Lecture Notes in Mathematics

Edited by A. Dold, B. Eckmann and F. Takens

1428

Karin Erdmann

Blocks of Tame Representation Type and Related Algebras

Springer-Verlag

Lecture Notes in Mathematics

Edited by A. Dold, B. Eckmann and F. Takens

1428

Karin Erdmann

Blocks of Tame Representation Type and Related Algebras

Springer-Verlag

Berlin Heidelberg New York London Paris Tokyo Hong Kong

Author

Karin Erdmann
Mathematical Institute, Oxford University,
24-29 St. Giles, Oxford OX1 3LB, England

Mathematics Subject Classification (1980): 20C20, 20C05, 20C15, 16A46, 16A48, 16A64

ISBN 3-540-52709-5 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-52709-5 Springer-Verlag New York Berlin Heidelberg

Printed in Germany

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210 – Printed on acid-free paper

Lecture Notes in Mathematics

For information about Vols. 1–1201 please contact your bookseller or Springer-Verlag

Vol. 1202: A. Dür, Möbius Functions, Incidence Algebras and Power Series Representations. XI, 134 pages. 1986.

Vol. 1203: Stochastic Processes and Their Applications. Proceedings, 1985. Edited by K. Itô and T. Hida. VI, 222 pages. 1986.

Vol. 1204: Séminaire de Probabilités XX, 1984/85. Proceedings. Edité par J. Azéma et M. Yor. V, 639 pages. 1986.

Vol. 1205: B.Z. Moroz, Analytic Arithmetic in Algebraic Number Fields. VII, 177 pages. 1986.

Vol. 1206: Probability and Analysis, Varenna (Como) 1985. Seminar. Edited by G. Letta and M. Pratelli. VIII, 280 pages. 1986.

Vol. 1207: P.H. Bérard, Spectral Geometry: Direct and Inverse Problems. With an Appendix by G. Besson. XIII, 272 pages. 1986.

Vol. 1208: S. Kaijser, J.W. Pelletier, Interpolation Functors and Duality. IV, 167 pages. 1986.

Vol. 1209: Differential Geometry, Peñíscola 1985. Proceedings. Edited by A.M. Naveira, A. Ferrández and F. Mascaró. VIII, 306 pages. 1986.

Vol. 1210: Probability Measures on Groups VIII. Proceedings, 1985. Edited by H. Heyer. X, 386 pages. 1986.

Vol. 1211: M.B. Sevryuk, Reversible Systems. V, 319 pages. 1986.

Vol. 1212: Stochastic Spatial Processes. Proceedings, 1984. Edited by P. Tautu. VIII, 311 pages. 1986.

Vol. 1213: L.G. Lewis, Jr., J.P. May, M. Steinberger, Equivariant Stable Homotopy Theory. IX, 538 pages. 1986.

Vol. 1214: Global Analysis – Studies and Applications II. Edited by Yu.G. Borisovich and Yu.E. Gliklikh. V, 275 pages. 1986.

Vol. 1215: Lectures in Probability and Statistics. Edited by G. del Pino and R. Rebolledo. V, 491 pages. 1986.

Vol. 1216: J. Kogan, Bifurcation of Extremals in Optimal Control. VIII, 106 pages. 1986.

Vol. 1217: Transformation Groups. Proceedings, 1985. Edited by S. Jackowski and K. Pawalowski. X, 396 pages. 1986.

Vol. 1218: Schrödinger Operators, Aarhus 1985. Seminar. Edited by E. Balslev. V, 222 pages. 1986.

Vol. 1219: R. Weissauer, Stabile Modulformen und Eisensteinreihen. III, 147 Seiten. 1986.

Vol. 1220: S[illegible]e d'Algèbre Paul Dubreil et Marie-Paule Malliavin. Proceedings, 1985. Edité par M.-P. Malliavin. IV, 200 pages. 1986.

Vol. 1221: Probability and Banach Spaces. Proceedings, 1985. Edited by J. Bastero and M. San Miguel. XI, 222 pages. 1986.

Vol. 1222: A. Katok, J.-M. Strelcyn, with the collaboration of F. Ledrappier and F. Przytycki, Invariant Manifolds, Entropy and Billiards; Smooth Maps with Singularities. VIII, 283 pages. 1986.

Vol. 1223: Differential Equations in Banach Spaces. Proceedings, 1985. Edited by A. Favini and E. Obrecht. VIII, 299 pages. 1986.

Vol. 1224: Nonlinear Diffusion Problems, Montecatini Terme 1985. Seminar. Edited by A. Fasano and M. Primicerio. VIII, 188 pages. 1986.

Vol. 1225: Inverse Problems, Montecatini Terme 1986. Seminar. Edited by G. Talenti. VIII, 204 pages. 1986.

Vol. 1226: A. Buium, Differential Function Fields and Moduli of Algebraic Varieties. IX, 146 pages. 1986.

Vol. 1227: H. Helson, The Spectral Theorem. VI, 104 pages. 1986.

Vol. 1228: Multigrid Methods II. Proceedings, 1985. Edited by W. Hackbusch and U. Trottenberg. VI, 336 pages. 1986.

Vol. 1229: O. Bratteli, Derivations, Dissipations and Group Actions on C*-algebras. IV, 277 pages. 1986.

Vol. 1230: Numerical Analysis. Proceedings, 1984. Edited by J.-P. Hennart. X, 234 pages. 1986.

Vol. 1231: E.-U. Gekeler, Drinfeld Modular Curves. XIV, 107 pages. 1986.

Vol. 1232: P.C. Schuur, Asymptotic Analysis of Soliton Problems. VIII, 180 pages. 1986.

Vol. 1233: Stability Problems for Stochastic Models. Proceedings, 1985. Edited by V.V. Kalashnikov, B. Penkov and V.M. Zolotarev. VI, 223 pages. 1986.

Vol. 1234: Combinatoire énumérative. Proceedings, 1985. Edité par G. Labelle et P. Leroux. XIV, 387 pages. 1986.

Vol. 1235: Séminaire de Théorie du Potentiel, Paris, No. 8. Directeurs: M. Brelot, G. Choquet et J. Deny. Rédacteurs: F. Hirsch et G. Mokobodzki. III, 209 pages. 1987.

Vol. 1236: Stochastic Partial Differential Equations and Applications. Proceedings, 1985. Edited by G. Da Prato and L. Tubaro. V, 257 pages. 1987.

Vol. 1237: Rational Approximation and its Applications in Mathematics and Physics. Proceedings, 1985. Edited by J. Gilewicz, M. Pindor and W. Siemaszko. XII, 350 pages. 1987.

Vol. 1238: M. Holz, K.-P. Podewski and K. Steffens, Injective Choice Functions. VI, 183 pages. 1987.

Vol. 1239: P. Vojta, Diophantine Approximations and Value Distribution Theory. X, 132 pages. 1987.

Vol. 1240: Number Theory, New York 1984–85. Seminar. Edited by D.V. Chudnovsky, G.V. Chudnovsky, H. Cohn and M.B. Nathanson. V, 324 pages. 1987.

Vol. 1241: L. Gårding, Singularities in Linear Wave Propagation. III, 125 pages. 1987.

Vol. 1242: Functional Analysis II, with Contributions by J. Hoffmann-Jørgensen et al. Edited by S. Kurepa, H. Kraljević and D. Butković. VII, 432 pages. 1987.

Vol. 1243: Non Commutative Harmonic Analysis and Lie Groups. Proceedings, 1985. Edited by J. Carmona, P. Delorme and M. Vergne. V, 309 pages. 1987.

Vol. 1244: W. Müller, Manifolds with Cusps of Rank One. XI, 158 pages. 1987.

Vol. 1245: S. Rallis, L-Functions and the Oscillator Representation. XVI, 239 pages. 1987.

Vol. 1246: Hodge Theory. Proceedings, 1985. Edited by E. Cattani, F. Guillén, A. Kaplan and F. Puerta. VII, 175 pages. 1987.

Vol. 1247: Séminaire de Probabilités XXI. Proceedings. Edité par J. Azéma, P.A. Meyer et M. Yor. IV, 579 pages. 1987.

Vol. 1248: Nonlinear Semigroups, Partial Differential Equations and Attractors. Proceedings, 1985. Edited by T.L. Gill and W.W. Zachary. IX, 185 pages. 1987.

Vol. 1249: I. van den Berg, Nonstandard Asymptotic Analysis. IX, 187 pages. 1987.

Vol. 1250: Stochastic Processes – Mathematics and Physics II. Proceedings 1985. Edited by S. Albeverio, Ph. Blanchard and L. Streit. VI, 359 pages. 1987.

Vol. 1251: Differential Geometric Methods in Mathematical Physics. Proceedings, 1985. Edited by P.L. García and A. Pérez-Rendón. VII, 300 pages. 1987.

Vol. 1252: T. Kaise, Représentations de Weil et GL_2 Algèbres de division et GL_n. VII, 203 pages. 1987.

Vol. 1253: J. Fischer, An Approach to the Selberg Trace Formula via the Selberg Zeta-Function. III, 184 pages. 1987.

Vol. 1254: S. Gelbart, I. Piatetski-Shapiro, S. Rallis. Explicit Constructions of Automorphic L-Functions. VI, 152 pages. 1987.

Vol. 1255: Differential Geometry and Differential Equations. Proceedings, 1985. Edited by C. Gu, M. Berger and R.L. Bryant. XII, 243 pages. 1987.

Vol. 1256: Pseudo-Differential Operators. Proceedings, 1986. Edited by H.O. Cordes, B. Gramsch and H. Widom. X, 479 pages. 1987.

Vol. 1257: X. Wang, On the C*-Algebras of Foliations in the Plane. V, 165 pages. 1987.

Vol. 1258: J. Weidmann, Spectral Theory of Ordinary Differential Operators. VI, 303 pages. 1987.

continued on page 319

Introduction

In these notes we shall study algebras which are associated to blocks of tame representation type. These are the 2-blocks whose defect groups are dihedral or semidihedral or (generalized) quaternion. Over the last few years, a range of new results on a class of algebras including such blocks have been obtained. The algebras are essentially defined in terms of their stable Auslander-Reiten quivers (which we shall describe later), and it has been proved that any such algebra is Morita equivalent to one of the algebras in a small list which is explicitly given by generators and relations. In particular, this describes tame blocks; and allows one to extend classical results on the arithmetic properties of these blocks.

The aim here is to provide a comprehensive account of these developments. We include new results, also some new proofs of known results; and the original work has been revised. We also present some general theory on algebras, including study of particular classes of algebras, which we think is important to understand the subject.

Suppose G is a finite group and K is a field of characteristic p; we assume that K is algebraically closed. The group algebra KG is a direct sum of indecomposable algebras, $KG = B_1 \oplus \ldots \oplus B_n$, and the B_i are the blocks of KG. Equivalently, the identity of KG is a sum of orthogonal centrally primitive idempotents e_i, and $B_i = e_iKG$.

One main topic in modular representation theory is the study of such blocks, as algebras, and their module categories. It is known that a block is a symmetric algebra and is in particular self-injective. When the prime p divides the group order, then the blocks of KG are usually not semisimple.

An analogue of the role played for KG by a Sylow p-subgroup of G is the *defect group of a block*. In our context, we define a defect group of a block B to be a minimal subgroup D of G such that every B-module is D-projective. (A module M is D-projective if M is isomorphic to a direct summand of $W \otimes_{KD} KG$ for some KD-module W.) The defect groups of a block form one conjugacy class of p-subgroups of G.

Considering a block as an algebra, it has a representation type. In general, an algebra is of *finite representation type* if it has finitely many isomorphism classes of indecomposable modules; otherwise it is *of infinite type*. An algebra of infinite type is *tame* if, roughly speaking, there is a good parametrization of the indecomposable modules; and it is *wild* if its module category is comparable with that of the free algebra in two generators, see I.4.2 and I.4.4. The representation type of blocks is characterized as follows, summarizing a number of results ([BD], [Hi], [Br_1]):

THEOREM *Consider the group algebra KG of a finite group G over a field K of characteristic p; or a block B of KG. Suppose D is a Sylow p-subgroup of G; or a defect group of B. Then the representation type of KG; or of B, is*

(i) finite if D is cyclic;

(ii) tame if p = 2 and D is dihedral or semidihedral or generalized quaternion;

(iii) wild, otherwise.

(For comparison, Maschke's theorem on semisimplicity corresponds to the case D = 1.) Blocks of finite type, from the module theoretic point of view, are now well-understood, thanks to [J], [K], but also [Gr_1], [M_2] and more recently [GR] and many others. Our interest lies in the study of tame blocks and their module cateories.

Historically, block theory started off with the functional approach, and this was developed by R. Brauer. There the object of study are characters, Brauer characters, numerical data such as k(B) (the number of irreducible characters of the block B) and l(B) (the number or irreducible Braer characters of B, that is, the simple B-modules). Much work has been done by Brauer and others; and one of the original motivations seems to have been classification problems for finite simple groups.(In fact, there are powerful results, for example [BS], [GW], [ABG]). Experience shows that these arithmetic data are to a large extent locally determined, that is, depend on p-local subgroups of the given group. In fact the defect groups of a block were discovered first in that context, but in a different form (determined by character values reduced modulo p). This suggests the following approach : Fix a p-group D and study arithmetic properties of arbitrary blocks with defect groups D.

There were results for particular Sylow p-groups of G, or defect groups of a block: An early paper of this type [B_2] deals with defect groups of order p, and these results have been generalized by Dade to arbitrary blocks with cyclic defect groups [Da_1]. Later, Brauer studied blocks whose defect groups are Klein 4-groups and dihedral groups, and Olsson obtained analogous results for semidihedral and quaternion defect groups. For cyclic defect groups, these arithmetic results include more or less complete information about decomposition numbers and Cartan matrices. For other types of defect groups, however, there were only partial results obtained by functional methods; even with hypotheses on the groups, as in [L_1].

Returning to the module approach, originally the methods used were based on Green correspondence, which exploits module categories of p-local subgroups. This works well for cyclic defect groups and in particular cases but is not as powerful in general.

Since then new discoveries on representations of finite dimensional algebras more generally have had an impact. For tame representation type, Ringel gave a classification of tame local algebras [R_1]. Going further in this direction, Donovan classified symmetric tame algebras with two simple modules [Do]; and using Brauer's arithmetical results on dihedral blocks, he obtained the following:

If B is a block with dihedral defect groups then the basic algebra of B belongs to a small list of algebras explicitly given by generators and relations (at least modulo the socle).

The hypotheses he was using, namely "tame, symmetric, elementary divisors of the Cartan matrix" seemed to be too weak to deal with larger numbers of simple modules.

In the meantime, Auslander and Reiten had discovered almost split sequences [$AR_{1,2}$] and they were used with great success in classification problems.

Concerning blocks, Gabriel and Riedtmann noticed that the graph structure of the stable Auslander-Reiten quiver is the same for all blocks with cyclic defect groups; it is a graph of the form $\mathbb{Z}A_m/e$ (here m is the order of the defect group, and e is the number of projective modules). In [GR] they classified all symmetric algebras, up to Morita equivalence whose stable Auslander-Reiten quiver is of the form $\mathbb{Z}A_n/k$, for arbitrary n and k; this includes an explicit description of the basic algebras by generators and relations. The list contains all blocks of finite type. (As in [Do], this work is independent of groups.)

Later, Riedtmann proved a general theorem which describes the graph structure of stable Auslander-Reiten components (and she used this result as an order principle to classify self-injective algebras of finite type). In the special case of group algebras, the graph structure of Auslander-Reiten components is even more restricted, by [W] (and [HPR]). This encourages one to try a strategy similar to [GR] also for tame blocks; and in fact, this approach is successful.

The algebras we study are defined as follows: Suppose Λ is a finite-dimensional algebra over a field K which is algebraically closed, of arbitrary characteristic. We say that Λ is of *dihedral (semidihedral, quaternion) type* if it satisfies the following conditions:

(1) Λ is tame, symmetric and indecomposable.

(2) The Cartan matrix of Λ is non-singular.

(3) the stable Auslander-Reiten quiver of Λ has the following components:

	dihedral type	semidihedral type	quaternion type
tubes	rank 1 and 3	rank ≤ 3	rank ≤ 2
	at most two 3-tubes	at most one 3-tube	
others	$\mathbb{Z}A_{\infty}^{\infty}/\Pi$	$\mathbb{Z}A_{\infty}^{\infty}$ and $\mathbb{Z}D_{\infty}$	

(For the dihedral type, we do not need the hypothesis that Λ is tame; this follows from the classification.) If B is a block with dihedral (semidihedral, quaternion) defect groups then B is an algebra of dihedral (semidihedral, quaternion) type, see (VI.1, VII.1, VIII.1).

The aim is to determine these algebras up to Morita equivalence, that is, to determine their basic algebras by generators and relations.

Comparing finite type and tame type, the difficulties are rather different. Given an algebra with stable Auslander-Reiten quiver $\mathbb{Z}A_m/k$; for fixed m and k it is possible with some combinatorics to determine all symmetric basic algebras explicitly by hand. The problem is to give a unified classification for arbitrary m, k. This was solved elegantly by Riedtmann with help of covering theory. On the other hand, for tame type any component of the stable Auslander-Reiten quiver is infinite; and the number of components of a given type is almost always infinite which means that one needs new methods again (and which makes regularity conditions, especially (2), necessary). The crucial observations are that for tame blocks the number of 3-tubes in the stable Auslander-Reiten quiver is finite, and that particular modules must lie at ends of components.

Condition (2) and the restriction to symmetric algebras is motivated by modular representation theory; in fact, symmetric algebras have been first studied by Brauer and Nesbitt [BN]. The importance in our context is that symmetric algebras are self-injective, and, moreover, that for any indecomposable projective module, the simple quotient and the socle are isomorphic. The other main property we use is that for symmetric algebras, Ω-periodicity coincides with τ-periodicity, where τ is the Auslander translation.

We will now describe the content of these notes in more detail. The first chapter is a general introduction into some representation theory of algebras. It includes a survey on Auslander-Reiten theory; and we also give an outline of some covering theory, in simple special cases, and we apply it to recognize wild algebras. Covering theory for representations of algebras has been developed and used extensively during the last years, especially for finite type (see for example [$Ri_{1,2,3}$], [G_3]); but also for tame type ([DS], [Sk]) and in general ([BG], [MP]).

In Chapter II we study special biserial algebras and also local "semidihedral algebras" and their representation theory. Special biserial algebras form an important class of algebras of tame or finite type. Well-known examples are blocks of group algebras with cyclic defect groups (see [Ku], [GR], [SW]), but also algebras appearing in the Gelfand-Ponomarev classification of Harish-Chandra modules over the Lorentz group [GP]. In fact, our results provide another instance: Blocks of group algebras with dihedral defect groups are also special biserial (modulo the socle); this property characterizes them amongst the blocks of infinite type.

The classification of indecomposable modules for special biserial algebras is essentially due to Gelfand-Ponomarev. In [GP] they have considered a special case, however the proof generalizes, see [R_2] (also [DF_3] and others). There are a number results on Auslander-Reiten theory for special biserial algebras, such as [BS]. More recently Butler and Ringel showed that the Gelfand-Ponamarev technique can be exploited to obtain certain irreducible maps [BR], and our account is based on this. There is also a new and elegant way to deal with the representation theory of special biserial algebras with covering theory, due to [DS].

For a long time, no explicit classification for the modules of the local semidihedral algebras Λ_m (see II.9) was known; but recently B. Crowley-Boevey has obtained a good parametrization [$CB_{2,3}$]. In fact, he has found a natural way to extend the idea of functorial filtration in [GP], and his work is of great importance in our context. We use his results to prove that the non-periodic stable Auslander-Reiten components of local semidihedral algebras are of tree class A_∞^∞ or D_∞. This is new; and it is essential later for determining the stable Auslander-Reiten quiver for arbitrary blocks with semidihedral defect groups.

In Chapter III we study tame local symmetric algebras in general. This is a revision of Ringel's work [R_1], restricted to the special case of symmetric algebras. We give a list of these algebras, by generators and relations. One reason for doing this is the observation that they are all either of dihedral type, or of semidihedral type, or else the simple module has Ω-period 4 (in which case, the algebra is very likely of quaternion type). Moreover, we use local tame algebras later to determine relations for arbitrary algebras of quaternion type. Another application is the structure of the centre of these algebras. We find that a local tame symmetric algebra which is not commutative must be of dimension 4k and its centre has dimension k+3. The interest of centres in modular representation theory arises from the fact that the dimension of the centre of a block B is the same as k(B); see also [CK].

In Chapter IV we collect general observations on modules with particular (small) Ω-period; we also determine all quivers for arbitrary tame local symmetric algebras with at most three simple modules. Then we come to a central part, namely to principles how to exploit the graph structure of the stable Auslander-Reiten quiver to information on the structure of projective modules. (A short illustration may be found in IV.3.8 where we determine all indecomposable symmetric algebras whose stable Auslander-Reiten quiver contains a component $\cong \mathbb{Z}\tilde{A}_{1,2}$.) The most important results here are methods to show that certain modules lie at ends of components; these go back to [E_4]. In IV.4 we give the proof of a theorem due to Butler and Ringel [BR], for arbitrary algebras; and in IV.5 we prove a generalization of (2.8),[E_4], valid for self-injective algebras. Then we study whether these modules lie at "a-ends" (components with tree class A_∞ or A_n), or at "d-ends" (components with tree class D_∞ or D_n or $\tilde{D}_n$). Our motivation is the study of algebras of dihedral (and semidihedral) type; here the only ends occur at tubes (and also at $\mathbb{Z}D_\infty$-components). Actually, since the results of these sections use only the local structure of an Auslander-Reiten component, they can also be used for self-injective algebras of finite type, of type A_n or D_n.

Chapter V deals mostly with group representations; the main aim is to determine the structure of the stable Auslander-Reiten quiver $\Gamma_s(B)$ for an arbitrary tame block B. We start with a survey on the relevant modular representation theory; then we determine the Morita equivalence classes for a few tame blocks of 2-local groups by hand; here we use a number of rather small general algebras, such as skew group rings and exploit the fact that the structure of these small algebras is rather restricted. (We note that these results also follow from more general theorems in [Ku], [KP], [$Pu_{1,2}$].) In the next section we summarize more generally methods for relating stable Auslander-Reiten components of group algebras for groups with those for p-local subgroups. This includes new results due to Kawata [$K_{1,2}$]. We use these techniques and earlier results to determine $\Gamma_s(B)$ for arbitrary tame blocks. In the last section we apply the theory of [AB] to determine the number of irreducible characters of tame blocks; the results are due to [$B_{2,3}$] and [O].

In Chapter VI we classify algebras of dihedral type. The proofs we give here are new; and we also include the algebras with two simple modules which have not been studied in [E_4]; and therefore we have a new approach (and improvement) of Donovan's results. Moreover, we have now simplified and generalized the definition, compared with [E_4] (where we assumed that each 3-tube is fixed by Ω). Therefore, we obtain a number of new algebras. To classify them, creates no problem.

In Chapter VII, we deal with algebras of quaternion type. We have a small list of algebras explicitly given by generators and relations, and we show that the basic algebra of an algebra of quaternion type belongs to this list. It is convenient for this type of algebras to work directly with the basic algebra since the hypotheses give immediately information about J^2 (modulo J^3). The treatment here is streamlined and shortened compared with [E_6]; we also include algebras which were omitted there; and the relations were not always correctly stated.

In Chapter VIII we study algebras of semidihedral type. We show that any such algebra which is basic belongs to a small list of algebras given explicitly by generators and relations. This is a different approach from that in [E_5]: we do not make assumptions about the number of simple modules. As an order principle we use various possible positions of simple modules in the stable Auslander-Reiten quiver and as for the dihedral type, modules at ends of components; however, the ends of $\mathbb{Z}D_\infty$-components increase the complexity of the problem considerably. We obtain as a corollary that the number of simple modules of these algebras is bounded by three; this is a new result. The work for this type of algebra is rather more involved than for the others; in fact, for part of the work one is close to algebras of finite type (or algebras with Euclidean components); this recalls how much more work the self-injective algebras of type D_n required, compared with A_n (see [$Ri_{2,3}$]). Parallel to this, one might compare the relative lengths of [ABG] and [GW].

The main aim of Chapter IX is to determine which of these algebras can occur as blocks, and to prove functional results. The proofs here are new (and simplified), continuing [E_9]; they do not assume results from [$B_{2,3}$], [O]. Instead, we use only formulae for dim $Z(\Lambda)$ from our list of algebras and $k(B)$, together with a few general principles. The results are a complete list of possible Cartan matrices for tame blocks, and for each of them the decomposition matrix (which is unique). We note that finding the decomposition matrix is very elementary; it also does not depend on defect groups. The method also determines without extra work generalized decomposition matrices D^j where j is a central involution of the defect group.

In Chapter X, we give a proof of the theorem by Brauer and Suzuki. As a further application we determine the Morita equivalence classes for a family of blocks for alternating groups. Moreover, we include some comments about misprints and errors in earlier work.

The families of algebras obtained in VI to VIII are listed at the end. We have adopted the following labelling: D, SD, Q denotes the type of algebra, and then we name all possible quivers; in most cases there is not more than one family per type of algebra with a given quiver, sometimes there are two (or three). Algebras which are isomorphic modulo the socle usually belong to the same family. In the tables, we do not give details on socle scalars, except for dihedral type; although in Chapter VI, VII we included some scalar transformations. The second part of the tables contains conditions to be satisfied for blocks, and the decomposition matrices, together with examples for blocks.

When determing relations, we tacitly use some standard arguments. Namely, there are automatic zero-relations satisfied by symmetric basic algebras, due to the fact that $e\Lambda f \cap \mathrm{soc}\,\Lambda = 0$ for orthogonal idempotents e, f. Moreover, to show that the given relations are sufficient, one verifies that dim KQ/I is the same as dim Λ where Λ is the algebra considered; and to show that an algebra KQ/I is symmetric one easily defines a symmetrizing form.

A number of problems remain open. Concerning the algebras, it is not proved whether the algebras in the list are tame. Possibly a new result by Bautista [B] could help. Going further, one would like to have a good parametrization for the indecomposable representations; perhaps there are refinements of the methods in [$CB_{2,3}$]. However, for quaternion type, it is still open at present how to deal with the local algebra. As for blocks, we ask whether there are examples in the remaining cases where the arithmetic conditions are satisfied. It may not be possible to answer this by general theory; bearing in mind that the similar question for blocks of finite type is also not answered.

There is a conjecture by Brauer, namely: Given a p-group D, there are only finitely many Cartan matrices which can occur for p-blocks with defect group D. A stronger question was asked by P. Donovan: Given a p-group D, is the number of Morita equivalence classes of p-blocks with defect group D then finite?

For tame blocks, we confirm Brauer's conjecture; on the other hand, we are not able to answer Donovan's question. For some families of blocks of quaternion type, there are infinitely many non-isomorphic algebras which are isomorphic modulo the socle.

The mathematical work for these notes and the writing was mostly done whilst I was visiting the University of Essen; I am grateful to Professor Michler for inviting me, and to the DFG for financial support.

The very first attempts started with the group theoretic approach via Green correspondence. However, J. Alperin pointed out to me that my work on Klein 4-groups contained an error (and I am indebted to him for this); so I had to revise my strategy, and it was possible to find a correct proof for the result using Auslander-Reiten theory. Thanks to the suggestion by Professor Michler, I continued,

first with the dihedral case and then with the other types, and this was a rather interesting project. I greatly appreciate various discussions with members of the groups in Essen and in Bielefeld and with B. Crowley-Boevey, L. Kovacs, M. Schaps and with A. Skowronski. Also, I should like to thank the Universities of Essen (and Oxford) for hospitality and for providing technical facilities.

TABLE OF CONTENTS

	Introduction	III
I.	Algebras, quivers, representation type, Auslander-Reiten theory coverings	1
I.1	Background	
I.2	Morita equivalence, basic algebras	
I.3	Symmetric algebras	
I.4	The representation type	
I.5	Algebras and quivers	
I.6	The quiver of a symmetric algebra	
I.7	Auslander-Reiten sequences	
I.8	The Auslander-Reiten quiver	
I.9	Some wild algebras	
I.10	Coverings	
II	Special biserial algebras and the local semidihedral algebra	47
II.1	Special bierial algebras, string algebras	
II.2	Words; strings, bands	
II.3	Indecomposable modules for special biserial algebras	
II.4	Irreducible maps for band modules	
II.5	Irreducible maps for string modules	
II.6	Auslander-Reiten sequences for special biserial algebras	
II.7	Auslander-Reiten components for some special biserial algebras	
II.8	Some algebras of finite type	
II.9	Indecomposable modules of the semidihedral algebra	
II.10	Auslander-Reiten components for the semidihedral algebra	
III	Tame symmetric local algebras	80
III.1	List of tame local symmetric algebras, characterizations	
III.2-12	Proofs	
III.13	The group algebras of the dihedral, semidihedral and quaternion 2-groups for char K = 2	

III.14 The centres of tame local symmetric algebras

III.15 The socles of tame local symmetric algebras

III.16 Recognizing the isomorphism type

III.17 The dihedral, semidihedral, quaternion 2-groups

IV More on modules, quivers, Auslander-Reiten sequences 93

IV.1 Modules and Ω, τ

IV.2 Quivers of tame symmetric algebras with at most three simple modules

IV.3 Exploiting the Auslander-Reiten quiver

IV.4-6 Ends of componentsI, II, III

V Stable Auslander-Reiten components for tame blocks 121

V.1 Modular group representations

V.2 Some tame blocks of 2-local groups

V.3 On the AR-quiver of group algebras

V.4 The stable AR-quiver for tame blocks

V.5 The number of irreducible characters for tame blocks

VI Algebras of dihedral type 159

VI.1 A class of symmetric algebras where hearts are decomposable

VI.2 Algebras of dihedral type

VI.3-5 The number of simple modules, the possible quivers

VI.6 Ω on arrows, composition factors

VI.7 Algebras with one simple module

VI.8 Algebras with two simple modules

VI.9 Algebras with three simple modules

VI.10 The converse

VII Algebras of quaternion type 181

VII.1-2 Definition, examples

VII.3 The number of simple modules, possible quivers, general properties

VII.6 Algebras with one simple module

VII.7 Algebras with two simple modules

VII.8 Algebras with three simple modules

VII.9 On the converse

VIII Algebras of semidihedral type 205

VIII.1-2 Definition, examples, outline of the strategy

VIII.3-4 Algebras with ≤ two simple modules

VIII.5 Simple modules with three predecessors and vertices with three arrows

VIII.6 Quivers with few arrows

VIII.7 Algebras where projectives have isomorphic hearts

VIII.8 End vertices, exceptions

VIII.9 Quivers with enough arrows

VIII.10 On the converse

IX Centres, blocks, decomposition numbers 263

IX.1 On the centres of algebras of dihedral, semidihedral, quaternion type

IX.2 Cartan matrices and decomposition numbers for tame blocks

IX.3 One simple module

IX.4 Two simple modules

IX.5 Cartan matrices and blocks, three simple modules

IX.6 Decomposition numbers for tame blocks with three simple modules

X Some applications 286

X.1 Quaternion Sylow 2-subgroups: The Theorem by Brauer and Suzuki

X.2 Relations for characters and recognizing the Morita equivalence class of a tame block

X.3 Tame blocks for alternating groups

X.4 Some corrections

Bibliography 307