

**Lecture Notes in  
Mathematics**

**1500**

**Jean-Pierre Serre**

**Lie Algebras  
and Lie Groups**



**Springer-Verlag**

Jean-Pierre Serre

# Lie Algebras and Lie Groups

1964 Lectures given at Harvard University

**Springer-Verlag**

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London Paris Tokyo

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Mathematics Subject Classification (1991): 17B

2nd edition

originally published by W. A. Benjamin, Inc., New York, 1965

ISBN 3-540-55008-9 Springer-Verlag Berlin Heidelberg New York

ISBN 0-387-55008-9 Springer-Verlag New York Berlin Heidelberg

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Printed in Germany

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.  
46/3140-543210 - Printed on acid-free paper

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# Part I – Lie Algebras

## Introduction

The main general theorems on Lie Algebras are covered, roughly the content of Bourbaki's Chapter I.

I have added some results on free Lie algebras, which are useful, both for Lie's theory itself (Campbell-Hausdorff formula) and for applications to pro- $p$ -groups.

Lack of time prevented me from including the more precise theory of semisimple Lie algebras (roots, weights, etc.); but, at least, I have given, as a last Chapter, the typical case of  $\mathfrak{sl}_n$ .

This part has been written with the help of F. Raggi and J. Tate. I want to thank them, and also Sue Golan, who did the typing for both parts.

Jean-Pierre Serre

Harvard, Fall 1964

# Chapter I. Lie Algebras: Definition and Examples

Let  $k$  be a commutative ring with unit element, and let  $A$  be a  $k$ -module, then  $A$  is said to be a  $k$ -algebra if there is given a  $k$ -bilinear map  $A \times A \rightarrow A$  (i.e., a  $k$ -homomorphism  $A \otimes_k A \rightarrow A$ ).

As usual we may define left, right and two-sided ideals and therefore quotients.

**Definition 1.** A *Lie algebra* over  $k$  is an algebra with the following properties:

- 1). The map  $A \otimes_k A \rightarrow A$  admits a factorization

$$A \otimes_k A \rightarrow \wedge^2 A \rightarrow A$$

i.e., if we denote the image of  $(x, y)$  under this map by  $[x, y]$  then the condition becomes

$$[x, x] = 0 \quad \text{for all } x \in k.$$

- 2).  $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$  (Jacobi's identity)

The condition 1) implies  $[x, y] = -[y, x]$ .

*Examples.* (i) Let  $k$  be a complete field with respect to an absolute value, let  $G$  be an analytic group over  $k$ , and let  $\mathfrak{g}$  be the set of tangent vectors to  $G$  at the origin. There is a natural structure of Lie algebra on  $\mathfrak{g}$ .

(For an algebraic analogue of this, see example (v) below.)

(ii) Let  $\mathfrak{g}$  be any  $k$ -module. Define  $[x, y] = 0$  for all  $x, y \in \mathfrak{g}$ . Such a  $\mathfrak{g}$  is called a *commutative* Lie algebra.

(ii') If in the preceding example we take  $\mathfrak{g} \oplus \wedge^2 \mathfrak{g}$  and define

$$\begin{aligned} [x, y] &= x \wedge y \\ [x, y \wedge z] &= 0 \\ [x \wedge y, z] &= 0 \\ [x \wedge y, z \wedge t] &= 0 \end{aligned}$$

for all  $x, y, z, t \in \mathfrak{g}$ , then  $\mathfrak{g} \oplus \wedge^2 \mathfrak{g}$  is a Lie algebra.

(iii) Let  $A$  be an associative algebra over  $k$  and define  $[x, y] = xy - yx$ ,  $x, y \in A$ . Clearly  $A$  with this product satisfies the axioms 1) and 2).

**Definition 2.** Let  $A$  be an algebra over  $k$ . A *derivation*  $D : A \rightarrow A$  is a  $k$ -linear map with the property  $D(x \cdot y) = Dx \cdot y + x \cdot Dy$ .

(iv) The set  $\text{Der}(A)$  of all derivations of an algebra  $A$  is a Lie algebra with the product  $[D, D'] = DD' - D'D$ .

We prove it by computation:



$$\begin{aligned}
[D, D'](x \cdot y) &= DD'(x \cdot y) - D'D(x \cdot y) \\
&= D(D'x \cdot y + x \cdot D'y) - D'(Dx \cdot y + x \cdot Dy) \\
&= DD'x \cdot y + D'x \cdot Dy + Dx \cdot D'y + x \cdot DD'y \\
&\quad - D'Dx \cdot y - Dx \cdot D'y - D'x \cdot Dy - x \cdot D'Dy \\
&= DD'x \cdot y + x \cdot DD'y - D'Dx \cdot y - x \cdot D'Dy \\
&= [D, D']x \cdot y + x \cdot [D, D']y .
\end{aligned}$$

**Theorem 3.** Let  $\mathfrak{g}$  be a Lie algebra. For any  $x \in \mathfrak{g}$  define a map  $\text{ad } x : \mathfrak{g} \rightarrow \mathfrak{g}$  by  $\text{ad } x(y) = [x, y]$ , then:

- 1)  $\text{ad } x$  is a derivation of  $\mathfrak{g}$ .
- 2) The map  $x \mapsto \text{ad } x$  is a Lie homomorphism of  $\mathfrak{g}$  into  $\text{Der}(\mathfrak{g})$ .

*Proof.*

$$\begin{aligned}
\text{ad } x[y, z] &= [x, [y, z]] \\
&= -[y, [z, x]] - [z, [x, y]] \\
&= [[x, y], z] + [y, [x, z]] \\
&= [\text{ad } x(y), z] + [y, \text{ad } x(z)] ,
\end{aligned}$$

hence, 1) is equivalent to the Jacobi identity. Now

$$\begin{aligned}
\text{ad}[x, y](z) &= [[x, y], z] \\
&= -[[y, z], x] - [[z, x], y] \\
&= [x, [y, z]] - [y, [x, z]] \\
&= \text{ad } x \text{ ad } y(z) - \text{ad } y \text{ ad } x(z) \\
&= [\text{ad } x, \text{ad } y](z) ,
\end{aligned}$$

hence 2) is also equivalent to the Jacobi identity.

(v) *The Lie algebra of an algebraic matrix group.*

Let  $k$  be a commutative ring and let  $A = M_n(k)$  be the algebra of  $n \times n$ -matrices over  $k$ .

Given a set of polynomials  $P_\alpha(X_{ij})$ ,  $1 \leq i, j \leq n$ , a zero of  $(P_\alpha)$  is a matrix  $x = (x_{ij})$  such that  $x_{ij} \in k$ ,  $P_\alpha(x_{ij}) = 0$  for all  $\alpha$ .

Let  $G(k)$  denote the set of zeroes of  $(P_\alpha)$  in  $A^* = \text{GL}_n(k)$ . If  $k'$  is any associative, commutative  $k$ -algebra we have analogously  $G(k') \subset M_n(k')$ .

**Definition 4.** The set  $(P_\alpha)$  defines an algebraic group over  $k$  if  $G(k')$  is a subgroup of  $\text{GL}_n(k')$  for all associative, commutative  $k$ -algebras  $k'$ .

The orthogonal group is an example of an algebraic group (equation:  ${}^tX \cdot X = 1$ , where  ${}^tX$  denotes the transpose of  $X$ ).

Now, let  $k'$  be the  $k$ -algebra which is free over  $k$  with basis  $\{1, \varepsilon\}$  where  $\varepsilon^2 = 0$ , i.e.,  $k' = k[\varepsilon]$ .

**Theorem 5.** Let  $\mathfrak{g}$  be the set of matrices  $X \in M_n(k)$  such that

$$1 + \varepsilon X \in G(k[\varepsilon]) .$$

Then  $\mathfrak{g}$  is a Lie subalgebra of  $M_n(k)$ .

We have to prove that  $X, Y \in \mathfrak{g}$  implies  $\lambda X + \mu Y \in \mathfrak{g}$ , if  $\lambda, \mu \in k$  and  $XY - YX \in \mathfrak{g}$ .

To prove that, note first that

$$P_\alpha(1 + \varepsilon X) = 0 \text{ for all } \alpha \iff X \in \mathfrak{g}$$

and, since  $\varepsilon^2 = 0$ , we have

$$P_\alpha(1 + \varepsilon X) = P_\alpha(1) + dP_\alpha(1)\varepsilon X .$$

But  $1 \in G(k)$ , i.e.  $P_\alpha(1) = 0$ ; therefore

$$P_\alpha(1 + \varepsilon X) = dP_\alpha(1)\varepsilon X .$$

Hence,  $\mathfrak{g}$  is a submodule of  $M_n(k)$ .

We introduce now an auxiliary algebra  $k''$  given by  $k'' = k[\varepsilon, \varepsilon', \varepsilon\varepsilon']$  where  $\varepsilon^2 = \varepsilon'^2 = 0$  and  $\varepsilon'\varepsilon = \varepsilon\varepsilon'$ , i.e.,  $k'' = k[\varepsilon] \otimes_k k[\varepsilon']$ .

Let  $X, Y \in \mathfrak{g}$ , so we have

$$\begin{aligned} g &= 1 + \varepsilon X \in G(k[\varepsilon]) \subset G(k'') \\ g' &= 1 + \varepsilon' Y \in G(k[\varepsilon']) \subset G(k'') \end{aligned}$$

$$\begin{aligned} gg' &= (1 + \varepsilon X)(1 + \varepsilon' Y) = 1 + \varepsilon X + \varepsilon' Y + \varepsilon\varepsilon' XY \\ g'g &= 1 + \varepsilon X + \varepsilon' Y + \varepsilon\varepsilon' YX . \end{aligned}$$

Write  $Z = [X, Y]$ ; we have

$$gg' = g'g(1 + \varepsilon\varepsilon'Z) .$$

Since  $gg', g'g \in G(k'')$ , it follows that

$$1 + \varepsilon\varepsilon'Z \in G(k'') .$$

But the subalgebra  $k[\varepsilon\varepsilon']$  of  $k''$  may be identified with  $k[\varepsilon]$ . It then follows that  $1 + \varepsilon Z \in G(k[\varepsilon])$ , hence  $Z \in \mathfrak{g}$ , q.e.d.

*Example.* The Lie algebra of the orthogonal group is the set of matrices  $X$  such that  $(1 + \varepsilon X)(1 + \varepsilon({}^t X)) = 1$ , i.e.,  $X + {}^t X = 0$ .

(vi) *Construction of Lie algebras from known ones.*

a) Let  $\mathfrak{g}$  be a Lie algebra and let  $J \subset \mathfrak{g}$  an ideal, then  $\mathfrak{g}/J$  is a Lie algebra.

b) Let  $(\mathfrak{g}_i)_{i \in I}$  be a family of Lie algebras, then  $\prod_{i \in I} \mathfrak{g}_i$  is a Lie algebra.

c) Suppose  $\mathfrak{g}$  is a Lie algebra,  $\mathfrak{a} \subset \mathfrak{g}$  is an ideal and  $\mathfrak{b}$  is a subalgebra of  $\mathfrak{g}$ , then  $\mathfrak{g}$  is called a *semidirect product* of  $\mathfrak{b}$  by  $\mathfrak{a}$  if the natural map  $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a}$

induces an isomorphism  $\mathfrak{b} \xrightarrow{\sim} \mathfrak{g}/\mathfrak{a}$ . If so, and if  $x \in \mathfrak{b}$ , then  $\text{ad } x$  maps  $\mathfrak{a}$  into  $\mathfrak{a}$  so that  $\text{ad}_{\mathfrak{a}} x \in \text{Der}(\mathfrak{a})$ , i.e., we have a Lie homomorphism  $\theta : \mathfrak{b} \rightarrow \text{Der}(\mathfrak{a})$ .

**Theorem 6.** *The structure of  $\mathfrak{g}$  is determined by  $\mathfrak{a}$ ,  $\mathfrak{b}$  and  $\theta$ , and these can be given arbitrarily.*

*Proof.* Since  $\mathfrak{g}$  is the direct sum of  $\mathfrak{a}$  and  $\mathfrak{b}$  as a  $k$ -module and since multiplication is bilinear and anticommutative we have to consider the product  $[x, y]$  in the following three cases:

$$\begin{aligned} x, y &\in \mathfrak{a} \\ x, y &\in \mathfrak{b} \\ x &\in \mathfrak{b}, y \in \mathfrak{a}. \end{aligned}$$

In the first case  $[x, y]$  is given in  $\mathfrak{a}$ , in the second one  $[x, y]$  is given in  $\mathfrak{b}$  and in the last one we have

$$[x, y] = \text{ad } x(y) = \theta(x)y .$$

Conversely, given the Lie algebras  $\mathfrak{a}$  and  $\mathfrak{b}$  and a Lie homomorphism

$$\theta : \mathfrak{b} \rightarrow \text{Der}(\mathfrak{a}) ,$$

we can construct a Lie algebra  $\mathfrak{g}$  which is a semidirect product of  $\mathfrak{b}$  by  $\mathfrak{a}$  in such a way that  $\theta(x) = \text{ad}_{\mathfrak{a}} x$ , where  $\text{ad}_{\mathfrak{a}} x$  is the restriction to  $\mathfrak{a}$  of  $\text{ad}_{\mathfrak{g}} x$ , for  $x \in \mathfrak{b}$ . One has to check that the Jacobi's identity

$$J(x, y, z) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

holds. There are essentially four cases to be considered:

- (a)  $x, y, z \in \mathfrak{a}$       - then  $J(x, y, z) = 0$  because  $\mathfrak{a}$  is a Lie algebra.
- (b)  $x, y \in \mathfrak{a}, z \in \mathfrak{b}$  -  $J(x, y, z) = 0 \iff \theta(z)$  is a derivation of  $\mathfrak{a}$ .
- (c)  $x \in \mathfrak{a}, y, z \in \mathfrak{b}$  -  $J(x, y, z) = 0 \iff \theta([y, z]) = \theta(y)\theta(z) - \theta(z)\theta(y)$ .
- (d)  $x, y, z \in \mathfrak{b}$       -  $J(x, y, z) = 0$  because  $\mathfrak{b}$  is a Lie algebra.

## Chapter II. Filtered Groups and Lie Algebras

### 1. Formulae on commutators

Let  $G$  be a group and let  $x, y, z \in G$ . We will use the following notations:

(i)  $x^y = y^{-1}xy$ , hence the map  $G \rightarrow G$  given by  $x \mapsto x^y$  is an automorphism of  $G$ , and we have the relation  $(x^y)^z = x^{yz}$ .

(ii)  $(x, y) = x^{-1}y^{-1}xy$  which is called the commutator of  $x$  and  $y$ .

**Proposition 1.1.** *We have the identities:*

$$(1) \quad xy = yx^y = yx(x, y), \quad x^y = x(x, y), \quad (x, x) = 1, \quad (y, x) = (x, y)^{-1}.$$

$$(2) \quad (x, yz) = (x, z)(x, y)^z.$$

$$(2') \quad (xy, z) = (x, z)^y(y, z).$$

$$(3) \quad (x^y, (y, z))(y^z, (z, x))(x^z, (x, y)) = 1.$$

*Proof.* (1) is trivial.

(2) From (i) and (1) we have

$$\begin{aligned} x(x, yz) &= x^{yz} \\ &= (x^y)^z \\ &= [x(x, y)]^z \\ &= x^z(x, y)^z = x(x, z)(x, y)^z \end{aligned}$$

and therefore  $(x, yz) = (x, z)(x, y)^z$ .

$$\begin{aligned} (2') \quad xy(xy, z) &= (xy)^z = x^z y^z \\ &= x(x, z)y(y, z) \\ &= xy(x, z)^y(y, z) \end{aligned}$$

and therefore  $(xy, z) = (x, z)^y(y, z)$ .

$$\begin{aligned} (3) \quad (x^y, (y, z)) &= y^{-1}x^{-1}yz^{-1}y^{-1}zyy^{-1}xyy^{-1}z^{-1}yz \\ &= y^{-1}x^{-1}yz^{-1}y^{-1}zxxz^{-1}yz. \end{aligned}$$

Put

$$\begin{aligned} u &= zxz^{-1}yz \\ v &= xyx^{-1}zx \\ w &= yzy^{-1}xy \end{aligned}$$

then  $(x^y, (y, z)) = w^{-1}u$ .

Analogously (by cyclic permutation)

$$\begin{aligned} (y^z, (z, x)) &= u^{-1}v \\ (z^x, (x, y)) &= v^{-1}w. \end{aligned}$$

Hence  $(x^y, (y, z))(y^z, (z, x))(z^x, (x, y)) = 1$  q.e.d.

Applications:

Let  $A, B$  be subgroups of a group  $G$  and let  $(A, B)$  denote the subgroup of  $G$  generated by the commutators  $(a, b)$  for all  $a \in A, b \in B$ .

If  $A, B, C$  are normal subgroups of  $G$ , then  $(A, B)$  is also normal and we have the relation

$$(A, (B, C)) \subset (B, (C, A))(C, (A, B))$$

which follows from 1.1(3).

## 2. Filtration on a group

**Definition 2.1.** A *filtration on a group*  $G$  is a map  $w : G \rightarrow \mathbf{R} \cup \{+\infty\}$  satisfying the following axioms:

- (1)  $w(1) = +\infty$ .
- (2)  $w(x) > 0$  for all  $x \in G$ .
- (3)  $w(xy^{-1}) \geq \inf\{w(x), w(y)\}$ .
- (4)  $w((x, y)) \geq w(x) + w(y)$ .

It follows from (3) that  $w(y^{-1}) = w(y)$ . If  $\lambda \in \mathbf{R}_+$  we define

$$G_\lambda = \{x \in G \mid w(x) \geq \lambda\}$$

$$G_\lambda^+ = \{x \in G \mid w(x) > \lambda\}.$$

The condition (3) shows that  $G_\lambda, G_\lambda^+$  are subgroups of  $G$ . Moreover, if  $x \in G_\lambda, y \in G$  then  $x^y \equiv x \pmod{G_\lambda^+}$  which follows from the relation

$$w((x, y)) \geq \lambda + w(y) > \lambda.$$

This also proves that  $G_\lambda$  is a normal subgroup of  $G$  and since  $G_\lambda^+ = \bigcup_{\mu > \lambda} G_\mu$  it follows that  $G_\lambda^+$  is also a normal subgroup of  $G$ .

The family  $\{G_\lambda\}$  (resp.  $\{G_\lambda^+\}$ ) is decreasing, i.e.,  $\lambda < \mu$  implies  $G_\lambda \supset G_\mu$  (resp.  $G_\lambda^+ \supset G_\mu^+$ ).

**Definition 2.2.** For all  $\alpha \geq 0$  we define

$$\text{gr}_\alpha G = G_\alpha / G_\alpha^+ \quad \text{and} \quad \text{gr} G = \sum_{\alpha} \text{gr}_\alpha G.$$

**Proposition 2.3.**

- 1)  $\text{gr}_\alpha G$  is an abelian group.
- 2) If  $x \in G_\alpha$  let  $\bar{x}$  be its image in  $\text{gr}_\alpha G$ ; one has  $\overline{(x^y)} = \bar{x}$  for all  $y \in G$ .

3) The map  $c_{\alpha,\beta} : G_\alpha \times G_\beta \rightarrow G_{\alpha+\beta}$  defined by  $x, y \mapsto (x, y)$  induces a bilinear map  $\bar{c}_{\alpha,\beta} : \text{gr}_\alpha G \times \text{gr}_\beta G \rightarrow \text{gr}_{\alpha+\beta} G$ .

4) The maps  $\bar{c}_{\alpha,\beta}$  can be extended by linearity to  $c : \text{gr} G \times \text{gr} G \rightarrow \text{gr} G$  and this defines a Lie algebra structure on  $\text{gr} G$ .

*Proof.* 1) It follows from 2.1(4).

2) It is already proved.

3) Let  $x, x' \in G_\alpha, y, y' \in G_\beta$ , then  $(x, y) \in G_{\alpha+\beta}$  and we have to prove that if  $u, v \in G_\alpha^+$  then  $(xu, y) \equiv (x, y) \pmod{G_{\alpha+\beta}^+}$ ,  $(x, yv) \equiv (x, y) \pmod{G_{\alpha+\beta}^+}$ .

Using 1.1(2') and (3) we have

$$\begin{aligned} \overline{(xu, y)} &= \overline{(x, y)^u} + \overline{(u, y)} = \overline{(x, y)} \\ \overline{(x, yv)} &= \overline{(x, v)} + \overline{(x, y)^v} = \overline{(x, y)} \\ \overline{(xx', y)} &= \overline{(x, y)^{x'}} + \overline{(x', y)} = \overline{(x, y)} + \overline{(x', y)} \\ \overline{(x, y'y)} &= \overline{(x, y)} + \overline{(x, y')^y} = \overline{(x, y)} + \overline{(x, y')} . \end{aligned}$$

This proves 3).

4) Let  $\xi \in \text{gr}_\alpha G, \eta \in \text{gr}_\beta G$  and choose elements  $x \in G_\alpha, x \in G_\beta$  such that  $\bar{x} = \xi, \bar{y} = \eta$ . Then we have  $\overline{(x, y)} = \bar{c}_{\alpha,\beta}(\xi, \eta)$ , which we also write  $[\xi, \eta]$ .

Now if  $\xi \in \text{gr} G$  then  $\xi = \sum_\alpha \xi_\alpha$  where  $\xi_\alpha \in \text{gr}_\alpha G$ . In order to prove that  $[\xi, \xi] = 0$ , it is sufficient to prove that  $[\xi_\alpha, \xi_\alpha] = 0$  and  $[\xi_\alpha, \xi_\beta] = -[\xi_\beta, \xi_\alpha]$ . Let  $x_\alpha \in G_\alpha$  such that  $\bar{x}_\alpha = \xi_\alpha$  for all  $\alpha$ . Then we have  $[\xi_\alpha, \xi_\alpha] = \overline{(x_\alpha, x_\alpha)} = \bar{1} = 0$ , and

$$[\xi_\alpha, \xi_\beta] = \overline{(x_\alpha, x_\beta)} = \overline{(x_\beta, x_\alpha)}^{-1} = -[\xi_\beta, \xi_\alpha] .$$

In order to prove the Jacobi identity  $J(\xi, \eta, \zeta) = 0$ , since  $J$  is trilinear, it is enough to consider the case  $\xi \in \text{gr}_\alpha G, \eta \in \text{gr}_\beta G$  and  $\zeta \in \text{gr}_\gamma G$ . Now using the Proposition 1.1(3) we have, for  $x \in G_\alpha, y \in G_\beta, z \in G_\gamma$  such that  $\bar{x} = \xi, \bar{y} = \eta, \bar{z} = \zeta$ .

$$J(\xi, \eta, \zeta) = \overline{(x^y, (y, z))(y^z, (z, x))(z^x, (x, y))} = \bar{1} = 0$$

because  $\overline{x^y} = \xi, \overline{y^z} = \eta, \overline{z^x} = \zeta$ . q.e.d.

### 3. Integral filtrations of a group

**Proposition 3.1.** *For any group  $G$  the following two objects are in a one-one correspondence:*

- 1) Filtrations  $w : G \rightarrow \mathbf{R} \cup \{+\infty\}$  such that  $w(G) \subset \mathbf{N} \cup \{+\infty\}$ .
- 2) Decreasing sequences  $\{G_n\}_{n \in \mathbf{N}}$  of subgroups of  $G$  such that
  - (i)  $G_1 = G$ .
  - (ii)  $(G_n, G_m) \subset G_{n+m}$ .

*Proof.* (1)  $\Rightarrow$  (2) is clear.

(2)  $\Rightarrow$  (1). Let  $x \in G$ , then we define a filtration  $w : G \rightarrow \mathbf{R} \cup \{+\infty\}$  by  $w(x) = \sup_{x \in G_n} \{n\}$ .

It is clear that  $w(1) = +\infty$ ,  $w(x) > 0$  for all  $x \in G$ , and  $w(x) = w(x^{-1})$ .

Now let  $w(x) = n$ ,  $w(y) = m$ , i.e.,  $x \in G_n$ ,  $y \in G_m$  and  $x \notin G_{n+1}$ ,  $y \notin G_{m+1}$ . Suppose  $n \leq m$ , then  $G_m \subset G_n$  and therefore  $xy^{-1} \in G_n$ , i.e.,

$$w(xy^{-1}) \geq \inf\{w(x), w(y)\} .$$

In case  $n = +\infty$  or  $m = +\infty$ , we have obviously this inequality.

Finally the inequality  $w((x, y)) \geq w(x) + w(y)$  follows from (ii). q.e.d.

*Example.* The descending central series of  $G$ .

Define  $G_1 = G$  and by induction  $G_{n+1} = (G, G_n)$ . Then the sequence  $\{G_n\}$  satisfies the conditions (i)–(ii) of (2) in the Proposition 3.1. Condition (i) is satisfied by definition, and we will prove (ii) by induction on  $n$  in the pair  $(G_n, G_m)$ .

Let first  $n = 1$ , then  $(G, G_m) \subset G_{m+1}$  by definition. Now suppose  $n > 1$ , then

$$\begin{aligned} (G_n, G_m) &= ((G, G_{n-1}), G_m) \subset (G, (G_{n-1}, G_m))(G_{n-1}, (G, G_m)) \\ &\subset (G, G_{n+m-1})(G_{n-1}, G_{m+1}) \\ &\subset G_{n+m} \cdot G_{n+m} = G_{n+m} . \end{aligned}$$

Conversely, if  $\{H_n\}$  is a decreasing sequence of subgroups of  $G$  which verifies (2), then  $H_n \supset G_n$  for all  $n$ . The proof of this is also by induction. Suppose  $n = 1$ , then by definition  $H_1 = G_1$ . Now if  $n \geq 1$ , we have

$$H_{n+1} \supset (H_1, H_n) \supset (G, G_n) = G_{n+1} .$$

#### 4. Filtrations in $GL(n)$

Let  $k$  be a field with an ultrametric absolute value  $|x| = a^{v(x)}$ . Let  $A_v$  be the ring of  $v$  and let  $\mathfrak{m}_v$  be the maximal ideal of  $A_v$ , let  $k(v) = A_v/\mathfrak{m}_v$ .

Let  $n$  be a positive integer and let  $G$  be the group of  $n \times n$ -matrices with coefficients in  $A_v$  such that  $g \equiv 1 \pmod{\mathfrak{m}_v}$ , i.e., if  $g = (g_{ij})$  then  $g_{ij} \equiv \delta_{ij} \pmod{\mathfrak{m}_v}$ .

If  $g \in G$  then  $g = 1 + x$  where  $x$  is a matrix with coefficients in  $\mathfrak{m}_v$ .

Clearly  $G$  is a group, because it can be described as

$$G = \text{Ker} \{ GL(n, A_v) \rightarrow GL(n, k(v)) \} .$$

Let  $X \in M_n(k)$ ,  $X = (x_{ij})$ , then define  $v(X) = \inf\{v(x_{ij})\}$ .

We can define a map  $w : G \rightarrow \mathbf{R} \cup \{+\infty\}$  by  $w(g) = v(x)$ , where  $g = 1 + x$ .

**Theorem 4.1.** *The map  $w$  is a filtration on  $G$ .*

*Proof.* The conditions  $w(1) = +\infty$  and  $w(g) > 0$  for all  $g \in G$  are obvious.

Let now  $G_\lambda = \{ g \in G \mid w(g) \geq \lambda \}$ . If  $\mathfrak{a}_\lambda$  is defined by

$$\mathfrak{a}_\lambda = \{ x \mid x \in \mathfrak{k}, v(x) \geq \lambda \},$$

the set  $G_\lambda$  is the kernel of the canonical homomorphism

$$\mathrm{GL}(n, A_v) \rightarrow \mathrm{GL}(n, A_v/\mathfrak{a}_\lambda).$$

Hence  $G_\lambda$  is a subgroup of  $G$ , and this proves condition (3).

To prove condition (4), i.e.,  $(G_\lambda, G_\mu) \subset G_{\lambda+\mu}$ , write  $g \in G_\lambda$ ,  $h \in G_\mu$  in the form:

$$g = 1 + x, \quad h = 1 + y.$$

One must check that  $hg \equiv gh \pmod{G_{\lambda+\mu}}$ . But

$$hg = 1 + x + y + yx$$

$$gh = 1 + x + y + xy$$

and the coefficients of  $xy$  and  $yx$  belong to  $\mathfrak{a}_{\lambda+\mu}$ . Hence  $hg$  and  $gh$  have the same image in  $\mathrm{GL}(n, A_v/\mathfrak{a}_{\lambda+\mu})$ , and they are congruent mod  $G_{\lambda+\mu}$ , q.e.d.

### Exercises

1. Determine the Lie algebra  $\mathrm{gr} G$ .
2. Prove that  $G = \varprojlim G/G_\lambda$  if  $\mathfrak{k}$  is complete.



# Chapter III. Universal Algebra of a Lie Algebra

## 1. Definition

Let  $k$  be a commutative ring and let  $\mathfrak{g}$  be a Lie algebra over  $k$ .

**Definition 1.1.** A *universal algebra* of  $\mathfrak{g}$  is a map  $\varepsilon : \mathfrak{g} \rightarrow U\mathfrak{g}$ , where  $U\mathfrak{g}$  is an associative algebra, with a unit satisfying the following properties:

1).  $\varepsilon$  is a Lie algebra homomorphism,

$$\text{(i.e., } \varepsilon \text{ is } k\text{-linear and } \varepsilon[x, y] = \varepsilon x \cdot \varepsilon y - \varepsilon y \cdot \varepsilon x).$$

2). If  $A$  is any associative algebra with a unit and  $\alpha : \mathfrak{g} \rightarrow A$  is any Lie algebra homomorphism, there is a unique homomorphism of associative algebras  $\varphi : U\mathfrak{g} \rightarrow A$  such that the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varepsilon} & U\mathfrak{g} \\ \alpha \downarrow & \swarrow \varphi & \\ & & A \end{array}$$

is commutative [i.e., there is an isomorphism

$$\text{Hom}_{\text{Lie}}(\mathfrak{g}, LA) \cong \text{Hom}_{\text{Ass}}(U\mathfrak{g}, A)$$

where  $LA$  is the Lie algebra associated to  $A$ , cf. Chap. I, example (iii).]

It is trivial that  $U\mathfrak{g}$ , if it exists, is unique (up to a unique isomorphism). To prove its existence, we use the *tensor algebra*  $T\mathfrak{g}$  of  $\mathfrak{g}$ , i.e.,  $T\mathfrak{g} = \sum_{n=0}^{\infty} T^n\mathfrak{g}$ , where  $T^n\mathfrak{g} = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g} = \bigotimes^n \mathfrak{g}$  for  $n \geq 0$ . For any associative algebra  $A$  with a unit, one has:  $\text{Hom}_{\text{Mod}}(\mathfrak{g}, A) = \text{Hom}_{\text{Ass}}(T\mathfrak{g}, A)$ .

Now let  $I$  be the two-sided ideal of  $T\mathfrak{g}$  generated by the elements of the form  $[x, y] - x \otimes y + y \otimes x$ ,  $x, y \in \mathfrak{g}$ .

Take  $U\mathfrak{g} = T\mathfrak{g}/I$ , then we have:

**Theorem 1.2.** *Let  $\varepsilon : \mathfrak{g} \rightarrow U\mathfrak{g}$  be the composition  $\mathfrak{g} \rightarrow T^1\mathfrak{g} \rightarrow T\mathfrak{g} \rightarrow U\mathfrak{g}$ . Then the pair  $(U\mathfrak{g}, \varepsilon)$  is a universal algebra of  $\mathfrak{g}$ .*

In fact, let  $\alpha$  be a Lie homomorphism of  $\mathfrak{g}$  into an associative algebra  $A$ . Since  $\alpha$  is  $k$ -linear, it extends to a unique homomorphism  $\psi : T\mathfrak{g} \rightarrow A$ . It is clear that  $\psi(I) = 0$ , hence  $\psi$  defines  $\varphi : U\mathfrak{g} \rightarrow A$ , and we have checked the universal property of  $U\mathfrak{g}$ .

*Remark.* Let  $E$  be a  $\mathfrak{g}$ -module (i.e., a  $k$ -module with a bilinear product  $\mathfrak{g} \times E \rightarrow E$  such that  $[x, y]e = x(ye) - y(x \cdot e)$  for  $x, y \in \mathfrak{g}$ ,  $e \in E$ ). The map  $\mathfrak{g} \rightarrow \text{End}(E, E)$  which defines the module structure of  $E$  is a Lie homomorphism. Hence it extends to an algebra homomorphism  $U\mathfrak{g} \rightarrow \text{End}(E, E)$  and  $E$  becomes a  $U\mathfrak{g}$ -left-module. It is easy to check that one obtains in this