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Alexander Kushkuley Zalman Balanov

Geometric Methods in Degree Theory for Equivariant Maps

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0. Introduction

0.1. The mapping degree.

In these notes we are concerned with the mapping degree problem in the presence of group symmetries.

It seems to be a truism to say that the mapping degree is one of the most important topological tools employed in the study of nonlinear problems. The solvability of equations, multiplicity results, structure of solutions, bifurcation phenomena, geometric and (co)homological characteristics of functionals – this is a rather incomplete list of the subjects where the mapping degree plays a very important role.

The basic principles of the degree theory in the finite-dimensional case have been worked out by Kronecker, Poincaré, Brouwer and Hopf. Even nowadays the famous Hopf Classification Theorem and the Brouwer Fixed Point Theorem remain as brilliant examples of how the mapping degree works within topology as well as in its applications.

Significant contributions to the degree theory have been done by Borsuk and Leray-Schauder in the early thirties. Borsuk established that the degree of an odd map of a finite-dimensional sphere into itself is odd. By the same token, he observed for the first time that symmetries can lead to the restriction of possible values of the mapping degree. On the other hand, Leray and Schauder have extended the classical finite-dimensional degree theory to the infinite-dimensional case, defining it for maps of the form $I + A$, where I is the identity operator and A is a compact operator. This work was especially important from the viewpoint of extending the "application area" for the mapping degree methods.

Since the times of Borsuk, Leray and Schauder many mathematicians have been involved in developing degree theory. We refer the reader to [St] for an excellent survey of related results as well as an extensive list of references. Although there are many schemes reducing the study of nonlinear problems to calculating the mapping degree, computing (or even estimating) the degree in a practical way remains an actual problem in general. From the thirties until these days the degree problem for *equivariant* maps is attracting a good deal of attention.

0.2. The mapping degree and group symmetries.

Recall that if X and Y are metric spaces and G is a topological group acting on X and Y then a continuous map $f : X \to Y$ is said to be equivariant if $fgx = gfx$ for all $x \in X$ and $g \in G$ (the oddness presents the simplest example of equivariance with respect to $Z_2 = \{\pm I\}$).

Why should anyone be interested in the degree of equivariant maps? Apparently, there are at least two reasons for that. First, group symmetries appear in nonlinear problems in a very natural way. For instance, if an elliptic equation is defined on a domain $\Omega \subset R^n$ invariant with respect to some subgroup $G \subset O(n)$, then G acts naturally on the corresponding Sobolev space, and the integral operator, say, B, associated with the equation is G-equivariant. In addition, the eigenspaces of $B'(0)$ are G-invariant and (usually) are of finite dimensions, so that the standard Lyapunov-Schmidt procedure reduces the bifurcation problem to studying G-equivariant maps in finite-dimensional G-spaces. Note also that in the above case, calculating degrees of the equivariant maps is in close connection with studying geometric and (co)homological characteristics (like genus, G-category, cup-length, etc.) of the invariant functionals associated with the initial equation (see, for instance, [Bar1]).

Another source of "real life" symmetries comes from autonomous ordinary differential equations. If one looks for periodic solutions then the S^1-action on a space of periodic functions (induced by the time translation) should be taken into account (see, for instance, [IMV]).

A more "academic" example of how degrees of equivariant maps appear as an appropriate subject to study, comes from group representation theory. Namely, the problem of classifying representations of a compact Lie group G up to G-equivariant homotopy equivalence leads to equivariant versions of the Hopf Theorem (see, for instance, [Di1]).

Except for "external" motivations for studying degrees of equivariant maps (a wide field of applications) there are also "internal" ones connected with the following two observations. As is well-known, the difficulties in degree calculations increase with increasing dimension. From this point of view the presence of symmetries allows in many cases to decrease the dimension of the problem in question. For example, if a finite group G acts smoothly and semi-freely on smooth, compact, connected, oriented n-dimensional G-manifolds M and N then one can reduce the computation of the degree of an equivariant map $f : M \to N$ to studying the behavior of f on the set M^G of G-fixed points of M only. Namely, assume M^G and N^G are connected. If $\dim M^G \neq \dim N^G$ then $\deg f = 0 \pmod{|G|}$; if $\dim M^G = \dim N^G$ and, in addition, M^G, N^G are oriented then $\deg f \equiv \alpha \cdot \deg f|M^G \pmod{|G|}$, where α is relatively prime to $|G|$ and modulo $|G|$ is uniquely determined by the actions of G on M and N (cf. Chapter 3).

In addition, in many cases one can get an important information on the degree of an equivariant map only from algebraic characteristics of the corresponding actions. For instance, if a finite p-group G acts orthogonally without G-fixed points on a finite-dimensional sphere S then for any map $f : S \to S$ commuting with the G-action the following relation holds: $\deg f \equiv 1 \pmod p$.

During the past sixty years, the degree problem of equivariant maps has been attacked using various methods. After Borsuk, the following development of the theory was mostly due to P.A. Smith and M.A. Krasnoselskii. Smith introduced a special cohomology theory on a category of Z_p-spheres for a prime p which, in particular, was used in order to express degrees of equivariant maps via the homological characteristics of the corresponding actions (the so-called Smith indices). and via the degrees of the restrictions of the maps in question to the relevant sets of fixed points (if defined). This gives rise to the so-called "homological approach". Krasnoselskii discovered a deep connection between the "degree" problem for equivariant maps of Z_p-spheres (p-arbitrary) and the problem of equivariant extension of maps – essentially, the equivariant homotopy types appeared for the first time as an appropriate context for study (the so-called "geometric approach").

These notes are an attempt to describe in detail some recent achievements of the *geometric approach* and to present a comparative (albeit unavoidably incomplete) study of the results obtained by geometric and homological methods.

Since the literature on degree theory for equivariant maps is still growing enormously we only mention four books and one survey relevant to our discussion. These are:

- Dold's book [Do1] on the topology behind the finite-dimensional degree theory;
- Bredon's book [Bre] on the equivariant topology background;
- Ize, Massabó and Vignoli's book [IMV2] where the geometric approach for linear S^1-actions has been worked out in details (actually, equivariant maps of spheres of different dimensions are studied in [IMV2]. See also [IV], where linear actions of arbitrary abelian groups are considered, and [IMV1] for more general constructions);
- Borisovich and Fomenko's survey [BF] on homological methods in a degree theory for equivariant maps;
- Bartsch's book [Bar1] on the connection between the Borsuk type theorems and variational problems with symmetries;

Of course, we should mention Krasnoselskii's paper [Kr1] as the starting point for our research.

0.3. The geometric approach.

1. In the geometric approach, the degrees of two equivariant maps from a topological (compact, closed, connected, oriented) n-dimensional manifold M to the oriented n-dimensional sphere S are compared using equivariant extension theorems. More precisely, we can define a cylindric action of a group G on the cylinder $C = M \times [0,1]$ by setting it to be trivial on the segment $[0,1]$. We can also define a conic action of G on a ball B bounded by the sphere S (via the radial extension). Let O be the center of the ball. For equivariant maps $\Phi, \Psi : M \to S$ an equivariant map $f_0 : M \times \{0,1\} \to S \subset B$ is obviously defined. Let $F : C \to B$ be an equivariant extension of f_0 and $K = F^{-1}(O)$. If orientations on C and B are properly chosen then there exist fundamental classes $O_K \in H_n(C, C\backslash K)$ and $O_O \in H_n(B, B\backslash\{O\})$ which determine the degree of F (as a map of manifolds with boundaries) by the formula $F_*(O_K) = (\deg F)O_O$ (cf. e.g. [Dol], p. 268). Note also, that $\deg F = \pm(\deg \Phi - \deg \Psi)$. Let G be finite and let $(H_1), ..., (H_\ell)$ be all the orbit types of the G-action on M (and hence on C). Without loss of generality one can assume that every $g \in G$ either changes (simultaneously) orientations on M and S or preserves them. Suppose the extension F satisfies the following conditions:

(α) $K = \bigcup_{j=1}^{l} T_j$, $T_s \cap T_p = \emptyset$ if $s \neq p$;

(β) $T_j = G(K_j)$ for some compact K_j, $j = 1, ..., \ell$;

(γ) $H_j(K_j) = K_j$;

(δ) $g(K_j) \cap h(K_j) = \emptyset$ if $gh^{-1} \notin H_j$.

Now using (α) and the additivity of the degree, one gets $\deg F = \sum_{j=1}^{\ell} \deg F_j$ where F_j is the restriction of F on a sufficiently small neighbourhood of T_j. Bearing in mind that F is equivariant and using (β)$-$(δ), we have $\deg F_j = a_j|G/H_j|$ where a_j is the degree of F in a small neighbourhood of K_j. From this we deduce the formula

$$\deg F = \pm(\deg \Phi - \deg \Psi) = \sum_{j=1}^{\ell} a_j|G/H_j| \,, \tag{0.1}$$

and so

$$\deg \Phi \equiv \deg \Psi \pmod{\mathrm{GCD}\{|G/H_j|_{j=1}^{\ell}\}}, \tag{0.2}$$

which is the typical "comparison principle" result in the geometric approach.

If now one wants to estimate the degree of an arbitrary equivariant map $\Phi : M \to S$ it suffices to find only one equivariant map Ψ whose degree is easy to calculate, and to use formula (0.2). In many cases it is not hard to find the appropriate

Ψ. For instance, if M and S coincide as the G-spaces then one can take the identity map for Ψ. If $S^G = \{x \in S \mid gx = x \text{ for all } g \in G\} \neq \emptyset$ then one can set $\Psi(x) \equiv \text{ pt } \in S^G$.

This approach has been first realized by M.A. Krasnoselskii [Kr1] for the case when $G = Z_p$ acts freely on a sphere.

2. The above discussion gives rise to the following problem. Let G be a compact Lie group, let X, Y be a couple of metric G-spaces, and let $A \subset X$ be a closed invariant subspace. What are conditions on X and Y which imply that an equivariant map $f : A \to Y$ has an equivariant extension over X? This problem was addressed by several authors, e.g. J. Jaworowski [Ja1, Ja2], R. Lashof [La], M. Madirimov [Mad1, Mad2] and others. All these authors used the reduction of the above problem to the problem of extending sections of fiber bundles associated with the maps in question. In this book we develop another general approach which is intuitively easier and allows us to obtain stronger extension results in certain cases. What is more important, this extension technique provides some means for controlling the extension map in a manner required by the "comparison principles" like the one described above. Again, the idea behind this approach can be traced back to the original paper of M.A. Krasnoselskii [Kr1].

The key to the extension results we are looking for is the following

Definition. Let a topological group H act on a metric space E. Let $D_0 \subset E$ be open in its closure D. Then D is said to be a *quasi-fundamental domain* of the H-action on E if the following conditions are satisfied:

(a) $H(D) = E$;

(b) $g(D_0) \cap h(D_0) = \emptyset$ $(g \neq h;\ g, h \in H)$;

(c) $E \backslash H(D_0) = H(D \backslash D_0)$.

If E is finite-dimensional and the following additional condition holds

(d) $\dim D = \dim E/H;\ \dim(D \backslash D_0) < \dim D;\ \dim H(D \backslash D_0) < \dim E$

then D will be called a *fundamental domain* for the H-action on E.

Note that, if H is a discrete group then one can set D_0 to be the interior of D. Hence the definition above naturally complies with the classical one (cf. e.g. [DFN], p. 169).

It turns out that a (quasi-)fundamental domain exists for any free action of a compact Lie group on a metric space.

Assume now that a compact Lie group G acts on a metric space X and let $A \subset X$ be a closed invariant subset such that the action of G on $X \backslash A$ is free. By the above observation there exists a (quasi-)fundamental domain $D^{(0)}$ of the G-space $X \backslash A$.

Let $D_0^{(0)}$ be the corresponding open subset of $D^{(0)}$ and let $X_1 = A \cup G(D^{(0)} \backslash D_0^{(0)})$. Applying the above observation once again to $X_1 \backslash A$ we get X_2, etc. So one has a closed invariant filtration $X = X_0 \supset X_1 \supset X_2 \ldots \supset A$. If $X \backslash A$ is finite-dimensional then this filtration is finite. Let Y be another metric G-space. It turns out that if $X \backslash A$ is finite dimensional then any equivariant map $A \to Y$ extends over X if for all $i = 1, 2, \ldots$ any equivariant map $X_i \to Y$ has a (non-equivariant) extension over $X_i \cup D^{(i-1)}$. The same is true for extensions of equivariant homotopies.

Combining the last argument with the standard induction over the orbit types (see, for instance, [Di1]) leads to a rather general equivariant version of the well-known Kuratowski-Dugundji Extension Theorem. In particular, if for any stationary subgroup H of the action of G on M one has dim $\{x \in M \mid hx = x$ for any $h \in H\} \leq n(H)$ and if the set $S^H = \{y \in S \mid hy = y$ for any $h \in H\}$ is locally and globally k-connected for each $k = 0, 1, 2, ..., n(H) - 1$, then the existence of an equivariant extension with properties (α) - (δ) required by the comparison principle follows immediately from the considerations above.

Using this scheme we strengthen the corresponding degree results by Krasnoselskii [Kr1], Zabrejko [Za1, Za2], Bowczyc [Bow1, Bow2], Dold [Do2], Daccach [Dac] and others.

To some extent, this approach can be characterized as "geometric equivariant obstruction theory without CW-complexes".

3. The next step of our program is to improve the general geometric approach in such a way that one could treat the following problems:

(a) to replace in formulas (0.1) and (0.2), a finite group (lengths of the orbits) by an arbitrary compact Lie group (Euler charactaristics of the orbits);

(b) to eliminate the connectedness conditions with respect to the sets S^H;

(c) to express explicitly the degree of an equivariant map via geometric characteristics of actions and degrees (if defined) of the restrictions of the given map to appropriate fixed point sets.

To this end, assuming G to be an arbitrary (not necessarily finite) compact Lie group, we impose the following additional conditions: M is a smooth G-manifold and S is a G-representation sphere. These assumptions allow us to take advantage of some standard (but important) tools from Riemannian G-geometry (invariant tubular neighborhood, normal slice, invariant foliations, etc.), algebraic topology (cap-product, Thom class, etc.) and piecewise linear topology. The main idea remains the same: to construct an equivariant extension F in such a way that the set $F^{-1}(0)$ is "computable". But now we provide F with more "delicate" properties than those formulated in $(\alpha) - (\delta)$ (for the precise formulation see Lemma 3.8).

Our approach is essentially based on the following three observations.

1) It is well-known (see, for instance, [Bre]) that if M is a compact smooth G-manifold and $N \subset M$ is a G-submanifold, then there exists an invariant tubular neighborhood of N in M. In particular, this means that there exists an invariant one-dimensional foliation around N. If now N is the union of all non-principal orbits for the action of G on M, then N is not a submanifold of M in general, so that it may happen that there does not exist a tubular neighborhood around N. It turns out, however, that there exists an invariant one-dimensional foliation around N in this case as well.

2) Let U be an oriented n-dimensional manifold, V an n-dimensional vector space and D its k-dimensional subspace. Assume W_1 is an open subset of V such that W_1 is contratible to $D \setminus 0$ and $W_1 \bigcup (V \setminus D) = V \setminus 0$. Let $f : U \to V$ be a continuous map such that $K = f^{-1}(0)$ is compact. Suppose, finally, that there exists an open subset $U_1 \subset f^{-1}(W_1)$ such that $U_1 \bigcup (U \setminus f^{-1}(D)) = U \setminus K$. Denote by τ_D^V the Thom class of D in V.

It turns out that $\deg_0 f = 0$ if $f^*(\tau_D^V) = 0$. In particular, under the above conditions $\deg_0 f = 0$ provided $H^{n-k}(U, U \setminus f^{-1}(D)) = 0$.

3) The last observation is concerned with "general position" theorems in the equivariant context. Let V be an orthogonal $(d+1)$-dimensional representation of a finite group G and B^{d+1} the unit ball in V. Let G act freely on a compact $(d-k)$-dimensional manifold X $(k \geq 1)$. For any finite set of linear subspaces $L_j \subset V$, $j = 1, \ldots, m$, there exists an equivariant map f from X to B^{d+1} such that $\dim f^{-1}(G(B^{d+1} \bigcap L_j)) \leq \dim L_j - k - 1$ for all $j = 1, 2, ..., m$, provided $\dim L_j \geq k$.

These three observations in compliance with the above mentioned equivariant extension technique based on the notion of fundamental domain lead to an essential strengthening of the comparison principle in directions (a) - (c). In particular, we generalize the corresponding results by Nirenberg [Ni2], Marzantowicz [Mar1], Wei Yue-Ding [We], Dancer [Dan], Lück [Lü], Komiya [Kom], Fadell, Husseini and Rabinowitz [FHR] and others; we also strengthen in certain cases the results by Ize, Massabo and Vignoli [IVM2, IV]; finally, we clarify the geometric nature of the results by Borisovich, Izrailevich and Fomenko (Schelokova) [Sc4, BF].

For the precise formulations of our results we refer the reader to Section 3.1. Below we present two corollaries which can be stated without additional explanations.

Let $G = T$ be a torus and let $\Phi, \Psi : M \to S$ be T-equivariant maps. Let $M_1, M_2, ..., M_m$ be the connected components of M^T .

(a) For each j, $\dim M_j = \dim S^T$, there exists an integer $\alpha_j = \alpha(M_j, S^n)$

completely defined by the G-actions on M and N such that

$$\deg \Phi - \deg \Psi = \sum_j \alpha_j \cdot (\deg \Phi|M_j - \deg \Psi|M_j);$$

(b) if $\dim M_i \neq \dim S^T$ for all i then $\deg \Phi = \deg \Psi$ is uniquely determined by the G-action on M and S.

Assume now a finite p-group G acts smoothly on compact, connected, oriented n-dimensiomal manifolds M and N. Suppose that $N^G \neq \emptyset$ and all the fixed point sets N^H, $(H) \in \mathrm{Or}(N)$, are connected and oriented. Let $\{M_i | i = 1, 2, ..., m\}$ be the set of connected componenets of M^G with $\dim M_i = \dim N^G$. Then for any equivariant map $f : M \to N$,

$$\deg f \equiv \sum_i \alpha_i \deg(f|M_i) \pmod p,$$

where the numbers α_i are uniquely determined modulo p. In particular, if $m = 0$ then $\deg f \equiv 0 \pmod p$.

It should be noticed that we are interested in equivariant maps of G-manifolds of the same dimension. Therefore, we are dealing with the Brouwer degrees only. At the same time, the authors of [IMV1, IMV2, IV] deal with equivariant maps of G-representation spheres of different dimensions, and calculate the so-called "equivariant degree" which coincides with the Brouwer degree if dimensions of the spheres coincide. From this point of view certain results obtained in [IVM1, IVM2, IV] are, of course, more general than those presented in our monograph. However, it is easy to see that one can use the methods developed in our monograpgh to study the above mentioned situation as well.

4. One of the natural applications of the stream of ideas discussed above is the so-called Equivariant Hopf Theorem.

Recall a classical theorem of H. Hopf (see, for instance, [Di2], p. 122). Let M be a closed, compact, connected, oriented n-dimensional manifold and S an oriented n-dimensional sphere. Hopf's theorem asserts that two continuous maps from M to S are homotopic iff their degrees are equal, and, in addition, that any integer can be realized as the degree of some map from M to S. Suppose now that a compact Lie group G acts on M and S. Classification of equivariant maps $M \to S$ up to equivariant homotopy can not be achieved in the same straightforward way as in the non-equivariant case. As an example, suppose that G is a finite group acting orthogonally on vector spaces V and W. Denote by $S(V)$ and $S(W)$ the correponding representation spheres. Suppose that for all subgroups $H \subset G$

the dimensions of fixed point sets V^H and W^H are equal. Consider the following statement :

(∗) *G-equivariant maps* $f_1, f_2 : S(V) \to S(W)$ *are equivariantly homotopic if and only if*

$$\deg(f_1|S(V)^H) = \deg(f_2|S(W)^H)$$

for all subgroups H *of* G.

Although this statement is not true in general (see [Ru]), there exists a rather general set of conditions on G-spaces V and W which ensure its validity. These conditions can be obtained as a corollary of the so-called Equivariant Hopf Theorem presented by tom Dieck in [Di1,Di2] and generalized by Tornehave [To] and Laitinen [Lai]. An equivariant cohomology theory has been used as the main tool in [Di1, Di2, Lai, To]. In these notes we discuss a more straightforward approach to the Equivariant Hopf Theorem based on combining the usual (non-equivariant) obstruction theory with the fundamental domain technique. In particular, this enables us to obtain *necessary and sufficient* conditions for statement (∗) and, in addition, to strengthen the results on equivariant homotopy classification obtained in [Di1,Di2,To,Lai].

5. Our final remark is concerned with the infinite dimensional aspect of the degree problem for equivariant maps.

In accordance with the classical approach by Leray and Schauder, to carry out the finite-dimensional results to completely continuous vector fields in Banach spaces one should solve the following problem. Let $\Phi = I + A$ be a completely continuous vector field defined on the closure of a bounded region Ω in a Banach space E. Let G be a compact Lie group. Assume Φ is equivariant with respect to a pair of linear representations of G in E. Given $\varepsilon > 0$ one should construct a finite dimensional operator $A_n : \overline{\Omega} \to E$ such that:

1) A_n is equivariant;

2) $||A - A_n|| < \varepsilon$.

This is not a difficult problem if one deals with vector fields which are equivariant with respect to one representation only. However, in the case of two representations the "co-existence" of equivariance with infinite dimension leads to a "conflict". Namely, even in the case when G is a cyclic group it may happen that given a finite-dimensional subspace $E^k \subset E$ there is no finite-dimensional subspace $E^d \supset E^k$, which is invariant with respect to the pair of G-representations *simultaneously*.

In these notes we develop a method, based on combining the classical Leray-Schauder technique with some ideas from the theory of gaps between linear subspaces, which allows us, in certain cases, to overcome this conflict.

0.4. Overview.

The book consists of five chapters.

The first chapter is devoted to studying the equivariant extension problem. In the first section we present auxiliary information from equivariant topology. In the second section we prove the existence of fundamental domains in a rather general situation. By means of this result we prove the Equivarint Kuratowski-Dugundji Theorem in the third section.

In the second chapter assuming M to be a closed, compact, connected, oriented, topological n-dimensional manifold, and S to be an oriented n-dimensional sphere we study degrees of maps $M \to S$ equivariant with respect to *topological* actions on M and S. The first section is devoted to the general comparison formula for degrees of equivariant maps (G is a finite group). Some special cases and generalizations of this formula (p-group actions, free actions, torus actions, etc.) are considered in the second section. We conclude the chapter with some counterexamples which show that our hypotheses are sharp in some respect.

In the third chapter we assume that G is an arbitrary compact Lie group, M is a smooth (closed, compact, connected, oriented) G-manifold and S is a G-representation sphere. Under these assumptions we get sharper results than those stated in Chapter 2. In particular, we remove the connectedness conditions with respect to the G-action of on S, and in many cases give precise restrictions on the possible values of degrees of equivariant maps from M to S.

This chapter is organized as follows. In the first section using our results from Chapters 1 and 2 we introduce some integer-valued characteristics connected with the actions of G on M and S. One may consider these characteristics as the geometric analogs of the so-called equivariance indices introduced by T. Fomenko (Schelokova) in [Sc4] (see also [BF], [Di1], [Di2]). In terms of these characteristics we formulate our main results and present some corollaries for p-group actions, torus actions, semi-free actions, abelian group actions, etc. Taking an arbitrary smooth manifold N instead of S and assuming a group G is acting on N so that $N^G \neq \emptyset$, we use some straightforward arguments in order to show that most of our results remain valid in this situation.

The second section is auxiliary. Here we present some properties of the cap-product and Thom class allowing us to deal with "bad" orbit types in M (those for which $\dim M^K > \dim S^K$).

In the third section we present the above mentioned invariant foliation and equivariant general position lemmas. These lemmas together with the "elimination" technique based on the usage of the Thom class are main ingredients in our approach. They come together in the forth section where we give a proof of one of our main degree results assuming G to be a finite group.

In the fifth section we extend our result to the case of an arbitrary compact Lie group actions. In the sixth section we consider equivariant maps from one G-manifold to another G-manifold without assuming the second manifold to be a sphere. As a particular case, we consider here abelian group actions.

The fourth chapter is devoted to the degree problem for completely continuous vector fields in Banach spaces. In the first section we develop some machinery for solving the conflict between equivariance and infinite dimensionality. In the second section we use this technique to get our degree results.

In the last chapter we present some applications of the methods developed in the previous chapters.

In the first section we consider a semi-linear elliptic boundary value problem which is associated with the corresponding linear problem of positive Fredholm index. Under some symmetry assumptions we prove the existence of solutions of arbitrarily large norm in the corresponding Hölder space. We follow the scheme by P. Rabinowitz [Ra1] (see also [Mar1]).

In the second section we give a lower estimate for the genus of the free part of a finite-dimensional sphere S with a compact Lie group action. To treat this problem we modify the well-known geometric aproach by M. Krasnoselskii [Kr1, KZ]. We apply the obtained result to the irreducible $SO(n)$-representations in spherical harmonics.

In the third section we present an equivariant version of the Hopf Theorem on the homotopy classification of mappings from a manifold to a sphere. Some illustrative examples are considered.

The fourth section is devoted to the Borsuk-Ulam type theorems on the non-existence of an equivariant mapping from an n-dimensional free G-sphere to an m-dimensional one if $n > m$; we consider a situation of non-free actions on manifolds.

In the fifth section we give an elementary proof of the well-known theorem of Atiyah-Tall [AT]:

Let V and W be two finite-dimensional orthogonal representations of a p-group G ($\dim V = \dim W$), and let $S(V)$ and $S(W)$ be the unit spheres in V and W respectively. Then there exists a G-equivariant map $f : S(V) \to S(W)$ with

$\deg f \neq 0 \pmod p$ iff irreducible components of V and W are conjugate in pairs by elements (possibly different) of the corresponding Galois group.

Certain questions concerning G-equivariant maps of G-manifolds related to this theorem are also discussed.

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