Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Author

Robert Silhol
Institut de Mathématiques, Université des Sciences
et Techniques du Languedoc
34060 Montpellier Cedex, France

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INTRODUCTION

These notes are centred on one question : *given a real algebraic surface* X *determine the topology of the real part* X(ℝ).

Of course, since, to quote Hartshorne, the guiding problem in algebraic geometry is the classification problem (and this goes for real algebraic geometry also), the latter is very present in these notes. In fact it is present to the point, that we have only obtained a precise answer to our original question when we have obtained a precise answer to the classification problem. In this sense one could say that the underlying theme (and even, the main theme) of these notes is the *classification problem of real algebraic surfaces*.

This second preoccupation has dictated the plan of these notes and to some extent the methods used. We explain this. If two algebraic varieties are real isomorphic, then they certainly are complex isomorphic. Hence, our starting point, the well known Enriques-Kodaira classification of complex algebraic surfaces, and the plan.

To be able to make the most of the knowledge accumulated on complex algebraic surfaces we have used an alternative definition for real algebraic varieties, explicitly, we define them as complex algebraic varieties with an antiholomorphic involution. Otherwise said, we consider real algebraic varieties as complex algebraic varieties with an action of the Galois group Gal(ℂ|ℝ) (in the projective case, our only preoccupation, the two definitions are equivalent - see I.§1). This is the foundation of all the methods used in these notes.

From this point of view real algebraic surfaces fall into two classes, those for which the Galois action on $H^*(X(\mathbb{C}),\mathbb{Z})$ determines $H^*(X(\mathbb{R}),\mathbb{Z}/2)$ and those for which the Galois action only gives bounds on the dimensions of $H^*(X(\mathbb{R}),\mathbb{Z}/2)$. We call the first *Galois-Maximal* or GM-*surfaces*, they include rational surfaces, abelian surfaces, K3 surfaces and surfaces in $\mathbb{P}^3$. In the second class lie ruled surfaces and in general, surfaces fibred on a curve (elliptic surfaces... etc...).

We have been able to solve the classification problem completely for rational surfaces (chap. III and VI), abelian surfaces (chap. IV) and K3 surfaces (chap. VIII). These are all GM-surfaces.

For non-GM-surfaces we have concentrated on ruled and elliptic surfaces. The basic method used in both cases has been to study the Galois action on the fibration. We have obtained in this way complete results for ruled surfaces (chap. V) and complete local results for elliptic surfaces (chap. VII). We have also applied these methods to rational ruled surfaces (chap. VI).

The ideas, behind these methods are not new. The idea to consider a real algebraic variety as a complex variety with an antiholomorphic involution goes back at least to F. Klein and the idea to consider Galois action on cohomology is implicitly in Comessatti (although, of course, in different terms). The deep relations, of which we have made an essential use, between the cup-product form on $H^2(X(\mathbb{C}),\mathbb{Z})$ and $H^*(X(\mathbb{R}),\mathbb{Z}/2)$ are originally due to Arnold, Rokhlin, Gudkov, Kraknov and Kharlamov (see chap. II). To these authors is also due the use in real algebraic geometry of the Smith sequence. Finally, in this direction, the concept of GM-variety is due to Krasnov.

The idea to study Galois action on fibrations, goes back to Comessatti who studied the Galois action on pencils of curves, but the methods we have used are far closer to the methods developed by Manin and Iskovskih, the results of whom we have made an essential use.

We have tried to include in these notes most of the results on surfaces of the above mentioned authors.

Because of its importance it may be useful here, to say a few words on our treatment of the classification problem. As is usual in algebraic geometry such a problem divides into a discrete part (finding the right invariants) and a continuous part (the moduli problem).

For the discrete part, the first idea one can have, to consider complex invariants plus topological invariants of the real part, does not give a complete set of invariants. This is already true, and well known, in the case of curves. For GM-surfaces the correct set of invariants turns out to be the invariants for the action of the antiholomorphic involution on $H^*(X(\mathbb{C}),\mathbb{Z})$ taken with the cup-product from and the Hodge decomposition (this includes all of the classical complex invariants). Note that for curves and abelian varieties, which are GM-varieties, this also gives the correct invariants.

We have taken these as our basic invariants even in the case of non-Galois-Maximal surfaces, in which case of course, they do not form a complete set of invariants.

For the continuous part, we have tried to build a Moduli space which classifies real algebraic surfaces with given invariants, up to real isomorphisms. This is somewhat different from considering the real part of the complex moduli space when it is known (for a discussion see chap. IV, §4). Even when a fine Moduli space does exist (for K3 surfaces for example) our method seems to give a more direct answer.

Among the results included in these notes, which are not specifically linked with the classification problem we should mention the results of chapter II giving bounds for the number of connected components and for the $h^1(X(\mathbb{R}),\mathbb{Z}/2)$ of a real algebraic surface and the results of chapter III on the subgroup of $H^1(X(\mathbb{R}),\mathbb{Z}/2)$ generated by algebraic cycles.

Among the subjects not treated in these notes the most important omission concerns surfaces of general type (they are only considered in a couple of examples). The main reason for this absence is that the study of complex surfaces of general type is still a field of active research and not all the questions we need an answer to (to apply our methods) have been solved. For specific families of surfaces of general type, for example double covers of rational surfaces, or surfaces fibred in curves of genus 2 (where we could have applied methods similar to those of chap. VII) the absence is essentially due to lack of time.

Another subject absent from these notes, is singular surfaces. The reasons for this omissions are more or less the same as the ones given for surfaces of general type.

A last subject omitted although originally planed to be included in these notes is the study of real automorphisms of surfaces and the parent problem of determining the number of distinct real structures on a complex surface.

Finally to end this introduction we would like to note that special attention has been given, throughout these notes, to examples. Indeed, we have tried to illustrate all of the important notions introduced in the text by adequate examples.

Prerequisites and notations

The general prerequisite for reading these notes, is a basic knowledge of algebraic geometry as exposed for example in Griffiths and Harris, Principles of Algebraic Geometry ([Gr & Ha]) chapters 0 and 1 or Hartshorne, Algebraic Geometry ([Ha]) chapters I and II, plus some knowledge on complex algebraic surfaces as exposed in [Gr & Ha] chapter 4 or [Ha] chapter V, or for more specific and precise results (but in this case we have tried to give complete references) Barth, Peters and Van de Ven [B & P & V], Beauville [Be], Shafarevich [Sha] or for K3-surfaces the Palaiseau Seminar [X].

On the other hand no knowledge of Real Algebraic Geometry is assumed. We expose the basic results needed in chapters I and II, and these notes can serve as an introduction to real algebraic geometry for non-specialists and graduate students in algebraic geometry.

The notations are the standard notations of algebraic geometry, for the less well known of these we have tried to give either definitions or precise references. There are some differences with the notations used by other authors. The most important are that, for reasons which will become clear in chapters I and II, we have used the Hodge numbers $h^{0,2}$ and $h^{0,1}$ in place of the geometric genus p_g and the irregularity q, $\chi(X(\mathbb{C}))$, topological Euler characteristic in place of $c_2(X)$ and $\chi(\mathcal{O}_X)$ for the Euler characteristic of the structure sheaf $\mathcal{O}_X$ (in place of $\chi(X)$ as used by some authors).

For notations concerning real algebraic geometry, we have systematically denoted G the Galois group $\mathrm{Gal}(\mathbb{C}|\mathbb{R})$ and S the generator of this group. This is why, although we consider in most cases S as an antiholomorphic involution on $X(\mathbb{C})$, we speak of the action of S on the groups $H^i(X(\mathbb{C}),\mathbb{Z})$, $H^i(X(\mathbb{C}),\mathbb{Q})$,... etc... (in place of S^*). For other important remarks on notations see I.(1.15).

TABLE OF CONTENTS

I. **PRELIMINARIES ON REAL ALGEBRAIC VARIETIES**.

1. Real structure on a complex algebraic variety.

We start by introducing the basic concepts we will be needing throughout these notes. We will introduce them in two different settings, complex analytic varieties on the one hand, schemes over $\mathbb{C}$ on the other. Of course we will not be using the full power of either of these theories but they provide a convenient frame for formulating some general definitions. The different definitions turn out to be equivalent when they concern the same objects, namely projective or quasi-projective algebraic varieties. We will take advantage of this and make constant use of the interplay between these two points of view.

Let f be a holomorphic function defined in a neighborhood of $z_0 \in \mathbb{C}^n$. Let $j_n : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ and $j : \mathbb{C} \longrightarrow \mathbb{C}$ denote complex conjugation. We defined f^σ, the conjugate of f, to be the holomorphic function $j \circ f \circ j_n$ defined in a neighbourhood of $\bar{z}_0 = j_n(z_0)$.

In other words, if $z_0 = (z_{0,1},\ldots,z_{0,n})$ and if f is defined by :

$$\sum_{k \in \mathbb{Z}^n} a_k(z_1-z_{0,1})^{k_1}\ldots(z_n- z_{0,n})^{k_n}$$

in a neighbourhood of z_0, then f^σ if defined by :

$$f^\sigma(z) = \sum_{k \in \mathbb{Z}^n} \bar{a}_k(z_1-\bar{z}_{0,1})^{k_1}\ldots(z_n-\bar{z}_{0,n})^{k_n}$$

(where we have used $\bar{\ }$'s to denote complexe conjugates) in a neighbourhood of $\bar{z}_0$.

Let $X(\mathbb{C})$ be a complex analytic variety in $\mathbb{C}^n$. We define the complex conjugate variety $X^\sigma(\mathbb{C})$ to be $X^\sigma(\mathbb{C}) = \{z \,/\, j_n(z) = \bar{z} \in X(\mathbb{C})\}$. If U is an open set in $\mathbb{C}^n$ such that $X(\mathbb{C}) \cap U$ is the common zero locus of holomorphic functions $f_1,\ldots,f_m$ then $X^\sigma(\mathbb{C}) \cap \bar{U}$ (where $\bar{U} = \{z \,/\, \bar{z} \in U\}$ is the common zero locus of $f_1^\sigma,\ldots,f_m^\sigma$. We note that if the f_i's are polynomials then obviously the f_i^σ's are also polynomials.

In a more general way we can define this notion of conjugate variety globally for analytic varieties. Let $(X(\mathbb{C}),\mathcal{O}_{X(\mathbb{C})})$ ($\mathcal{O}_{X(\mathbb{C})}$ the sheaf of holomorphic functions) be a complexe analytic variety. We define the complexe conjugate variety to be $(X(\mathbb{C}),\bar{\mathcal{O}}_{X(\mathbb{C})})$, where $\bar{\mathcal{O}}_{X(\mathbb{C})}$ is the sheaf of antiholomorphic functions on $X(\mathbb{C})$.

To see that these two definitions are compatible assume $X(\mathbb{C}) \subset \mathbb{C}^n$. Then via j_n we can identify, as point sets, $X(\mathbb{C})$ and $X^\sigma(\mathbb{C})$. This identification identifies $\mathcal{O}_{X^\sigma(\mathbb{C})}$ with $\bar{\mathcal{O}}_{X(\mathbb{C})}$. The equivalence of the two definitions follows from this .

If $X(\mathbb{C})$ is a complex manifold we can reformulate the second definition in the following way : let (U_i,φ_i) be an atlas defining the complex structure on $X(\mathbb{C})$. We define $X^\sigma(\mathbb{C})$, the complex conjugate manifold of $X(\mathbb{C})$, to be defined by the atlas $(U_i,j_n\circ\varphi_i)$ where again $j_n : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is complex conjugation.

Of these three definitions the most useful, which we write separately, will be :

(1.1) Définition : Let $(X(\mathbb{C}),\mathcal{O}_{X(\mathbb{C})})$ be a complex analytic variety. We will call $(X(\mathbb{C}),\bar{\mathcal{O}}_{X(\mathbb{C})})$, where $\bar{\mathcal{O}}_{X(\mathbb{C})}$ is the sheaf of anti-holomorphic function on $X(\mathbb{C})$, the complex conjugate variety of $X(\mathbb{C})$. When the sheaf $\mathcal{O}_{X(\mathbb{C})}$ is understood we will write simply $X(\mathbb{C})$ for the original complex analytic variety and $X^\sigma(\mathbb{C})$ for the complex conjugate variety.

Our second point of view is the following : let X be a scheme over $\mathbb{C}$ and let $j : \mathbb{C} \longrightarrow \mathbb{C}$ be again complex conjugation. To X we can associate its complex conjugate scheme X^σ defined by composing the structural morphism $X \longrightarrow \mathrm{Spec}(\mathbb{C})$ with $j^* : \mathrm{Spec}(\mathbb{C}) \longrightarrow \mathrm{Spec}(\mathbb{C})$.

We note that if $\mathcal{O}_X$ (resp. $\mathcal{O}_{X^\sigma}$) is the sheaf of regular functions on X (resp. X^σ) then $\mathcal{O}_{X^\sigma} = \bar{\mathcal{O}}_X$ where $\bar{\mathcal{O}}_X(U) = \{j\circ f \ / \ f \in \mathcal{O}_X(U)\}$. This shows that this last definition is compatible with definition (1.1) .

In particular if X is projective (or quasi-projective) defined in some $\mathbb{P}^n(\mathbb{C})$ by polynomial equations $p_1(z) = \ldots = p_m(z) = 0$ then X^σ is defined by $p_1^\sigma(z) = \ldots = p_m^\sigma(z) = 0$ where p_i^σ is the conjugate polynomial of p_i.

If X is of finite type over $\mathbb{C}$, $X(\mathbb{C})$, its set of complex valued points, has a natural analytic structure. If $X^\sigma(\mathbb{C})$ is the set of complex points of X^σ, then, by the above, $X^\sigma(\mathbb{C})$ is the conjugate variety of $X(\mathbb{C})$ in the sense of definition (1.1).

(1.2) <u>Definition and Proposition</u> : <u>Let</u> X <u>be a scheme over</u> $\mathbb{C}$. <u>We will say that</u> (X,S) <u>or simply</u> S <u>is a real structure on</u> X, <u>if</u> S <u>is an involution on</u> X <u>such that the diagram</u>

$$\begin{array}{ccc} X & \xrightarrow{S} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{j^*} & \mathrm{Spec}(\mathbb{C}) \end{array}$$

(<u>where</u> $j : \mathbb{C} \longrightarrow \mathbb{C}$ <u>is complex conjugation)</u> <u>commutes</u>.
<u>If</u> (X,S) <u>is a real structure on</u> X <u>and</u> $\mathcal{O}_X$ <u>is the structure sheaf then for any open set</u> U <u>of</u> X <u>the morphism</u> :

$$\begin{array}{rcl} \Gamma(U,\mathcal{O}_X) & \longrightarrow & \Gamma(S(U),\mathcal{O}_X) \\ f & \longmapsto & j\circ f\circ S = f^S \end{array}$$

<u>is an isomorphism of rings.</u>

<u>Proof</u> : By the definition, a real structure on X is a descent datum relative to the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$, in the sense of Grothendieck [Gr_2]. With this it is easy to check that, if S is a real structure,

then $f \longmapsto f\circ S$ induces, for every open set U, an isomorphism $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X^\sigma}(S(U))$, hence the last assertion of (1.2).

Let X be of finite type over $\mathbb{C}$ and consider $X(\mathbb{C})$, the set of complex points with its natural complex analytic structure. If (X,S) is a real structure, then S <u>restricted to</u> $X(\mathbb{C})$ <u>is just an anti-holomorphic involution on</u> $X(\mathbb{C})$. We have a partial converse to this :

(1.3) <u>Proposition</u> : <u>If</u> X <u>is a projective variety over</u> $\mathbb{C}$ <u>then</u> X <u>has a real structure if and only if there exists an anti-holomorphic involution on</u> $X(\mathbb{C})$, <u>the set of complex points of</u> X.

<u>Proof</u> : We only need to prove the "if" part. If X is projective then clearly the conjugate variety is also projective. Let S be an anti-holomorphic involution on $X(\mathbb{C})$ and $\sigma : X(\mathbb{C}) \longrightarrow X^\sigma(\mathbb{C})$ the canonical map induced by the identity on the point sets. The map $S\circ\sigma$ is holomorphic, hence, since X is projective, algebraic by GAGA [Se_1]. In other words, S induces a continuous involution on X. Identifying X and X^σ, as point sets we can write that S induces, for all open sets U of X, an isomorphism $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X^\sigma}(S(U))$. Since by definition the map $f \mapsto j\circ f$ identifies $\mathcal{O}_{X^\sigma}(S(U))$ and $\mathcal{O}_X(S(U))$, we see that S satisfies the conditions of (1.2).

Let X be a scheme over $\mathbb{C}$. We will say that X has a real model if there exists a scheme X_o over $\mathbb{R}$ such that $X \cong X_o \times_{\mathbb{R}} \mathbb{C}$. We will say, in such a case, that X_o is a <u>real model</u> of X.

If a scheme over $\mathbb{C}$ has a real model then the action of the Galois group $Gal(\mathbb{C}|\mathbb{R})$ on X, defines in a natural way a real structure on X. We have a converse to this :

(1.4) <u>Proposition</u> : <u>Let</u> X <u>be a projective or quasi-projective scheme over</u> $\mathbb{C}$. X <u>has a real model if and only if</u> X <u>admits a real structure</u> (X,S). <u>More precisely there exists a real structure</u> (X,S) <u>on</u> X <u>if and only if there exists a real model</u> X_o <u>for</u> X <u>and an isomorphism</u>

$\varphi : X \longrightarrow X_o \times_{\mathbb{R}} \mathbb{C}$ such that $S = \varphi^{-1} \circ \sigma \circ \varphi$, where σ is induced by complex conjugation in $X_o \times_{\mathbb{R}} \mathbb{C}$. For a fixed (X,S), φ and X_o are unique up to real isomorphism.

Proof : This is nothing but a reformulation, in our special case, of a well known theorem of Weil (see [We_1] or [Gr_2] Exp. 190 Théorème 3).

The trivial example given by $\mathbb{P}^1_{\mathbb{R}}$ and the curve defined in $\mathbb{P}^2$ by $x^2+y^2+z^2 = 0$ shows that complex varieties can have different non real-isomorphic real models. (1.4) implies in such a case that they have different real structures. If we have two real structures (X,S) and (X,S'), (1.4) implies that they correspond to a same real model X_o if and only if there exists a complex automorphism φ of X such that :

(1.5) $$S = \varphi^{-1} \circ S' \circ \varphi .$$

As a consequence, we will say that two real structures (X,S) and (X',S') are isomorphic or real equivalent if there exists an isomorphism $\varphi : X \longrightarrow X'$ such that S, S' and φ verify (1.5).

Let A be a set and G a group operating on A. We will denote A^G the set of fixed points of A under the action of G. Let X_o be a real algebraic variety (or more generally a scheme over $\mathbb{R}$). Let $G = \mathrm{Gal}(\mathbb{C}|\mathbb{R})$. We have :

(1.6) $$X_o(\mathbb{R}) = X_o(\mathbb{C})^G .$$

Let (X,S) be a real structure on a complex algebraic variety. We will call $X(\mathbb{C})^G$ (where, as always, $G = \mathrm{Gal}(\mathbb{C}|\mathbb{R})$ acts on $X(\mathbb{C})$ through S) the real part of X. We will write $X(\mathbb{R})$ or $(X,S)(\mathbb{R})$, if the emphasis is on S, for this real part.

We will assume until the end of this § that X is a complex algebraic variety and that X has a real structure (X,S).