Lecture Notes in Mathematics

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Jean-Pierre Serre

Lie Algebras and Lie Groups

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Part I – Lie Algebras

Introduction

The main general theorems on Lie Algebras are covered, roughly the content of Bourbaki's Chapter I.

I have added some results on free Lie algebras, which are useful, both for Lie's theory itself (Campbell-Hausdorff formula) and for applications to pro-p-groups.

Lack of time prevented me from including the more precise theory of semisimple Lie algebras (roots, weights, etc.); but, at least, I have given, as a last Chapter, the typical case of $\mathfrak{sl}_n$.

This part has been written with the help of F. Raggi and J. Tate. I want to thank them, and also Sue Golan, who did the typing for both parts.

Jean-Pierre Serre

Harvard, Fall 1964

Chapter I. Lie Algebras: Definition and Examples

Let k be a commutative ring with unit element, and let A be a k-module, then A is said to be a k-algebra if there is given a k-bilinear map $A \times A \to A$ (i.e., a k-homomorphism $A \otimes_k A \to A$).

As usual we may define left, right and two-sided ideals and therefore quotients.

Definition 1. A *Lie algebra* over k is an algebra with the following properties:

1). The map $A \otimes_k A \to A$ admits a factorization

$$A \otimes_k A \to \bigwedge^2 A \to A$$

i.e., if we denote the image of (x, y) under this map by $[x, y]$ then the condition becomes

$$[x, x] = 0 \qquad \text{for all } x \in k.$$

2). $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (Jacobi's identity)

The condition 1) implies $[x, y] = -[y, x]$.

Examples. (i) Let k be a complete field with respect to an absolute value, let G be an analytic group over k, and let $\mathfrak{g}$ be the set of tangent vectors to G at the origin. There is a natural structure of Lie algebra on $\mathfrak{g}$.

(For an algebraic analogue of this, see example (v) below.)

(ii) Let $\mathfrak{g}$ be any k-module. Define $[x, y] = 0$ for all $x, y \in \mathfrak{g}$. Such a $\mathfrak{g}$ is called a *commutative* Lie algebra.

(ii′) If in the preceding example we take $\mathfrak{g} \oplus \bigwedge^2 \mathfrak{g}$ and define

$$\begin{aligned} [x, y] &= x \wedge y \\ [x, y \wedge z] &= 0 \\ [x \wedge y, z] &= 0 \\ [x \wedge y, z \wedge t] &= 0 \end{aligned}$$

for all $x, y, z, t \in \mathfrak{g}$, then $\mathfrak{g} \oplus \bigwedge^2 \mathfrak{g}$ is a Lie algebra.

(iii) Let A be an associative algebra over k and define $[x, y] = xy - yx$, $x, y \in A$. Clearly A with this product satisfies the axioms 1) and 2).

Definition 2. Let A be an algebra over k. A *derivation* $D : A \to A$ is a k-linear map with the property $D(x \cdot y) = Dx \cdot y + x \cdot Dy$.

(iv) The set $\mathrm{Der}(A)$ of all derivations of an algebra A is a Lie algebra with the product $[D, D'] = DD' - D'D$.

We prove it by computation:

$$\begin{aligned}[D,D'](x\cdot y) &= DD'(x\cdot y) - D'D(x\cdot y)\\ &= D(D'x\cdot y + x\cdot D'y) - D'(Dx\cdot y + x\cdot Dy)\\ &= DD'x\cdot y + D'x\cdot Dy + Dx\cdot D'y + x\cdot DD'y\\ &\quad - D'Dx\cdot y - Dx\cdot D'y - D'x\cdot Dy - x\cdot D'Dy\\ &= DD'x\cdot y + x\cdot DD'y - D'Dx\cdot y - x\cdot D'Dy\\ &= [D,D']x\cdot y + x\cdot [D,D']y\ .\end{aligned}$$

Theorem 3. *Let* $\mathfrak{g}$ *be a Lie algebra. For any* $x\in\mathfrak{g}$ *define a map* $\operatorname{ad}x:\mathfrak{g}\to\mathfrak{g}$ *by* $\operatorname{ad}x(y)=[x,y]$, *then:*

1) $\operatorname{ad}x$ *is a derivation of* $\mathfrak{g}$.

2) *The map* $x\mapsto\operatorname{ad}x$ *is a Lie homomorphism of* $\mathfrak{g}$ *into* $\operatorname{Der}(\mathfrak{g})$.

Proof.

$$\begin{aligned}\operatorname{ad}x[y,z] &= [x,[y,z]]\\ &= -[y,[z,x]] - [z,[x,y]]\\ &= [[x,y],z] + [y,[x,z]]\\ &= [\operatorname{ad}x(y),z] + [y,\operatorname{ad}x(z)]\ ,\end{aligned}$$

hence, 1) is equivalent to the Jacobi identity. Now

$$\begin{aligned}\operatorname{ad}[x,y](z) &= [[x,y],z]\\ &= -[[y,z],x] - [[z,x],y]\\ &= [x,[y,z]] - [y,[x,z]]\\ &= \operatorname{ad}x\operatorname{ad}y(z) - \operatorname{ad}y\operatorname{ad}x(z)\\ &= [\operatorname{ad}x,\operatorname{ad}y](z)\ ,\end{aligned}$$

hence 2) is also equivalent to the Jacobi identity.

(v) *The Lie algebra of an algebraic matrix group.*

Let k be a commutative ring and let $A=M_n(k)$ be the algebra of $n\times n$-matrices over k.

Given a set of polynomials $P_\alpha(X_{ij})$, $1\le i,j\le n$, a zero of (P_α) is a matrix $x=(x_{ij})$ such that $x_{ij}\in k$, $P_\alpha(x_{ij})=0$ for all α.

Let $G(k)$ denote the set of zeroes of (P_α) in $A^*=\mathrm{GL}_n(k)$. If k' is any associative, commutative k-algebra we have analogously $G(k')\subset M_n(k')$.

Definition 4. The set (P_α) defines an algebraic group over k if $G(k')$ is a subgroup of $\mathrm{GL}_n(k')$ for all associative, commutative k-algebras k'.

The orthogonal group is an example of an algebraic group (equation: ${}^tX\cdot X=1$, where ${}^tX$ denotes the transpose of X).

Now, let k' be the k-algebra which is free over k with basis $\{1,\varepsilon\}$ where $\varepsilon^2=0$, i.e., $k'=k[\varepsilon]$.

Theorem 5. *Let $\mathfrak{g}$ be the set of matrices $X \in M_n(k)$ such that*

$$1 + \varepsilon X \in G(k[\varepsilon]) \; .$$

Then $\mathfrak{g}$ is a Lie subalgebra of $M_n(k)$.

We have to prove that $X, Y \in \mathfrak{g}$ implies $\lambda X + \mu Y \in \mathfrak{g}$, if $\lambda, \mu \in k$ and $XY - YX \in \mathfrak{g}$.

To prove that, note first that

$$P_\alpha(1 + \varepsilon X) = 0 \text{ for all } \alpha \quad \Longleftrightarrow \quad X \in \mathfrak{g}$$

and, since $\varepsilon^2 = 0$, we have

$$P_\alpha(1 + \varepsilon X) = P_\alpha(1) + dP_\alpha(1)\varepsilon X \; .$$

But $1 \in G(k)$, i.e. $P_\alpha(1) = 0$; therefore

$$P_\alpha(1 + \varepsilon X) = dP_\alpha(1)\varepsilon X \; .$$

Hence, $\mathfrak{g}$ is a submodule of $M_n(k)$.

We introduce now an auxiliary algebra k'' given by $k'' = k[\varepsilon, \varepsilon', \varepsilon\varepsilon']$ where $\varepsilon^2 = \varepsilon'^2 = 0$ and $\varepsilon'\varepsilon = \varepsilon\varepsilon'$, i.e., $k'' = k[\varepsilon] \otimes_k k[\varepsilon']$.

Let $X, Y \in \mathfrak{g}$, so we have

$$g = 1 + \varepsilon X \in G(k[\varepsilon]) \subset G(k'')$$
$$g' = 1 + \varepsilon' Y \in G(k[\varepsilon']) \subset G(k'')$$

$gg' = (1 + \varepsilon X)(1 + \varepsilon' Y) = 1 + \varepsilon X + \varepsilon' Y + \varepsilon\varepsilon' XY$

$g'g = 1 + \varepsilon X + \varepsilon' Y + \varepsilon\varepsilon' YX$.

Write $Z = [X, Y]$; we have

$$gg' = g'g(1 + \varepsilon\varepsilon' Z) \; .$$

Since $gg', g'g \in G(k'')$, it follows that

$$1 + \varepsilon\varepsilon' Z \in G(k'') \; .$$

But the subalgebra $k[\varepsilon\varepsilon']$ of k'' may be identified with $k[\varepsilon]$. It then follows that $1 + \varepsilon Z \in G(k[\varepsilon])$, hence $Z \in \mathfrak{g}$, q.e.d.

Example. The Lie algebra of the orthogonal group is the set of matrices X such that $(1 + \varepsilon X)(1 + \varepsilon({}^tX)) = 1$, i.e., $X + {}^tX = 0$.

(vi) *Construction of Lie algebras from known ones.*

a) Let $\mathfrak{g}$ be a Lie algebra and let $J \subset \mathfrak{g}$ an ideal, then $\mathfrak{g}/J$ is a Lie algebra.

b) Let $(\mathfrak{g}_i)_{i \in I}$ be a family of Lie algebras, then $\prod_{i \in I} \mathfrak{g}_i$ is a Lie algebra.

c) Suppose $\mathfrak{g}$ is a Lie algebra, $\mathfrak{a} \subset \mathfrak{g}$ is an ideal and $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$, then $\mathfrak{g}$ is called a *semidirect product* of $\mathfrak{b}$ by $\mathfrak{a}$ if the natural map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$

induces an isomorphism $\mathfrak{b} \xrightarrow{\sim} \mathfrak{g}/\mathfrak{a}$. If so, and if $x \in \mathfrak{b}$, then $\operatorname{ad} x$ maps $\mathfrak{a}$ into $\mathfrak{a}$ so that $\operatorname{ad}_{\mathfrak{a}} x \in \operatorname{Der}(\mathfrak{a})$, i.e., we have a Lie homomorphism $\theta : \mathfrak{b} \to \operatorname{Der}(\mathfrak{a})$.

Theorem 6. *The structure of $\mathfrak{g}$ is determined by $\mathfrak{a}$, $\mathfrak{b}$ and θ, and these can be given arbitrarily.*

Proof. Since $\mathfrak{g}$ is the direct sum of $\mathfrak{a}$ and $\mathfrak{b}$ as a k-module and since multiplication is bilinear and anticommutative we have to consider the product $[x, y]$ in the following three cases:

$$x, y \in \mathfrak{a}$$
$$x, y \in \mathfrak{b}$$
$$x \in \mathfrak{b},\ y \in \mathfrak{a}\ .$$

In the first case $[x, y]$ is given in $\mathfrak{a}$, in the second one $[x, y]$ is given in $\mathfrak{b}$ and in the last one we have

$$[x, y] = \operatorname{ad} x(y) = \theta(x)y\ .$$

Conversely, given the Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$ and a Lie homomorphism

$$\theta : \mathfrak{b} \to \operatorname{Der}(\mathfrak{a})\ ,$$

we can construct a Lie algebra $\mathfrak{g}$ which is a semidirect product of $\mathfrak{b}$ by $\mathfrak{a}$ in such a way that $\theta(x) = \operatorname{ad}_{\mathfrak{a}} x$, where $\operatorname{ad}_{\mathfrak{a}} x$ is the restriction to $\mathfrak{a}$ of $\operatorname{ad}_{\mathfrak{g}} x$, for $x \in \mathfrak{b}$. One has to check that the Jacobi's identity

$$J(x, y, z) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

holds. There are essentially four cases to be considered:

(a) $x, y, z \in \mathfrak{a}$ – then $J(x, y, z) = 0$ because $\mathfrak{a}$ is a Lie algebra.

(b) $x, y \in \mathfrak{a}$, $z \in \mathfrak{b}$ – $J(x, y, z) = 0 \iff \theta(z)$ is a derivation of $\mathfrak{a}$.

(c) $x \in \mathfrak{a}$, $y, z \in \mathfrak{b}$ – $J(x, y, z) = 0 \iff \theta([y, z]) = \theta(y)\theta(z) - \theta(z)\theta(y)$.

(d) $x, y, z \in \mathfrak{b}$ – $J(x, y, z) = 0$ because $\mathfrak{b}$ is a Lie algebra.

Chapter II. Filtered Groups and Lie Algebras

1. Formulae on commutators

Let G be a group and let $x, y, z \in G$. We will use the following notations:

(i) $x^y = y^{-1}xy$, hence the map $G \to G$ given by $x \mapsto x^y$ is an automorphism of G, and we have the relation $(x^y)^z = x^{yz}$.

(ii) $(x, y) = x^{-1}y^{-1}xy$ which is called the commutator of x and y.

Proposition 1.1. *We have the identities*:

(1) $xy = yx^y = yx(x,y),\ x^y = x(x,y),\ (x,x) = 1,\ (y,x) = (x,y)^{-1}$.

(2) $(x, yz) = (x,z)(x,y)^z$.

(2′) $(xy, z) = (x,z)^y(y,z)$.

(3) $(x^y, (y,z))(y^z, (z,x))(x^z, (x,y)) = 1$.

Proof. (1) is trivial.

(2) From (i) and (1) we have

$$\begin{aligned} x(x,yz) &= x^{yz} \\ &= (x^y)^z \\ &= [x(x,y)]^z \\ &= x^z(x,y)^z = x(x,z)(x,y)^z \end{aligned}$$

and therefore $(x,yz) = (x,z)(x,y)^z$.

(2′)
$$\begin{aligned} xy(xy,z) = (xy)^z &= x^z y^z \\ &= x(x,z)y(y,z) \\ &= xy(x,z)^y(y,z) \end{aligned}$$

and therefore $(xy,z) = (x,z)^y(y,z)$.

(3)
$$\begin{aligned} (x^y,(y,z)) &= y^{-1}x^{-1}yz^{-1}y^{-1}zyy^{-1}xyy^{-1}z^{-1}yz \\ &= y^{-1}x^{-1}yz^{-1}y^{-1}zxz^{-1}yz\ . \end{aligned}$$

Put

$$\begin{aligned} u &= zxz^{-1}yz \\ v &= xyx^{-1}zx \\ w &= yzy^{-1}xy \end{aligned}$$

then $(x^y,(y,z)) = w^{-1}u$.

Analogously (by cyclic permutation)

$$\begin{aligned} (y^z,(z,x)) &= u^{-1}v \\ (z^x,(x,y)) &= v^{-1}w\ . \end{aligned}$$

Hence $(x^y,(y,z))(y^z,(z,x))(z^x,(x,y)) = 1$ q.e.d.

Applications:

Let A, B be subgroups of a group G and let (A, B) denote the subgroup of G generated by the commutators (a, b) for all $a \in A$, $b \in B$.

If A, B, C are normal subgroups of G, then (A, B) is also normal and we have the relation

$$(A,(B,C)) \subset (B,(C,A))(C,(A,B))$$

which follows from 1.1(3).

2. Filtration on a group

Definition 2.1. A *filtration on a group* G is a map $w : G \to \mathbf{R} \cup \{+\infty\}$ satisfying the following axioms:

(1) $w(1) = +\infty$.

(2) $w(x) > 0$ for all $x \in G$.

(3) $w(xy^{-1}) \geq \inf\{w(x), w(y)\}$.

(4) $w((x,y)) \geq w(x) + w(y)$.

It follows from (3) that $w(y^{-1}) = w(y)$. If $\lambda \in \mathbf{R}_+$ we define

$$G_\lambda = \{\, x \in G \mid w(x) \geq \lambda \,\}$$
$$G_\lambda^+ = \{\, x \in G \mid w(x) > \lambda \,\} .$$

The condition (3) shows that G_λ, G_λ^+ are subgroups of G. Moreover, if $x \in G_\lambda$, $y \in G$ then $x^y \equiv x \pmod{G_\lambda^+}$ which follows from the relation

$$w((x,y)) \geq \lambda + w(y) > \lambda .$$

This also proves that G_λ is a normal subgroup of G and since $G_\lambda^+ = \bigcup_{\mu>\lambda} G_\mu$ it follows that G_λ^+ is also a normal subgroup of G.

The family $\{G_\lambda\}$ (resp. $\{G_\lambda^+\}$) is decreasing, i.e., $\lambda < \mu$ implies $G_\lambda \supset G_\mu$ (resp. $G_\lambda^+ \supset G_\mu^+$).

Definition 2.2. For all $\alpha \geq 0$ we define

$$\operatorname{gr}_\alpha G = G_\alpha / G_\alpha^+ \quad \text{and} \quad \operatorname{gr} G = \sum_\alpha \operatorname{gr}_\alpha G .$$

Proposition 2.3.

1) $\operatorname{gr}_\alpha G$ *is an abelian group.*

2) *If* $x \in G_\alpha$ *let* $\bar{x}$ *be its image in* $\operatorname{gr}_\alpha G$; *one has* $\overline{(x^y)} = \bar{x}$ *for all* $y \in G$.

3) *The map* $c_{\alpha,\beta} : G_\alpha \times G_\beta \to G_{\alpha+\beta}$ *defined by* $x, y \mapsto (x,y)$ *induces a bilinear map* $\bar{c}_{\alpha,\beta} : \mathrm{gr}_\alpha G \times \mathrm{gr}_\beta G \to \mathrm{gr}_{\alpha+\beta} G$.

4) *The maps* $\bar{c}_{\alpha,\beta}$ *can be extended by linearity to* $c : \mathrm{gr}\, G \times \mathrm{gr}\, G \to \mathrm{gr}\, G$ *and this defines a Lie algebra structure on* $\mathrm{gr}\, G$.

Proof. 1) It follows from 2.1(4).

2) It is already proved.

3) Let $x, x' \in G_\alpha$, $y, y' \in G_\beta$, then $(x,y) \in G_{\alpha+\beta}$ and we have to prove that if $u, v \in G_\alpha^+$ then $(xu, y) \equiv (x,y) \bmod G_{\alpha+\beta}^+$, $(x, yv) \equiv (x,y) \bmod G_{\alpha+\beta}^+$. Using 1.1(2′) and (3) we have

$$\begin{aligned}
\overline{(xu,y)} &= \overline{(x,y)^u} + \overline{(u,y)} = \overline{(x,y)} \\
\overline{(x,yv)} &= \overline{(x,v)} + \overline{(x,y)^v} = \overline{(x,y)} \\
\overline{(xx',y)} &= \overline{(x,y)^{x'}} + \overline{(x',y)} = \overline{(x,y)} + \overline{(x',y)} \\
\overline{(x,y'y)} &= \overline{(x,y)} + \overline{(x,y')^y} = \overline{(x,y)} + \overline{(x,y')} \; .
\end{aligned}$$

This proves 3).

4) Let $\xi \in \mathrm{gr}_\alpha G$, $\eta \in \mathrm{gr}_\beta G$ and choose elements $x \in G_\alpha$, $x \in G_\beta$ such that $\bar{x} = \xi$, $\bar{y} = \eta$. Then we have $\overline{(x,y)} = \bar{c}_{\alpha,\beta}(\xi,\eta)$, which we also write $[\xi,\eta]$.

Now if $\xi \in \mathrm{gr}\, G$ then $\xi = \sum_\alpha \xi_\alpha$ where $\xi_\alpha \in \mathrm{gr}_\alpha G$. In order to prove that $[\xi,\xi] = 0$, it is sufficient to prove that $[\xi_\alpha,\xi_\alpha] = 0$ and $[\xi_\alpha,\xi_\beta] = -[\xi_\beta,\xi_\alpha]$. Let $x_\alpha \in G_\alpha$ such that $\bar{x}_\alpha = \xi_\alpha$ for all α. Then we have $[\xi_\alpha,\xi_\alpha] = \overline{(x_\alpha,x_\alpha)} = \bar{1} = 0$, and

$$[\xi_\alpha,\xi_\beta] = \overline{(x_\alpha,x_\beta)} = \overline{(x_\beta,x_\alpha)}^{-1} = -[\xi_\beta,\xi_\alpha] \; .$$

In order to prove the Jacobi identity $J(\xi,\eta,\zeta) = 0$, since J is trilinear, it is enough to consider the case $\xi \in \mathrm{gr}_\alpha G$, $\eta \in \mathrm{gr}_\beta G$ and $\zeta \in \mathrm{gr}_\gamma G$. Now using the Proposition 1.1(3) we have, for $x \in G_\alpha$, $y \in G_\beta$, $z \in G_\gamma$ such that $\bar{x} = \xi$, $\bar{y} = \eta$, $\bar{z} = \zeta$.

$$J(\xi,\eta,\zeta) = \overline{(x^y,(y,z))(y^z,(z,x))(z^x,(x,y))} = \bar{1} = 0$$

because $\overline{x^y} = \xi$, $\overline{y^z} = \eta$, $\overline{z^x} = \zeta$. q.e.d.

3. Integral filtrations of a group

Proposition 3.1. *For any group G the following two objects are in a one-one correspondence*:

1) *Filtrations* $w : G \to \mathbf{R} \cup \{+\infty\}$ *such that* $w(G) \subset \mathbf{N} \cup \{+\infty\}$.

2) *Decreasing sequences* $\{G_n\}_{n \in \mathbf{N}}$ *of subgroups of* G *such that*
   (i) $G_1 = G$.
   (ii) $(G_n, G_m) \subset G_{n+m}$.

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). Let $x \in G$, then we define a filtration $w : G \to \mathbf{R} \cup \{+\infty\}$ by $w(x) = \sup_{x \in G_n}\{n\}$.

It is clear that $w(1) = +\infty$, $w(x) > 0$ for all $x \in G$, and $w(x) = w(x^{-1})$.

Now let $w(x) = n$, $w(y) = m$, i.e., $x \in G_n$, $y \in G_m$ and $x \notin G_{n+1}$, $y \notin G_{m+1}$. Suppose $n \leq m$, then $G_m \subset G_n$ and therefore $xy^{-1} \in G_n$, i.e.,

$$w(xy^{-1}) \geq \inf\{w(x), w(y)\} .$$

In case $n = +\infty$ or $m = +\infty$, we have obviously this inequality.

Finally the inequality $w((x,y)) \geq w(x) + w(y)$ follows from (ii). q.e.d.

Example. The descending central series of G.

Define $G_1 = G$ and by induction $G_{n+1} = (G, G_n)$. Then the sequence $\{G_n\}$ satisfies the conditions (i)–(ii) of (2) in the Proposition 3.1. Condition (i) is satisfied by definition, and we will prove (ii) by induction on n in the pair (G_n, G_m).

Let first $n = 1$, then $(G, G_m) \subset G_{m+1}$ by definition. Now suppose $n > 1$, then

$$\begin{aligned}(G_n, G_m) = ((G, G_{n-1}), G_m) &\subset (G, (G_{n-1}, G_m))(G_{n-1}, (G, G_m)) \\ &\subset (G, G_{n+m-1})(G_{n-1}, G_{m+1}) \\ &\subset G_{n+m} \cdot G_{n+m} = G_{n+m} .\end{aligned}$$

Conversely, if $\{H_n\}$ is a decreasing sequence of subgroups of G which verifies (2), then $H_n \supset G_n$ for all n. The proof of this is also by induction. Suppose $n = 1$, then by definition $H_1 = G_1$. Now if $n \geq 1$, we have

$$H_{n+1} \supset (H_1, H_n) \supset (G, G_n) = G_{n+1} .$$

4. Filtrations in GL(n)

Let k be a field with an ultrametric absolute value $|x| = a^{v(x)}$. Let A_v be the ring of v and let $\mathfrak{m}_v$ be the maximal ideal of A_v, let $k(v) = A_v/\mathfrak{m}_v$.

Let n be a positive integer and let G be the group of $n \times n$-matrices with coefficients in A_v such that $g \equiv 1 \bmod \mathfrak{m}_v$, i.e., if $g = (g_{ij})$ then $g_{ij} \equiv \delta_{ij} \bmod \mathfrak{m}_v$.

If $g \in G$ then $g = 1 + x$ where x is a matrix with coefficients in $\mathfrak{m}_v$.

Clearly G is a group, because it can be described as

$$G = \mathrm{Ker}\{\ \mathrm{GL}(n, A_v) \to \mathrm{GL}(n, k(v))\ \} .$$

Let $X \in M_n(k)$, $X = (x_{ij})$, then define $v(X) = \inf\{v(x_{ij})\}$.

We can define a map $w : G \to \mathbf{R} \cup \{+\infty\}$ by $w(g) = v(x)$, where $g = 1 + x$.

Theorem 4.1. *The map w is a filtration on G.*

Proof. The conditions $w(1) = +\infty$ and $w(g) > 0$ for all $g \in G$ are obvious.
Let now $G_\lambda = \{\, g \in G \mid w(g) \geq \lambda \,\}$. If $\mathfrak{a}_\lambda$ is defined by

$$\mathfrak{a}_\lambda = \{\, x \mid x \in k,\ v(x) \geq \lambda \,\} ,$$

the set G_λ is the kernel of the canonical homomorphism

$$\mathrm{GL}(n, A_v) \to \mathrm{GL}(n, A_v/\mathfrak{a}_\lambda) .$$

Hence G_λ is a subgroup of G, and this proves condition (3).

To prove condition (4), i.e., $(G_\lambda, G_\mu) \subset G_{\lambda+\mu}$, write $g \in G_\lambda$, $h \in G_\mu$ in the form:

$$g = 1 + x , \qquad h = 1 + y .$$

One must check that $hg \equiv gh \pmod{G_{\lambda+\mu}}$. But

$$hg = 1 + x + y + yx$$
$$gh = 1 + x + y + xy$$

and the coefficients of xy and yx belong to $\mathfrak{a}_{\lambda+\mu}$. Hence hg and gh have the same image in $\mathrm{GL}(n, A_v/\mathfrak{a}_{\lambda+\mu})$, and they are congruent $\bmod G_{\lambda+\mu}$, q.e.d.

Exercises

1. Determine the Lie algebra $\mathrm{gr}\, G$.
2. Prove that $G = \varprojlim G/G_\lambda$ if k is complete.

Chapter III. Universal Algebra of a Lie Algebra

1. Definition

Let k be a commutative ring and let $\mathfrak{g}$ be a Lie algebra over k.

Definition 1.1. A *universal algebra* of $\mathfrak{g}$ is a map $\varepsilon : \mathfrak{g} \to U\mathfrak{g}$, where $U\mathfrak{g}$ is an associative algebra, with a unit satisfying the following properties:

1). ε is a Lie algebra homomorphism,

$$\text{(i.e., } \varepsilon \text{ is } k\text{-linear and } \varepsilon[x,y] = \varepsilon x \cdot \varepsilon y - \varepsilon y \cdot \varepsilon x).$$

2). If A is any associative algebra with a unit and $\alpha : \mathfrak{g} \to A$ is any Lie algebra homomorphism, there is a unique homomorphism of associative algebras $\varphi : U\mathfrak{g} \to A$ such that the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varepsilon} & U\mathfrak{g} \\ {\scriptstyle\alpha}\downarrow & \swarrow {\scriptstyle\varphi} & \\ A & & \end{array}$$

is commutative [i.e., there is an isomorphism

$$\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, LA) \cong \mathrm{Hom}_{\mathrm{Ass}}(U\mathfrak{g}, A)$$

where LA is the Lie algebra associated to A, cf. Chap. I, example (iii).]

It is trivial that $U\mathfrak{g}$, if it exists, is unique (up to a unique isomorphism). To prove its existence, we use the *tensor algebra* $T\mathfrak{g}$ of $\mathfrak{g}$, i.e., $T\mathfrak{g} = \sum_{n=0}^{\infty} T^n\mathfrak{g}$, where $T^n\mathfrak{g} = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g} = \bigotimes^n \mathfrak{g}$ for $n \geq 0$. For any associative algebra A with a unit, one has: $\mathrm{Hom}_{\mathrm{Mod}}(\mathfrak{g}, A) = \mathrm{Hom}_{\mathrm{Ass}}(T\mathfrak{g}, A)$.

Now let I be the two-sided ideal of $T\mathfrak{g}$ generated by the elements of the form $[x,y] - x \otimes y + y \otimes x,\ \ x, y \in \mathfrak{g}$.

Take $U\mathfrak{g} = T\mathfrak{g}/I$, then we have:

Theorem 1.2. *Let $\varepsilon : \mathfrak{g} \to U\mathfrak{g}$ be the composition $\mathfrak{g} \to T^1\mathfrak{g} \to T\mathfrak{g} \to U\mathfrak{g}$. Then the pair $(U\mathfrak{g}, \varepsilon)$ is a universal algebra of $\mathfrak{g}$.*

In fact, let α be a Lie homomorphism of $\mathfrak{g}$ into an associative algebra A. Since α is k-linear, it extends to a unique homomorphism $\psi : T\mathfrak{g} \to A$. It is clear that $\psi(I) = 0$, hence ψ defines $\varphi : U\mathfrak{g} \to A$, and we have checked the universal property of $U\mathfrak{g}$.

Remark. Let E be a $\mathfrak{g}$-module (i.e., a k-module with a bilinear product $\mathfrak{g} \times E \to E$ such that $[x,y]e = x(ye) - y(x \cdot e)$ for $x, y \in \mathfrak{g}$, $e \in E$). The map $\mathfrak{g} \to \mathrm{End}(E,E)$ which defines the module structure of E is a Lie homomorphism. Hence it extends to an algebra homomorphism $U\mathfrak{g} \to \mathrm{End}(E,E)$ and E becomes a $U\mathfrak{g}$-*left-module*. It is easy to check that one obtains in this