) positive (negative) prolongational limit set of x is given by

$$J^+(x) = \bigcap_{t \in R} \{D^+(x)\} \qquad (J^-(x) = \bigcap_{t \in R} \{D^-(xt)\}).$$

It is known and easy to verify that $L^+(x) \subset J^+(x)$. Further, if X is a Hausdorff space, then $D^+(x) = C^+(x) \cup J^+(x)$.

A closed invariant set M is said to be <u>positively</u> D-<u>stable</u> if $D^+(M) = M$. (The theory of prolongation and D-stability is due to Ura (see [11], [12], and [13]). Ura [11] refers to D-stability as <u>stability</u> and to Liapunov stability as L-<u>stability</u>.)

It is easy to verify that if X is locally compact and a closed invariant set M is stable (i.e. stable in Liapunov's sense as defined above), it is also D-stable. The converse is false.

The following theorem, which we use several times in this paper, is due to Ura [11].

THEOREM (URA). Let (X,π) be a dynamical system on a locally compact space X, and let M be a compact subset of X. Then M is positively stable if and only if it is positively D-stable.

REMARK. The statement "X is locally compact" is used in the Bourbaki sense throughout this paper, i.e. X is assumed to be a Hausdorff space.

3. FLOWS OF CHARACTERISTIC 0^+

Before discussing flows of characteristic 0^+, we prove a lemma and a proposition concerning flows in general.

LEMMA 1. Let (X,π) be any dynamical system. If $x \in X$ and $y_1, y_2 \in L^+(x)$, then $y_1 \in D^+(y_2)$ and $y_2 \in D^+(y_1)$.

<u>Proof</u>. We note that

$$D^+(y_1) = \bigcap_{N\in\eta(y_1)} \overline{\{C^+(N)\}},$$

where $\eta(y_1)$ denotes the neighborhood filter of y_1. Since $y_1, y_2 \in L^+(x)$, for each $N \in \eta(y_1)$ and $M \in \eta(y_2)$, there exist $t_1, t_2 \in R^+$ with $xt_1 \in N$ and $(xt_1)t_2 \in M$. Hence $y_2 \in \overline{C^+(N)}$, and consequently, $y_2 \in D^+(y_1)$. Similarly, $y_1 \in D^+(y_2)$.

PROPOSITION 3.1. Let (X,π) be a dynamical system on a normal (and Hausdorff) connected topological space X. If a closed invariant subset F of X is globally + asymptotically stable, then F is connected.

<u>Proof</u>. Suppose F is not connected. Then there exist two non-empty disjoint closed sets F_1 and F_2 such that $F = F_1 \cup F_2$. Since X is normal, there exist two disjoint open neighborhoods U_1 and U_2 of F_1 and F_2, respectively. On the other hand, since F is positively stable, corresponding to the neighborhood $U = U_1 \cup U_2$ of F, there is an open neighborhood V of F such that $C^+(V) \subset U$. Therefore, if we let $V_i = V \cap U_i$, $i = 1,2$, then for each $x \in V_i$, $C^+(x) \subset U_i$ since $C^+(x)$ is connected. Thus, $L^+(x) \subset F_i$, i.e. $V_i \subset A^+(F_i)$ since $U_i \cap F = \emptyset$, $i \neq j$. Hence, we have shown that F_1 and F_2 are positive attractors; consequently $A^+(F_1)$ and $A^+(F_2)$ are open, since the boundary of each is closed and invariant. But this contradicts the assumption that X is connected, since $X = A^+(F) = A^+(F_1) \cup A^+(F_2)$, where $A^+(F_1)$ and $A^+(F_2)$ are clearly

non-empty disjoint open sets. This completes the proof of Proposition 3.1.

DEFINITION 3.1. A dynamical system (X,π) is said to have characteristic 0^+ if and only if $D^+(x) = \overline{C^+(x)}$ for all $x \in X$.

The above definition is equivalent to saying that (X,π) has characteristic 0^+ if and only if every closed positively invariant subset of X is positively D-stable.

It follows that if the phase space X of a flow of characteristic 0^+ is a Hausdorff space, then $D^+(x) = C^+(x) \cup L^+(x)$, for all $x \in X$.

LEMMA 2. Let (X,π) be a flow of characteristic 0^+. If $x \in X$ such that $L^-(x) \neq \emptyset$, then $x \in L^-(x)$.

Proof. Suppose $L^-(x) \neq \emptyset$ and let $y \in L^-(x)$. Then, $y \in D^-(x)$, and hence $x \in D^+(y) = \overline{C^+(y)}$. On the other hand, $y \in L^-(x)$ implies that $\overline{C^+(y)} \subset L^-(x)$, since $L^-(x)$ is a closed invariant set. Therefore, $x \in L^-(x)$.

PROPOSITION 3.2. Let (X,π) be a flow of characteristic 0^+ on a connected locally compact space X. If M is a compact positively invariant subset of X and M is a positive attractor, then M is globally + asymptotically stable.

Proof. Since M is a closed positively invariant set, we have $D^+(M) = M$. Therefore, M is positively stable, by Ura's Theorem. It is sufficient to show that $\partial A^+(M) = \emptyset$. Suppose that $\partial A^+(M) \neq \emptyset$, and let $x \in \partial A^+(M)$. Let $\eta_A(x)$ be the trace of the neighborhood filter $\eta(x)$ of x on $A \equiv A^+(M)$. Then, for each $N_A \in \eta_A(x)$, $\emptyset \neq L^+(N_A) \subset M$. Since M is compact, the cluster set of the filter base $\{L^+(N_A) | N_A \in \eta_A(x)\}$ is a non-empty subset of M; hence $J^+(x) \cap M \neq \emptyset$. However, this contradicts the assumption that (X,π) has characteristic 0^+, since $\partial A^+(M)$ is a closed invariant set disjoint with M. Therefore, $\partial A^+(M) = \emptyset$ and the proof of Proposition 3.2 is complete.

4. FLOWS OF CHARACTERISTIC 0^+ ON THE PLANE

Throughout this section, we assume the phase space to be the plane R^2 and (R^2,π) to be a fixed flow of characteristic 0^+. We let S denote the set of rest points of this flow.

LEMMA 3. For each $x \in X$, if $L^+(x) \neq \emptyset$, then $L^+(x)$ is either a periodic orbit or it consists of a single rest point.

Proof. If $L^+(x)$ contains a rest point s_0, then $L^+(x) = \{s_0\}$. For, $y \in L^+(x)$ implies that $y \in D^+(s_0) = \{s_0\}$, by Lemma 1. Suppose that $L^+(x)$ consists of regular points only. Then, to complete the proof of the lemma, it is sufficient to prove that $L^+(x)$ is compact. We note that if $y \in L^+(x)$, then $\overline{C^+(y)} = L^+(x)$. For, $z \in L^+(x)$ implies that $z \in D^+(y) = \overline{C^+(y)}$. Also, $\overline{C^+(y)} \subset L^+(x)$ since $L^+(x)$ is a closed invariant set, and hence $\overline{C^+(y)} = L^+(x)$. Since $\overline{C^+(y)} \subset \overline{C(y)} \subset L^+(x)$, we have $\overline{C(y)} = L^+(x)$. Therefore, $L^+(x)$ is a minimal set (cf. p. 26 of [6]). We recall that if M is a minimal subset of R^2 which is not compact, then for each $m \in M$, $L^{\pm}(m) = \emptyset$ (cf. p. 37 of [6]). Suppose that $L^+(x)$ is not compact, and let y_1 and y_2 be two distinct points in $L^+(x)$. Then,

$y_1 \in D^+(y_2) = C^+(y_2)$ and $y_2 \in D^+(y_1) = C^+(y_1)$. But, if t_1 and t_2 are positive numbers such that $y_1 = y_2 t_1$ and $y_2 = y_1 t_2$, then $y_1 = y_1(t_1+t_2)$; this shows that $C^+(y_1)$ is a periodic orbit. Hence, $L^+(x)$ is a periodic orbit, since $L^+(x) = C^+(y_1)$, as it is a minimal set; thus contradicting the assumption that $L^+(x)$ is not compact.

For a proof of the following theorem see Bhatia [5].

THEOREM (BHATIA). A flow F on a metric space X is dispersive if and only if for each $x \in X$, $D^+(x) = C^+(x)$ and there are no rest points or periodic orbits.

THEOREM 4.1. If $S = \emptyset$, then the flow (R^2, π) is parallelizable.

<u>Proof</u>. We note that for each $x \in R^2$, $L^+(x) = \emptyset$, and hence $D^+(x) = C^+(x) = C^+(x)$. For, if $L^+(x) \neq \emptyset$, then by Lemma 3, it must be a periodic orbit since it consists of regular points only. But this is impossible since the bounded component of a periodic orbit contains a rest point. Thus, the proof of our assertion follows from Bhatia's Theorem, stated above (cf. Auslander [2]) and the fact that the notions of parallelizability and dispersiveness are equivalent for a flow on the plane (see Antosiewicz and Dugundji [1]).

THEOREM 4.2. If R^2 contains a periodic point, then S is a singleton. Further, if $S = \{s_0\}$, then one of the following holds.

1. s_0 is a global Poincaré center.

2. s_0 is a local Poincaré center. The neighborhood N of s_0, consisting of s_0 and periodic orbits surrounding s_0, is a