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## 1. INTRODUCTION

In this paper the term *manifold* will be used to mean a finite-dimensional topological m-manifold $M^m$ --possibly with boundary $\partial M^m$. By a *PL manifold* we will mean a piecewise linear manifold as in [16]. In a very broad sense the purpose of this paper is to investigate questions of the following type: *If* $f: M \to N$ *is a map between manifolds, when can* f *be approximated by "nice" maps?* The "nice" maps referred to include homeomorphisms, approximate fibrations, and block bundles. The key idea common to all of these results is a very carefully controlled engulfing. This is established very early in Section 3 and is then used at crucial points throughout the paper. This entire line of research is a natural outgrowth of the approximation results of [4] and [7]. The statements and proof of the theorems are influenced heavily by the corresponding Q-manifold results of [6].

For the most part all spaces in this paper will be locally compact, separable and metric. A map $f: X \to Y$ (i.e., a continuous function) is *proper* provided that $f^{-1}(C)$ is compact, for all compact $C \subset Y$. If $\alpha$ is an open cover of Y, then a proper map $f: X \to Y$ is said to be an $\alpha$-*fibration* (or, f has the $\alpha$-*lifting property*) if for all maps $F: Z \times [0, 1] \to Y$ and $\tilde{F}_o: Z \to X$ for which $f\tilde{F}_o = F_o$, there is a map $G: Z \times [0, 1] \to X$ such that $G_o = \tilde{F}_o$ and fG is $\alpha$-close to F. This latter statement means that given any $(z, t) \in Z \times [0, 1]$ there is a $U \in \alpha$ containing both $fG(z, t)$ and $F(z, t)$. Finally a proper map $p: X \to Y$ is said to be an *approximate fibration* provided that it has the $\alpha$-lifting property, for all open covers $\alpha$ of Y. We refer the reader to [8], where the notion of an approximate fibration was introduced and some of its basic properties were established.

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Our first result is concerned with the homotopy detection of those maps which are close to approximate fibrations.

THEOREM 1. *Let* B *be a space which is locally polyhedral, let* $\alpha$ *be an open cover of* B, *and let* $m \geq 5$ *be an integer. There exists an open cover* $\beta$ *of* B *so that if* $M^m$ *is a manifold* $(\partial M = \emptyset)$ *and* $f: M \to B$ *is a* $\beta$*-fibration, then* f *is* $\alpha$*-close to an approximate fibration* $p: M \to B$.

There are two variations of this result which follow from the same proof. The first assumes that $\partial M \neq \emptyset$ and f is already an approximate fibration over an open set in B which contains $f(\partial M)$. The conclusion is that the approximate fibration $p: M \to B$ can be chosen to agree with f on $\partial M$. The second variation assumes that $f|\partial M: \partial M \to B$ is the projection map of a locally trivial bundle or a block bundle over arbitrarily small simplices. The conclusion is that p can be chosen to agree with f on $\partial M$.

The proof of Theorem 1, which is given in Section 6, uses a handle lemma from Section 5 that is established by engulfing and torus geometry. The idea of the proof is to construct maps $f_i: M \to B$ which are close to f and which are $\beta_i$-fibrations, where $\{\beta_i\}$ is a sequence of open covers of B whose mesh converges to 0. The $f_i$ are additionally chosen so that $\lim f_i = p: M \to B$ is defined and is our desired approximate fibration.

In [15] there is constructed an example of an approximate fibration $p: M^6 \to S^1$ $(\partial M = \emptyset)$ whose fiber is not homotopy equivalent to any finite complex. Perhaps the following result will be useful in determining all such fibers (up to homotopy type).

COROLLARY. *A polyhedron* K *is homotopy equivalent to the (homotopy) fiber of some approximate fibration* $p: M^m \to S^1$ $(m \geq 5$ *and* $\partial M = \emptyset)$ *if and only if* $K \times S^1$ *is homotopy equivalent to some compact* m*-manifold without boundary.*

The following notation will be used throughout this paper. If $\alpha$ is an open cover of Y, then a homotopy $h_t: X \to Y$ is said to be an $\alpha$*-homotopy* provided that each set $\{h_t(x) \mid 0 \le t \le 1\}$ lies in some element of $\alpha$. A proper map $f\colon X \to Y$ is said to be an $\alpha$-*equivalence* if there is a proper map $g\colon Y \to X$ and proper homotopies $\phi_t\colon gf \simeq id_X$, $\theta_t\colon fg \simeq id_Y$ such that $f\phi_t\colon X \to Y$ and $\theta_t\colon Y \to Y$ are $\alpha$-homotopies. We write this as

$$\phi_t\colon gf \overset{f^{-1}(\alpha)}{\simeq} id \qquad \text{and} \qquad \theta_t\colon fg \overset{\alpha}{\simeq} id,$$

where $f^{-1}(\alpha)$ denotes the open cover of $X$ defined by $f^{-1}(\alpha) = \{f^{-1}(U) \mid U \in \alpha\}$. Finally if $Y$ has a specified metric and $\varepsilon > 0$ is given, then we will also use $\varepsilon$ to denote the open cover of $Y$ by balls of diameter $< \varepsilon$. This convention means that we have also defined the notions of $\varepsilon$-homotopy and $\varepsilon$-homotopy equivalence.

It might be of some interest to point out another application of Theorem 1. In [7] the following $\alpha$-Approximation Theorem was established: *For every manifold* $N^m$, $m \ge 5$, *and open cover* $\alpha$ *of* $N$, *there exists an open cover* $\beta$ *of* $N$ *such that if* $M^m$ *is a manifold and* $f\colon M \to N$ *is a* $\beta$-*equivalence which is a homeomorphism from* $\partial M$ *to* $\partial N$, *then* $f$ *is* $\alpha$-*homotopic* rel $\partial M$ *to a homeomorphism.* Here is another proof of this result based on Theorem 1 above and Edwards' Approximation Theorem [9]. For simplicity assume that $\partial M = \emptyset = \partial N$. First by use of Theorem 1 we may assume that $f$ is an approximate fibration. Next it easily follows that $f$ is CE, i.e., the inverse image of each point has property $UV^\infty$ in $M$. Finally we use Edwards' Approximation Theorem to approximate $f$ by a homeomorphism. One advantage of this proof over the proof given in [7] is that surgery theory is completely avoided, i.e., we do not need the fact that any homotopy $I^i \times T^{m-i}$ is a real $I^i \times T^{m-i}$ rel $\partial$.

Let $F^m$ be a compact manifold for which the Whitehead group, $Wh(F)$, vanishes. We use $\mathcal{S}(F)$ to denote the set of equivalence classes of the form $[f]$, where $f\colon M^m \to F$ is a homotopy equivalence of a compact manifold to $F$ which is a homeomorphism from $\partial M$ to $\partial F$. Another such map, $f'\colon M' \to F$, is defined to be *equivalent* to $f$ provided that there exists a homeomorphism $h\colon M \to M'$ for which $hf' \simeq f$. Note that via use of the s-cobordism theorem,

$S(F)$ is just the set of homotopy-topological structures on $F$ rel $\partial$ [17, p.265]. If $T^n$ is the n-torus and $e: T^n \to T^n$ is any standard finite cover, then there is a transfer map $e^{\#}: S(T^n \times F) \to S(T^n \times F)$ defined by $e^{\#}([f]) = [\tilde{f}]$, where $\tilde{f}$ comes from the pullback diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & T^n \times F \\ \downarrow & & \downarrow e \times id \\ M & \xrightarrow{f} & T^n \times F \end{array} .$$

We use $S_o(T^n \times F)$ to denote those elements of $S(T^n \times F)$ that are invariant under any of these transfer maps.

Our metric on euclidean n-space $R^n$ is the one derived from the norm $\|x\| = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$. In the following statement $p$ will denote projection to $R^n$.

THEOREM 2. *Let* $n \geq 0$ *be an integer. For every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *so that if* $f: M \to R^n \times F$ *is a* $p^{-1}(\delta)$*-equivalence for which* $f|\partial M: \partial M \to R^n \times \partial F$ *is a homeomorphism, where* $M^{m+n}$ *is a manifold,* $F^m$ *is as above and* $m + n \geq 5$, *then there is an element* $\sigma(f)$ *of* $S_o(T_n \times F)$ *which vanishes if and only if* $f$ *is* $p^{-1}(\varepsilon)$*-homotopic to a homeomorphism.*

The proof of Theorem 2, which is given in Section 8, is purely geometric in nature. It uses only engulfing and torus geometry. The utility of this result is that many times a geometric problem is encountered which gives rise to a $p^{-1}(\delta)$-equivalence $f: M \to R^n \times F$ as above; thus an obstruction is encountered in $S_o(T^n \times F)$. In certain cases surgery theory tells us that $S_o(T^n \times F)$ vanishes. For example it follows from [17, p.285] that if $F^m$ is a $K(\pi, 1)$ with $\pi$ poly Z and $m + n \geq 5$, then $S(T^n \times F) = 0$. So in particular $S_o(T^n \times F) = 0$. The following result deals with simply connected fibers.†

THEOREM 3. *If* $F^m$ *is a compact simply connected manifold and* $m + n \geq 5$, *then* $S_o(T^n \times F)$ *is in* 1-1 *correspondence with the set of homotopy classes*

$$[F_o, \partial F_o; G/TOP, *],$$

*where* $F_o$ *is the manifold obtained by deleting an interior point of* $F$.

Recall that G/TOP is the homotopy fiber of the natural map of classifying spaces, BTOP → BG. It is an H-space and the functor $X \mapsto [X; G/TOP]$ is a contravariant functor to abelian groups. The proof of Theorem 3, which is given in Section 9, uses the surgery exact sequence [17, p.269].

Combining Theorem 3 with the remarks made above about fibers $F$ which are $K(\pi, 1)$'s we have the following corollary. It is used in Appendix 2 to show that locally homotopically unknotted embeddings are locally flat.

COROLLARY. *If* $F$ *is a sphere, then* $S_o(T^n \times F) = 0$.

Here is a generalization of the result of [4] on approximating maps into bundles by homeomorphisms.

THEOREM 4. *Let* $B^n$ *be a manifold, let* $\alpha$ *be an open cover of* $B$, *and let* $m + n \geq 5$ *be an integer. There exists an open cover* $\beta$ *of* $B$ *so that if* $M^{m+n}$ *is a manifold,* $p: E^{m+n} \to B$ *is a fiber bundle with fiber a compact manifold* $F$, *and* $f: M \to E$ *is a* $p^{-1}(\beta)$-*equivalence which is a homeomorphism from* $\partial M$ *to* $\partial E$, *then* $f$ *is* $p^{-1}(\alpha)$-*homotopic* (rel $\partial M$) *to a homeomorphism provided that* $S_o(T^i \times I^j \times F) = 0$, *for* $i + j = n$.

The idea of the proof (see Section 10) is to locally work through a handle decomposition of $B$. Starting with the handles of index 0 we use Theorem 2

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†The author is indebted to L. C. Siebenmann for providing him with this computation.

as a handle lemma to deform $f$ to a map which is a homeomorphism over these handles. Then we inductively continue through the handles of positive index.

From the remarks following Theorem 2 we conclude that Theorem 4 is true if $F$ is a $K(\pi, 1)$ with $\pi$ poly Z. We also point out that Theorem 4 is true if we merely assume that $S_o(T^i \times I^j \times F) = 0$ for $j$ ranging through the indices of a given handle decomposition of $B$. For example if $B$ is a sphere, it has a handle decomposition with handles of only index $0$ and $\dim B$, so we only need $S_o(T^i \times I^j \times F) = 0$ for $j = 0$ and $j = \dim B$. This is somewhat different from the procedure of [7] or [20], in which the degree of the approximation depended on the size of the handles in the target.

Finally we mention that the techniques of this paper have important applications to the homotopy detection of local flatness and to the homotopy detection of those maps which are close to block bundle maps. More specifically in Appendix 2 we sketch a proof that a locally homotopically unknotted manifold pair is locally flat. This is essentially what appears in [5], but now we capture some extra dimensions that eluded us at the time. Results of this type are not new any longer and we refer the reader to Appendix 2 for a fuller account of the literature on the subject. In Appendix 3 we show how the approximation techniques of this paper can be used to prove that approximate fibrations can be approximated by block bundles provided that a $\pi_1$-condition on the fiber is satisfied. Again this is not a new result--it having been done recently by Quinn [18].

The key idea in this paper is an engulfing trick, which might best be described as a "shuffle." To acquaint the reader immediately with this we sketch a quick proof of the $\alpha$-Approximation Theorem of [7]. This is done in Appendix 1 and we recommend that it be read first before anything else is attempted.

## 2. PRELIMINARIES

In this section we will introduce some additional notation and remind the reader of some well-known results from the literature which will be used throughout this paper.

Recall from Section 1 that $R^n$ denotes euclidean n-space, i.e., $R^n = R \times \cdots \times R$ (n times). In $R^n$ we consider n-cells of the form $B_r^n = [-r, r]^n$. We use $S^1$ to denote the set of complex numbers of absolute value 1, and the n-torus is $T^n = S^1 \times S^1 \times \cdots \times S^1$. We always let $I = [0, 1]$, and the standard n-cell is $I^n = [0, 1] \times \cdots \times [0, 1]$.

Expanding on the definition of an $\alpha$-equivalence which was given in Section 1 let $f: X \to Y$ be proper, let $C \subset Y$ be closed, and let $\alpha$ be an open cover of Y. Then f is said to be an *$\alpha$-equivalence over* C if there is a proper map $g: C \to X$ and proper homotopies $\phi_t: gf|f^{-1}(C) \overset{f^{-1}(\alpha)}{\simeq} \text{id}$, $\theta_t: fg \overset{\alpha}{\simeq} \text{id}$. This means that $\phi_t: f^{-1}(C) \to X$ and $\theta_t: C \to Y$ are proper homotopies such that $\phi_o = \text{id}$, $\phi_1$ is the inclusion $f^{-1}(C) \hookrightarrow X$, $\phi_t$ is an $f^{-1}(\alpha)$-homotopy, $\theta_o = \text{id}$, $\theta_1$ is the inclusion $C \hookrightarrow Y$, and $\theta_t$ is an $\alpha$-homotopy. Note that if $C \subset G \subset Y$, where C is closed and G is open, then for all sufficiently fine open covers $\alpha$ of Y, any $\alpha$-equivalence $f: X \to Y$ restricts to an $(\alpha \cap G)$-equivalence over C, $f|f^{-1}(G): f^{-1}(G) \to G$. (Here $\alpha \cap G = \{U \cap G | U \in \alpha\}$.

We write "f = g over C" to mean that $f^{-1}(C) = g^{-1}(C)$ and $f = g$ on $f^{-1}(C)$. In general we say that f has property P over C whenever $f|f^{-1}(C)$ has property P. In some instances when the meaning is clear we will simply write f when the restriction $f|$ is intended. Finally, a homotopy $f_t: X \to Y$ is said to be an *$\alpha$-homotopy over* $C \subset Y$ if $f_t|f_o^{-1}(C): f_o^{-1}(C) \to Y$ is an $\alpha$-homotopy.

The following result tells us how to detect $\alpha$-equivalences locally. It is established in [6].

PROPOSITION 2.1. *Let* B *be a space and let* $\gamma$ *be an open cover of* B. *For every open cover* $\alpha$ *of* B *there exists an open cover* $\beta$ *of* B *so that if* $f: X \to Y$ *and* $p: Y \to B$ *are proper maps such that* X, Y *are* ANRs *and* f *is a* $p^{-1}(\beta)$*-equivalence over the closure of each element of* $\gamma$, *then* f *is a* $p^{-1}(\alpha)$*-equivalence.*

REMARKS. 1. There is a similarly-worded local version of this result whose conclusion states that f is a $p^{-1}(\alpha)$-equivalence over some closed set $C \subset Y$. 2. The above result is also true if we merely assume that X, Y are separable metric ANRs and f, p are no longer necessarily proper. For this to make sense we also have to drop the requirement that the map g and homotopies $\phi_t$, $\theta_t$ of the definition are proper.

We now expand on the definition of an $\alpha$-fibration which was given in Section 1. Let $f: X \to Y$ be proper, let $C \subset Y$ be closed, and let $\alpha$ be an open cover of Y. Then f is said to be an $\alpha$-*fibration over* C if for all maps $F: Z \times I \to C$ and $\tilde{F}_0: Z \to X$ for which $f\tilde{F}_0 = F_0$, there is a map $G: Z \times I \to X$ such that $G_0 = \tilde{F}_0$ and fG is $\alpha$-close to F. Note that if $f: X \to Y$ is an $\alpha$-fibration and $C \subset G \subset Y$, where C is closed and G is open, then the restriction $f|f^{-1}(G): f^{-1}(G) \to G$ is an $(\alpha \cap G)$-fibration over C provided that $\alpha$ is sufficiently fine.

Here is an analogue of Proposition 2.1 for $\alpha$-fibrations. Again see [6] for a proof.

PROPOSITION 2.2. *Let* B *be an* ANR *and let* $\gamma$ *be an open cover of* B. *For every open cover* $\alpha$ *of* B *there exists an open cover* $\beta$ *of* B *so that if* X *is an* ANR *and* $f: X \to B$ *is a* $\beta$*-fibration over the closure of each element of* $\gamma$, *then* f *is an* $\alpha$*-fibration.*

REMARK. As in Proposition 2.1 there is a similarly-worded local version of this result whose conclusion states that f is an $\alpha$-fibration over some closed set $C \subset B$.

The following result gives us another point of view from which we can recognize $p^{-1}(\alpha)$-equivalences. See [6] for a proof.

PROPOSITION 2.3. *Let* B *be an* ANR *and let* $\alpha$ *be an open cover of* B. *There exists an open cover* $\beta$ *of* B *so that if* $p\colon E \to B$ *is a Hurewicz fibration,* X *is arbitrary, and* $f\colon X \to E$ *is a homotopy equivalence for which* $pf\colon X \to B$ *is a* $\beta$*-fibration, then* f *is a* $p^{-1}(\alpha)$*-equivalence.*

## 3. ENGULFING

In this section we will establish Theorem 3.6, which is the main engulfing result which will be needed in the sequel. It is deduced rather formally from an engulfing result of [22] which is stated below in Lemma 3.1. We will first need a definition. A polyhedron $P$ in a manifold $M$ is said to be *locally polyhedral in* $M$ if for each $x \in P$ there is an open set $U \subset M$ of $x$ and a triangulation of $U$ as a PL manifold so that $U \cap P$ is a subpolyhedron of this triangulation. Here is Proposition 2.3 of [22].

LEMMA 3.1. *Let* $M^m$ *be a manifold,* $\partial M = \emptyset$, *let* $p \geq 0$ *be an integer, let* $P \subset M$ *be a polyhedron which is closed and locally polyhedral in* $M$, $\dim P \leq m - 3$, *and let* $P_o \subset P$ *be a subpolyhedron so that* $Q = \overline{P - P_o}$ *is compact and* $\dim Q \leq p$. *Also let* $U_o \subset U_1 \subset \cdots \subset U_p$ *and* $M_o \subset M_1 \subset \cdots \subset M_p = M$ *be non-null open subsets of* $M$ *such that* $P \subset M_o$, $P_o \subset U_o$, $U_i \subset M_i$, *and such that for* $0 \leq i \leq p - 1$, *all maps* $(K, L) \to (M_i, U_i)$ *of a finite simplicial pair of dimension* $\leq p - i$ *are homotopic in* $(M_{i+1}, U_i)$ *to a map* $(K, L) \to (U_{i+1}, U_i)$. *Then there exists an isotopy, with compact support, of* $id_M$ *to a homeomorphism* $h: M \to M$ *for which* $P \subset h(U_p)$.

We now derive an easy consequence of Lemma 3.1 which will be more directly useful to us. For notation let $u, v: [0, \infty) \to (-2, 2) \subset R$ be maps so that we have $v(s) < u(s)$, for all $s \geq 2$.

We also use $\Gamma(u)$ and $\Gamma(v)$ to denote the graphs under $u$ and $v$. That is, $\Gamma(u)$ and $\Gamma(v)$ are the subsets of $[0, \infty) \times R$ defined by

$$\Gamma(u) = \{(s, t) | -\infty < t \leq u(s)\},$$

$$\Gamma(v) = \{(s, t) | -\infty < t \leq v(s)\}.$$

LEMMA 3.2. *For each manifold* $M^m$, $\partial M = \emptyset$, *there exists an* $\varepsilon = \varepsilon(m) > 0$ *so that if* $f: M \to [0, \infty) \times R$ *is an* $\varepsilon$*-fibration over* $[0, 4] \times [-4, 4]$, *and* $P \subset M$ *is a polyhedron which is closed and locally polyhedral in* $M$, $\dim P \leq m - 3$, *then there exists an isotopy, which is supported on* $f^{-1}([0,3] \times [-3,3])$, *of* $\mathrm{id}_M$ *to a homeomorphism* $h: M \to M$ *which satisfies*
$P \cap f^{-1}(\Gamma(v)) \subset hf^{-1}(\Gamma(u))$.

Proof. Without loss of generality assume that $P \subset f^{-1}(\Gamma(v))$. Choose a closed subpolyhedron $P_o$ of $P$ in $f^{-1}([2, \infty) \times R)$ so that $P \cap f^{-1}([2.5, \infty) \times R)$ lies in $P_o$. Choose continuous functions $u_i: [0, \infty) \to (-2, 2)$, $v_i: [0, \infty) \to (-2, 2)$, $0 \leq i \leq m$, so that

(1) $v(s) < u_o(s) < v_o(s) < \cdots < u_m(s) < v_m(s)$, for all $s \geq 2$,

(2) $u_o < u_1 < \cdots < u_m = u$,

(3) $v < v_o < \cdots < v_m$,

(4) $u_i < v_i$, for all $i$.

It should be pointed out that the choice of the $u_i$ and $v_i$ depend only on $u$ and $v$, and the $\varepsilon$ must be calculated in terms of these choices. We are assuming that $f$ is an $\varepsilon$-fibration over $[0, 4] \times [-4, 4]$, thus for each $i$ there is a homotopy of the identity on $(f^{-1}(\Gamma(v_i)), f^{-1}(\Gamma(u_i)))$ to a mapping into $(f^{-1}(\Gamma(u_{i+1})), f^{-1}(\Gamma(u_i)))$, with the homotopy taking place in $(f^{-1}(\Gamma(v_{i+1})), f^{-1}(\Gamma(u_i)))$. Also we have $P_o \subset f^{-1}(\Gamma(u_o))$. Now a quick application of Lemma 3.1 gives our desired homeomorphism $h: M \to M$. □

The following result is a generalization of Lemma 3.2. Its proof is based on the proof of Theorem 2.1 of [22]. For notation we still have the maps $u, v: [0, \infty) \to (-2, 2)$ of Lemma 3.2.

LEMMA 3.3. *For each manifold* $M^m$, $\partial M = \emptyset$ *and* $m \geq 5$, *there exists an* $\varepsilon = \varepsilon(m) > 0$ *so that if* $f: M \to [0, \infty) \times R$ *is an* $\varepsilon$*-fibration over* $[0,4] \times [-4,4]$, *then there is an isotopy, which is supported on* $f^{-1}([0, 3] \times [-3, 3])$, *of* $\mathrm{id}_M$ *to a homeomorphism* $h: M \to M$ *which satisfies* $f^{-1}(\Gamma(v)) \subset h(f^{-1}(\Gamma(u))$.

Proof. Choose maps $v_1, v_2: [0, \infty) \to (-2, 2)$ so that $v < v_1 < v_2$ and $v_2(s) < u(s)$ for all $s \geq 2$. If $P^{m-3} \subset M$ is a polyhedron which is closed and locally polyhedral, then by Lemma 3.2 there is a homeomorphism $h_1: M \to M$ which is supported on $f^{-1}([0, 2] \times [-2, 2])$ and for which $P \cap f^{-1}(\Gamma(v_2))$ lies in $h_1(f^{-1}(\Gamma(u))$. If $Q^2 \subset M - P$ is a polyhedron which is closed and locally polyhedral in $M$, then Lemma 3.2 again provides us with a homeomorphism $h_2: M \to M$ which is supported on $f^{-1}(\Gamma(v_1) \cap ([0, 3] \times [-3, 3]))$ and for which $h_2(Q \cap f^{-1}([0, 2.1] \times [-2.1, \infty))$ lies in $f^{-1}(\Gamma(v))$.

According to Proposition 2.7 of [22] we can choose $P$ and $Q$ to be "topological dual skeleta" in $M$. This means that for every $\delta > 0$ we can choose $P$ and $Q$ so that if $C \subset M - P$ is a compactum, then there are homeomorphisms of $M$, with compact support and which are $\delta$-close to $id_M$, which take $C$ as close to $Q$ as we want. The compactum that we have in mind is the closure of

$$h_1(M - f^{-1}(\Gamma(u))) \cap f^{-1}(\Gamma(v_2)),$$

which we call $C$. Let $h_3: M \to M$ be a homeomorphism which is supported on $f^{-1}([0, 3] \times [-3, 3])$, which is $\delta$-close to $id_M$, and which takes $C$ close to $Q$. It is easy to see that with an appropriate choice of $\delta$, $h = h_2 h_3 h_1$ fulfills our requirements. □

We now deduce the following "radial" engulfing result from Lemma 3.3. For notation $B$ will be a finite-dimensional polyhedron which will act as a parameter space. Also $B$ will have the metric topology determined by a fixed triangulation, and $B \times R$ will have the metric which is the product of the metric on $B$ with the standard metric on $R$.

LEMMA 3.4. *For every integer* $m \geq 5$ *and* $\varepsilon > 0$ *there exists a* $\delta > 0$ *so that if* $M^m$ *is a manifold,* $\partial M = \emptyset$, *and* $f: M \to B \times R$ *is a* $\delta$-*fibration over* $B \times [-4, 4]$, *then there exists a homeomorphism* $h: M \to M$ *which is supported on* $f^{-1}(B \times [-3, 3])$ *and which satisfies* $f^{-1}(B \times (-\infty, 1]) \subset hf^{-1}(B \times (-\infty, 0))$.

*Moreover if* $p_B$ = proj: $B \times R \to B$, *then there is an isotopy* $h_t\colon id_M \simeq h$ *which is supported on* $f^{-1}(B \times [-3, 3])$ *and which is a* $(p_B f)^{-1}(\varepsilon)$*-homotopy.*

**Proof.** Choose a fine triangulation of $B$ whose mesh depends on $\varepsilon$. Once this is done the $\delta$ is calculated so that the following constructions make sense. For each vertex $v$ of $B$ let $C_v$ be a small closed neighborhood of $v$ and let $\tilde{C}_v$ be an open set containing $C_v$ so that the $\tilde{C}_v$'s are pairwise-disjoint. We are going to apply Lemma 3.3 to each composition

$$(q \times id)f\colon f^{-1}(\tilde{C}_v \times R) \xrightarrow{f} \tilde{C}_v \times R \xrightarrow{q\times id} [0, \infty) \times R,$$

where $q\colon \tilde{C}_v \to [0, \infty)$ is a proper map. To see how Lemma 3.3 applies first note that we may take $q\colon \tilde{C}_v \times [0, \infty)$ to be a proper retraction. We can choose $\delta$ small enough so that $f\colon f^{-1}(\tilde{C}_v \times R) \to \tilde{C}_v \times R$ is a $\delta$-fibration over $q^{-1}([0, 4] \times [-4, 4])$, yet $(q \times id)f\colon f^{-1}(\tilde{C}_v \times R) \to [0, \infty) \times R$ might not be a $\delta$-fibration over $[0, 4] \times [-4, 4]$. However in the proof of Lemma 3.3 we did not need the full strength of the $\varepsilon$-lifting property over $[0, 4] \times [-4, 4]$. We only needed the $\varepsilon$-lifting property for homotopies which move only in the R-direction. It is easy to see that this is true for the map $(q \times id)f$: $f^{-1}(\tilde{C}_v \times R) \to \tilde{C}_v \times R$. Now applying Lemma 3.3 to $(q \times id)f\colon f^{-1}(\tilde{C}_v \times R) \to \tilde{C}_v \times R$ we obtain a homeomorphism $h(v)\colon M \to M$ which is supported on $f^{-1}(\tilde{C}_v \times [-2, 2])$ and for which $f^{-1}(C_v \times (-\infty, 1.5]) \subset h(v)f^{-1}(\tilde{C}_v \times (-\infty, 0))$. The $h(v)$'s then compose to yield a homeomorphism $h^0\colon M \to M$ such that

$$f^{-1}((\cup C_v) \times (-\infty, 1.5]) \subset h^0 f^{-1}(B \times (-\infty, 0)).$$

This completes the first step of the construction.

The effect of $h^0$ was to deal with the 0-skeleton. The next step is to show how to deal with the 1-skeleton. Let $\sigma$ be a 1-simplex in $B$ with vertices $v_1$, $v_2$, and let $C_\sigma$ be a closed set containing the closure of $\sigma - (C_{v_1} \cup C_{v_2})$ in its interior. Let $\tilde{C}_\sigma$ be a slightly larger open set containing $C_\sigma$. This can be done so that $\tilde{C}_\sigma$'s are pairwise disjoint and so that $\tilde{C}_\sigma$ lies in a small neighborhood of $\sigma$. Again using Lemma 3.3 there is a homeomorphism $h(\sigma)\colon M \to M$ which is supported on $f^{-1}(\tilde{C}_\sigma \times [-2.5, 2.5])$ and for which $f^{-1}((C_\sigma \cup [\tilde{C}_\sigma \cap (C'_{v_1} \cup C'_{v_2})]) \times (-\infty, 1.4])$ lies in

$$h(\sigma)f^{-1}[(\tilde{C}_\sigma \times (-\infty, -2)) \cup ((\tilde{C}_\sigma \cap (C_{v_1} \cup C_{v_2})) \times (-\infty, 1.5))],$$

where $C'_{v_1}$ and $C'_{v_2}$ are slightly smaller neighborhoods of $v_1$ and $v_2$, respectively. Do this for each $\sigma$ and then compose the $h(\sigma)$'s to obtain a homeomorphism $h^1\colon M \to M$. We note that $h^1 h^0 f^{-1}(B \times (-\infty, 0))$ contains $f^{-1}(B^1 \times (-\infty, 1.4])$, where $B^1$ is the 1-skeleton of $B$. If we continue to work through the skeleta of $B$ in this manner we eventually obtain our desired homeomorphism $h\colon M \to M$. The $(p_B f)^{-1}(\varepsilon)$-homotopy $h_t\colon \mathrm{id} \simeq h$ is clear from the construction because $\mathrm{diam}(\Delta) << \varepsilon$, for all simplices $\Delta$ in the triangulation of $B$. □

We now generalize Lemma 3.4. For notation $B$ will still be a finite dimensional polyhedron which acts as a parameter space and $\theta\colon R \to R$ will be a homeomorphism which is supported on $[-1, 1]$.

LEMMA 3.5. *For every integer* $m \geq 5$ *and* $\varepsilon > 0$ *there exists a* $\delta > 0$ *so that if* $M^m$ *is a manifold,* $\partial M = \emptyset$, *and* $f\colon M \to B \times R$ *is a* $\delta$*-fibration over* $B \times [-4, 4]$, *then there exists a homeomorphism* $\tilde{\theta}\colon M \to M$ *which is supported on* $f^{-1}(B \times [-2, 2])$ *and for which* $d(f\tilde{\theta}, (\mathrm{id}_B \times \theta)f) < \varepsilon$. *Moreover we can construct* $\tilde{\theta}$ *so that there is a* $(p_B f)^{-1}(\varepsilon)$*-isotopy of* $\tilde{\theta}$ *to* id *which is supported on* $f^{-1}(B \times [-2, 2])$, *where* $p_B = \mathrm{proj}\colon B \times R \to B$.

Proof. Choose a fine partition of $[-1, 1]$, $-1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ By stacking together the homeomorphisms of Lemma 3.4 we will construct a homeomorphism $\tilde{\theta}\colon M \to M$ which is supported on $f^{-1}(B \times [-2, 2])$, which satisfies $d(p_B f\tilde{\theta}, p_B f) < \varepsilon/2$, and which also satisfies

$$f^{-1}(B \times (-\infty, \theta(x_{i-1})]) \subset \tilde{\theta}f^{-1}(B \times (-\infty, x_i]) \subset f^{-1}(B \times (-\infty, \theta(x_i)]),$$

for $1 \leq i \leq n - 1$. If the support of $\tilde{\theta}$ is sufficiently close to