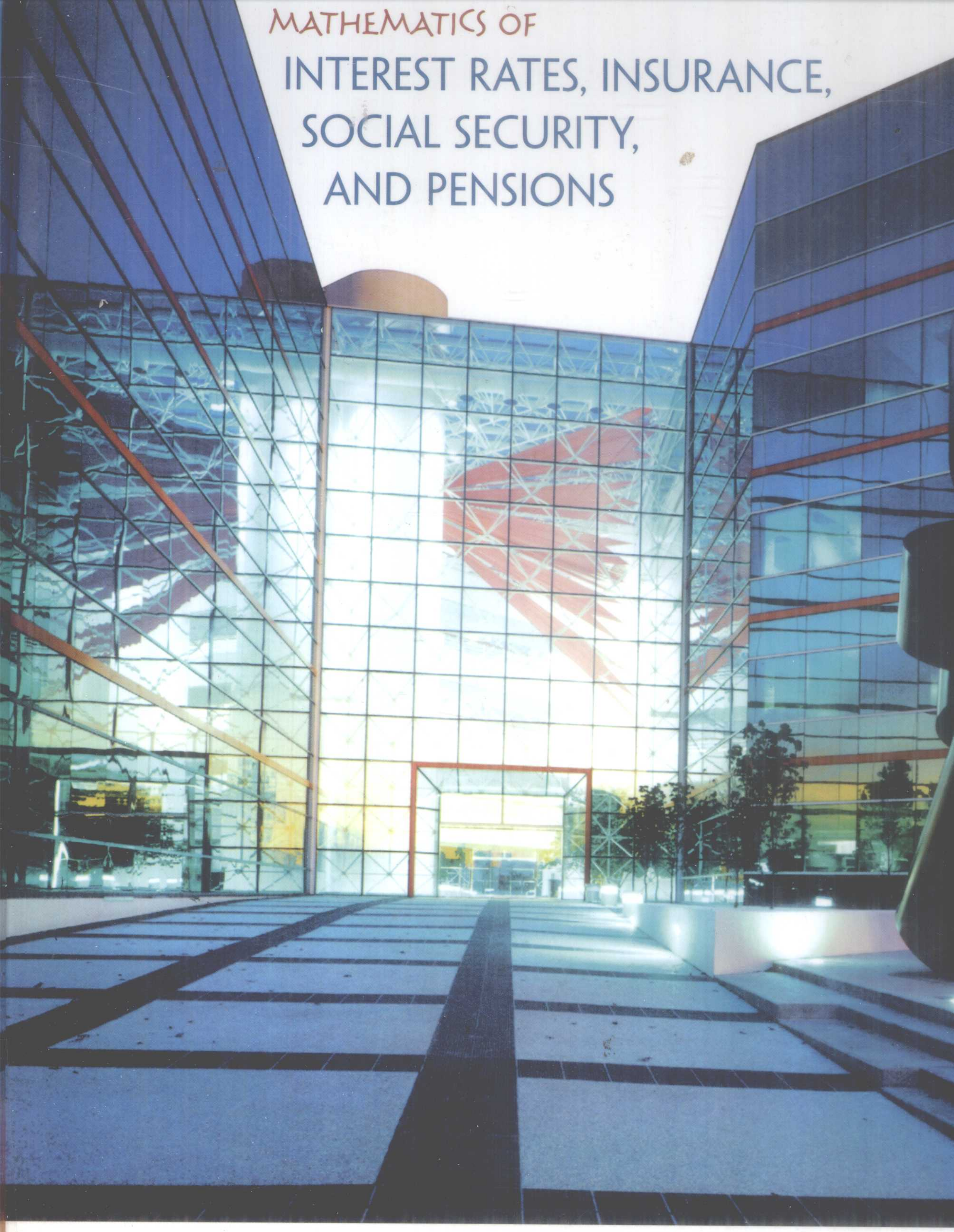
MATHEMATICS OF

INTEREST RATES, INSURANCE, SOCIAL SECURITY, AND PENSIONS



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Mathematics of Interest Rates, Insurance, Social Security, and Pensions

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To my grandson, Mark Steven Pitts.

He showed an affinity to mathematics at a very early age.

Preface

This book is intended for individuals whose career paths may include the need for mathematics of finance, insurance, breakeven analyses, and retirement planning via Social Security and private pensions. It is intended to be a helpful adjunct to business concentrations such as:

- Accounting, since many accounting firms have included retirement planning on a consulting basis.
- Finance and Financial services, in that the lines of separation between banks, insurance companies, and brokerage houses are no longer firmly delineated.
- Management, to be able to "converse" in the mathematics of all segments of a business.
- Marketing services, in order to be "literate" in the vocabulary of finance, insurance, and pensions for professional and personal use.
- Computer Information Systems, in order to know specific mathematical concepts so as to be able to develop appropriate algorithms for solution.
- Economics, in order to extend the use of financial mathematics to insurance and pensions.

The book can also be used for those students whose concentrations are in the Liberal Arts. It will help to make them "literate" in the vocabulary of finance, insurance, and pensions and to be able to utilize the appropriate mathematics for professional and personal use.

The prerequisite preparation that is necessary is at least a solid foundation in high school algebra. Knowledge of the contents of this book can be useful for personal financial planning as well as for the business uses cited above.

Chapter 1 discusses concepts that involve Simple Interest and Simple Discount. Chapter 2 introduces Practical Applications of Simple Interest, including the determination of short-term rates of return on investments. Chapter 3 is a presentation of Compound Interest and includes the determination of time-weighted, compounded rates-of-return on long-term investments. Chapter 4 discusses the concepts of Simple Annuities, where the frequency of payments and the frequency of compounding are the same. In Chapter 5, the concepts of annuities are extended to include deferred annuities, complex annuities (where the annual frequency of payments and the annual frequency of compounding are different), annuities in perpetuity, and annuities where the periodic payments vary geometrically. Additional practical applications of annuities are included in Chapter 6 (Bonds). In order to assist in the optimization of investment portfolios, elements of linear programming are introduced in Chapter 7. After a brief introduction to optimization concepts with two variables, the use of the **Solver** tool in Microsoft Excel is shown for optimization with more than two variables. Advanced topics of Capital Rationing and Working Capital Management are included to show how the **Solver** can be used in advanced topics of finance. Breakeven Models for cost-revenue, supply-demand, and financial considerations such as the effect of commissions on purchases and early retirement decisions are discussed in Chapter 8. Chapter 9 introduces concepts of life insurance through life annuities, net annual premiums, and terminal reserves. Elements of the mathematics of Social Security are presented in Chapter 10, and Chapter 11 introduces you to the elements of the mathematics associated with private pensions.

This is a better book because of many useful comments and suggestions received from colleagues and correspondents Kevin Charlwood, Washburn University; Jeffrey Forrest, Slippery Rock University; Joseph Greene, Augusta State University; Kerri McMillan, Clemson University; Yujin Shen, The Richard Stockton College of New Jersey; Thomas Springer, Florida Atlantic University, and Jianzhong Su, University of Texas at Arlington.

I wish to give special thanks to Dr. Hsi Li, Professor of Finance, Dr. Alan Olinksy, Professor of Mathematics, and Dr. Phyllis Schumacher, Professor of Mathematics, all of Bryant College, Smithfield, R. I., for their kind suggestions relative to the material. I also wish to give special thanks to Mr. William Pitts, Vice-President, Strategic Planning, Textron Financial Corporation, Providence, R.I., and to Mr. Paul Merlino, Esq. of Lamoriello and Company, Warwick, R.I for comments relative to private pensions. I would especially thank my daughter, Dr. Robin Muksian Schutt, Associate Professor of English, New England Institute of Technology, Warwick R.I., for commenting on the readability of the manuscript.

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Contents

CHAPTER 1

Simple Interest and Discount

The payment of interest is the cost for borrowing money. If an individual borrows an amount of money from a lender (e.g., bank, finance company, credit union, individual), the amount of the repayment is greater than the amount borrowed. The difference between the two amounts is the cost of borrowing—called interest. If an individual deposits money in a savings institution (i.e., bank, credit union, savings and loan association), that institution has, in effect, borrowed the money from the depositor and pays the depositor interest. This interest is the cost to the savings institution for having use of the depositor's money. If an individual invests money in corporate bonds, the corporation, in effect, has borrowed the money from the investor and pays interest for the use of that money. Since this interest decreases profit, it is a cost to the corporation for having use of the money.

Simple interest is the interest earned only on the borrowed amount. If a corporate bond pays a fixed interest rate annually, that rate and face value (the amount "borrowed") determine the amount of interest of the bond. The interest rate is a percent of the principal amount. The interest charged for automobile financing, for example, is based on the amount of the automobile cost that must be borrowed. Interest charges in business transactions, on the other hand, are based on the amount owed after partial payments are made. The concepts of simple interest are discussed in this chapter and Chapter 2. **Compound interest** is interest that is earned on interest. Simple interest concepts are the foundation for compound interest, which will be discussed in Chapters 3, 4, and 5.

1.1 SIMPLE INTEREST

If money is placed in a savings account to earn interest or if interest must be paid on money that is borrowed, the amount of interest depends upon three factors:

1. The principal,
2. The interest rate, and
3. The length of time.

Note that the interest rate and time must be in compatible units. If interest rates are specified as monthly, time must be the number of months. If interest rates are specified as annual, time must be the number of years. The usual specifications of interest rates are annual.

All stated interest rates in this book are annual rates unless indicated to be different.

Then, letting:

P be the principal, the amount earning interest,

r be the annual interest rate in %,

t be the number of years, and

I be the interest,

simple interest is defined by

$$I = Prt. \tag{1.1-1}$$

EXAMPLE 1.1.1 What is the simple interest on \$1,000 at 6% for a) 2 years, and b) 6 months?

Solution The principal is 1,000 and 6% converts to 0.06.

a) For 2 years, $t = 2$ and from Equation (1.1-1)

$$\begin{aligned} I &= Prt \\ &= (1{,}000)(0.06)(2) = \$120. \end{aligned}$$

b) For 6 months, $t = \frac{1}{2}$ of 1 year (0.5 years). From Equation (1.1-1)

$$\begin{aligned} I &= Prt \\ &= (1{,}000)(0.06)(0.5) = \$30. \quad \blacksquare \end{aligned}$$

Interest is earned or paid for the exact amount of time that the principal is outstanding. Frequently, the exact time involves a number of days only, or a number of days in conjunction with a number of months or years. Since it is customary to express interest rates on an annual basis, the number of days must be converted to a fraction of a year. Two methods for this conversion are used, and the respective interests are named ordinary interest and exact interest.

ORDINARY INTEREST

A lending institution may use ordinary interest when interest is paid to the institution. The method assumes a 360-day year. Letting:

N be the exact number of days, and

t_O be ordinary time (hence "ordinary" interest),

$$t_O = \frac{N}{360}. \tag{1.1-2}$$

EXACT INTEREST

A savings institution may use exact interest when the institution pays the interest. The method uses a 365-day year. Letting:

N be the exact number of days, and
t_e be exact time (hence "exact" interest)

$$t_e = \frac{N}{365}. \tag{1.1-3}$$

EXAMPLE 1.1.2 For a principal of $1,800, a 7% interest rate, and a time of 92 days, find a) the ordinary interest and b) the exact interest.

Solution

a) For ordinary interest, Equation (1.1-1) becomes

$$I = Prt_o,$$

and from Equation (1.1-2),

$$t_o = \frac{92}{360}.$$

Then,

$$I = (1{,}800)(0.07)\left(\frac{92}{360}\right) = 32.20,$$

and the ordinary interest is $32.20.

b) For exact interest, Equation (1.1-1) becomes

$$I = Prt_e,$$

and from Equation (1.1-3),

$$t_e = \frac{92}{365}.$$

Then,

$$I = (1{,}800)(0.07)\left(\frac{92}{365}\right) = 31.76,$$

and the exact interest is $31.76. ■

Given a starting date for earning or paying interest, on what date will a specific number of days occur? Another way of asking the question is how many days are there between two calendar dates? The most convenient solution is to count the number of days in conjunction with a calendar, but in so doing, *the first date is excluded and the last date is included.* Thus, if time begins on March 14, 92 days later occurs on a date as follows:

Month	Days in the Month	Cumulative Days
Starting on March, 14	17	
April	30	47
May	31	78
Ending on June, 14	14	92

Then, June 14 is 92 days following March 14. If available, a table of Julian Dates will facilitate the determination of dates. Such a table is supplied as Table A-1 in the Appendix. The months are listed across the top, and the day of the month is listed down the left. Thus, for March 14, find March at the top and 14 at the left. The intersection of these two factors indicates March 14 to be the 73rd day of the year. Then, 92 days later would be the 165th day (73 + 92) of the year which is June (at the top) 14 (at the left). For the number of days between March 14 and June 14, find June 14 to be the 165th day and March 14 to be the 73rd day. Then, the number of days between the two dates is 92 (165 – 73). For leap years, 1 would be added to each date after February 28.

EXAMPLE 1.1.3 A $1,500 loan is initiated on April 19 and is to be repaid on July 7. What will be the ordinary interest if the interest rate is 10%?

Solution Since this is ordinary interest, 360 is used for the number of days in a year. From Equations (1.1-1) and (1.1-2),

$$I = Prt_O,$$

and

$$t_O = \frac{N}{360}.$$

Using a table of Julian Dates, Table A-1, July 7 is the 188th day and April 19 is the 109th day. Thus, $N = 188 - 109 = 79$ days, and

$$t_O = \frac{79}{360}.$$

Then,

$$I = (1{,}500)(0.10)\left(\frac{79}{360}\right) = 32.92,$$

and the ordinary interest is $32.92. ■

EXAMPLE 1.1.4 How many days will there be between August 11 and May 8 of the following year? (Use 365-day years.)

Solution Since two calendar years are involved, the number of days in each must be determined. August 11 is the 223rd day and the number of days in the first year is 365 – 223 or 142. May 8 is the 128th day of the year and is the number of days in the following year. Then, the number of days between August 11 and May 8 is 142 + 128, or 270 days. To simplify the determination of the number of days between calendar dates in two consecutive years, let D_1 be the number of the date in the first year, and D_2 be the number of the date in the following year. The number of days, N, between the two dates can be determined by

$$N = 365 - (D_1 - D_2). \tag{1.1-4}$$

For the dates above, $N = 365 - (223 - 128) = 365 - 95 = 270$ days. ■

The definition of simple interest, Equation (1.1-1) involves four parameters—I, P, r, t. Given any three of the parameters, the 4th may be determined algebraically. From Equation (1.1-1), $I = Prt$. Given r, t, and I, $P = \frac{I}{rt}$; given P, t, and I, $r = \frac{I}{Pt}$; and given P, r, and I, $t = \frac{I}{Pr}$. These modified forms of Equation (1.1-1) should not be memorized as such. The algebraic manipulations are quite simple and are shown in Example 1.1.5.

EXAMPLE 1.1.5 a) What principal will earn \$50 interest in five years at a simple interest rate of 6%? b) What simple interest rate is necessary for \$1,000 to earn \$90 interest in 18 months? c) How long will it take to earn \$110 interest with a principal of \$1,000 at a simple interest rate of 5%?

Solution

a) The given information is: $I = 50$, $r = 0.06$, and $t = 5$.
From Equation (1.1-1), $I = Prt$ and substituting the given values gives

$$50 = P(0.06)(5), \text{ or}$$

$$50 = P(0.3).$$

Dividing both sides by 0.3 results in $P = \$166.67$.

b) The given information is: $P = 1{,}000$, $I = 90$, and $t = 18$ months. From the definition of simple interest, $I = Prt$, and substituting the given values, gives

$$90 = (1{,}000)(r)\left(\frac{18}{12}\right), \text{ or}$$

$$90 = 1{,}500r.$$

Dividing both sides by 1,500 gives $r = 0.06$, or 6% per year since the time was indicated as $\frac{18}{12}$ years.

c) The given information is: $I = 110$, $P = 1{,}000$, and $r = 0.05$.
From the definition of simple interest, $I = Prt$ and substituting the given values yields

$$110 = (1{,}000)(0.05)t, \text{ or}$$

$$110 = 50t.$$

Dividing both sides by 50 gives $t = 2.2$ years since the interest rate is assumed to be annual. Now, 0.2 years = (0.2)(360) days if it is ordinary time and (0.2)(365) days if it is exact time, or 72 days and 73 days respectively. Thus, $t_o = 2$ years and 72 days or $t_e = 2$ years and 73 days. ■

If time is indicated as a number of months, then the number of years is obtained by dividing the number of months by 12. That is, 3 months is $\frac{3}{12}$ years and reduces to $\frac{1}{4}$ of a year. A longer period such as 15 months is $\frac{15}{12}$ years, which is $1\frac{1}{4}$ years or one year and three months.

PROBLEM SET 1.1

– 1. Determine the simple interest on $1,000 at 5% for the following:

a) 4 months, b) 7 months, c) 9 months, d) 15 months, e) 27 months, f) 3 years, g) 5 years, and h) 10 years.

2. Determine the simple interest on $2,000 at 8% for 142 days using a) ordinary time, and b) exact time.

3. If a loan is initiated on June 2, of a 365-day year, determine the due date if the loan is for a) 90 days, b) 180 days, and c) 270 days.

4. Determine the number of days between

a) January 4 and September 8 of a 366-day year.

b) April 29, yyyy and January 12 of the following year. Assume yyyy is a leap year.

c) October 16, yyyy and February 26 of the following year. Assume both years are 365-day years.

d) November 12, yyyy and March 3 of the following year. Assume the "following year" is a leap year.

5. Determine the ordinary interests on a $5,000 loan at 9% for

a) 90 days, b) 180 days, and c) 270 days.

6. Determine the exact interests on $2,500 at 5% for the number of days determined for each part in Problem 4.

Complete the following for Problems 7–11.

	I	P	r	t
7.	$1,000	-----	7.25%	3 years
8.	$100	$2,000	-----	6 months
9.	$75	$1,000	-----	1 year
10.	$250	$5,000	8.0%	-----
– 11.	$100	$1,500	10.0%	-----

12. If the simple interest on $5,000 is $1,105 for an 8.5% interest rate, determine the a) ordinary and b) exact times in years and days.

1.2 FUTURE VALUE AND PRESENT VALUE

The "future amount" and the "present value" of money refer to "growth" of money because money has a time-value. The following are the respective definitions.

1. The **future value** is the amount to which a present value will grow, given the interest rate and time.

2. The **present value** is the amount necessary in order that a specified future value is realized, given the interest rate and time. Time is measured from the original maturity date backwards to the date on which the present value is being determined.

FUTURE VALUE

Given the amount of interest, the future value (e.g., the maturity value of a loan) is the sum of the principal and the interest. Letting:

S be the future amount, and

I be the interest,

the future amount is defined by

$$S = P + I. \tag{1.2-1}$$

In the preceding section, simple interest was defined by $I = Prt$. Substituting Equation (1.1-1) into Equation (1.2-1) gives

$$S = P + Prt.$$

and factoring P gives

$$S = P(1 + rt). \tag{1.2-2}$$

As indicated by Equations (1.2-1) and (1.2-2), interest is added at the end of an interest earning time period. This concept is shown in Figure 1-1.

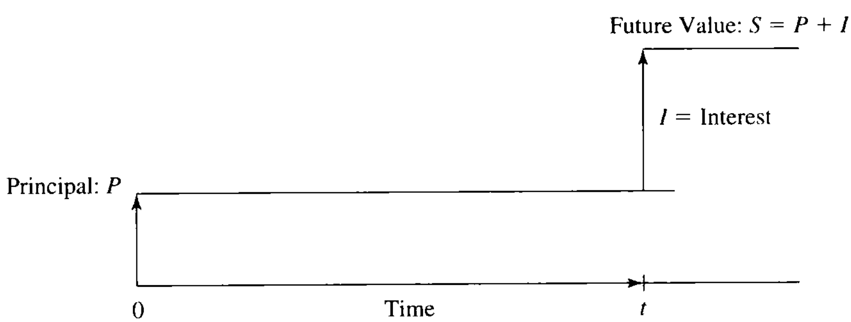

FIGURE 1-1 Future Amount at Simple Interest

Equation (1.2-2) allows the future amount to be determined without first computing the interest. If knowing the amount of the interest is necessary, Equation (1.2-1) may be rearranged to give

$$I = S - P. \tag{1.2-3}$$

The use of Equations (1.1-1) and (1.2-1), in combination, in order to determine a future value is a discretionary matter. However, the form of Equation (1.2-2) is necessary in later chapters; therefore, familiarity with that form for a future amount is recommended.