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Phillip A. Griffiths
Exterior Differential Systems and the Calculus of Variations

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LIST OF COMMONLY USED NOTATIONS

(Note: The references for the undefined terms used below may be found in the index.)

$A^*(X)$	Exterior algebra of smooth differential forms on a manifold X
$\{\Sigma\}$	Algebraic ideal in $A^*(X)$ generated by a set Σ of forms on X
(I,ω)	Exterior differential system with independence condition
$V(I,\omega)$	Set of integral manifolds of (I,ω)
$T_N(V(I,\omega))$	Tangent space to $V(I,\omega)$ at N
$(I,\omega;\varphi)$	Variational Problem (cf. Chapter I, Sec. a)
$\Phi: V(I,\omega) \to \mathbb{R}$	Functional on $V(I,\omega)$
$\delta\Phi: T_N(I,\omega) \to \mathbb{R}$	Differential of Φ
$V(I,\omega;[A,B]))$	Subset of $V(I,\omega)$ given by endpoint conditions
$T_N(V(I,\omega;[A,B]))$	Tangent space to $V(I,\omega;[A,B])$
$\equiv \bmod I$	Congruence modulo an ideal $I \subset A^*(X)$
$\overline{\equiv}$	Congruence modulo the image of $I \wedge I \to A^*(X)$ (cf. (II.b.4))
$\mathbb{P}E$	Projectivization of a vector space E
θ_N	Restriction of $\theta \in A^*(X)$ to a submanifold $N \subset X$
$d\theta = \Theta$	Exterior derivative of a differential form; little θ is frequently denoted by capital Θ
$F(\cdot)$	Frame manifold
$L_v\varphi$	Lie derivative of a form φ along a vector field v
Y	Momentum space associated to $(I,\omega;\varphi)$
Q	Reduced momentum space associated to $(I,\omega;\varphi)$

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INTRODUCTION

This monograph is a revised and expanded version of lecture notes from a class given at Harvard University, Nankai University, and the Graduate School of the Academia Sinica during the academic year 1981-82. The objective was to present the formalism, together with numerous illustrative examples, of the calculus of variations for functionals whose domain of definition consists of integral manifolds of an exterior differential system. This includes as a special case the Lagrange problem of analyzing classical functionals with arbitrary (i.e., non-holonomic as well as holonomic) constraints. A secondary objective was to illustrate in practice some aspects of the theory of exterior differential systems. In fact, even though the calculus of variations is a venerable subject about which it is hard to say something new, (1) we feel that utilizing techniques from exterior differential systems such as Cauchy characteristics, the derived flag, and prolongation allows a systematic treatment of the subject in greater generality than customary and sheds new light on even the classical Lagrange problem.

As indicated by the table of contents the text is divided into four chapters, with most of the general theory being presented in the first and last. We break somewhat with current tradition in that an unusually large amount of space is devoted to examples. Perhaps even more of a break (or is it a regression?) is the special concern given to the explicit integration of the Euler-Lagrange equations, Jacobi equations, Hamilton-Jacobi equations, etc. in these examples—in a word we want to get out *formulas*. Much of the middle two chapters are devoted to methods for doing this; again the theory of exterior differential systems provides an effective computational tool. (2)

For reasons of space, and even moreso because the several variable theory is incomplete at several crucial points, the discussion is restricted to the case of one independent variable; i.e., we consider functionals defined on integral *curves* of an exterior differential system.

We will now describe an example that may help motivate developing the theory in such generality. Let $\gamma \subset \mathbb{E}^n$ be a smooth curve given parametrically by its position vector $x(s) \in \mathbb{E}^n$ viewed as a function of arclength. It is well-known that in general γ has curvatures $\kappa_1(s),\ldots,\kappa_{n-1}(s)$ that are Euclidean invariants and that uniquely determine γ up to a rigid motion (when $n = 3$ these are the usual curvature and torsion). We consider a functional

$$\Phi(\gamma) = \int_\gamma L(\kappa_1(s),\ldots,\kappa_{n-1}(s))\, ds \qquad (1)$$

and ask standard questions such as i) find the Euler-Lagrange equations and explicitly integrate them if possible; ii) find the Jacobi equations and information on conjugate points; and iii) if $L = L(\kappa_1,\ldots,\kappa_r)$ depends only on the first r curvatures and if the matrix $\|\partial^2 L/\partial\kappa_i\partial\kappa_j\|_{1\le i,j\le r} > 0$, then show that a solution to the Euler-Lagrange equations having no conjugate points is a local minimum for (1). It is clear that this problem may be set up in coordinates as a classical higher order variational problem, and it is equally clear that in this formulation the resulting computations will be quite lengthy. Alternatively, we may consider the Frénet frame associated to γ as a curve N in the group $E(n)$ of Euclidean motions. Then N is an integral manifold of a left invariant exterior differential system (I,ω) on $E(n)$, and (1) may be viewed as an invariant functional defined on *any* integral manifold of (I,ω). Once the general formalism of the calculus of variations is in place for functionals defined only on integral manifolds of differential systems, we may hope that in examples such as this the theory should provide an effective computational tool. For instance, it is known that the classical theory o rigid body motion extends to Lagrangians defined by left-invariant metrics on any Lie group (theory of Kirilov-Kostant-Souriau; cf. [50] and [61]), and it is reasonable to try to further extend this theory to invariant functionals defined only on integral manifolds of invariant exterior differential systems and apply the result to the study of (1). This will be done in Chapter III. (3)

We shall now describe in more detail some of the contents of this monograph, where we refer to the text for explanation of notations and undefined terms (there is an index at the end).

Chapter 0 is preliminary and is intended only for reference. (It is suggested that the reader begin with Chapter I.) In it are first collected some terminology and notations from standard manifold theory. Next there is a very brief description of the language of jet manifolds and of moving frames. The former provides a useful formalism for introducing derivatives as new variables (cf. [31], [38], [43], and [62]). The latter is especially relevant due to the fact that a general curve in many homogeneous spaces G/H have a "Frénet frame"; i.e., a canonical lifting to G (cf. [34], [44]), and consequently the aforementioned analysis of the functional (1) may be expected to reflect rather general phenomena. Finally, in Chapter 0 we record some of the definitions and elementary facts from the theory of exterior differential systems. Again this is only meant to establish language; the more substantial aspects of the theory are introduced as needed during the text.[4]

In Chapter I we explain the basic setup and derive the main equations of the theory, the Euler-Lagrange equations. Assume given an exterior differential system (I,ω) on a manifold X and denote by $V(I,\omega)$ the set of integral manifolds $N\subset X$ of (I,ω). For an example in addition to the Frenet liftings mentioned above, we consider the Lagrange problem:[5] Let $J^1(\mathbb{R},\mathbb{R}^m)$ denote the space of 1-jets of maps from $\mathbb{R}$ to $\mathbb{R}^m$. On $J^1(\mathbb{R},\mathbb{R}^m)$ we have a natural coordinate system $(x;y^1,\dots,y^m;\dot{y}^1,\dots,\dot{y}^m)$ and canonical differential ideal I_0 generated by the Pfaffian forms

$$\theta^\alpha = dy^\alpha - \dot{y}^\alpha\, dx \qquad \alpha = 1,\dots,m\,.$$ [6]

Setting $\omega = dx$, $V(I_0,\omega)$ consists of 1-jets $x \to \left(x, y(x), \frac{dy(x)}{dx}\right)$ of parametrized curves in $\mathbb{R}^m$. Let $X\subset J^1(\mathbb{R},\mathbb{R}^m)$ be a submanifold and let (I,ω) be the restriction of (I_0,ω) to X. We may think of X as defined by equations

$$g^\rho(x,y,\dot{y}) = 0\,, \tag{2}$$

and then $V(I,\omega)$ consists of 1-jets of parametrized curves that satisfy the constraints

$$g^\rho\left(x, y(x), \frac{dy(x)}{dx}\right) = 0\,.$$

A special case is when the constraints (2) are of the form

$$g^\rho_\alpha(y)\dot{y}^\alpha = 0\,.$$

Then they correspond to the sub-bundle $W^* = \text{span}\{g^\rho_\alpha(y)dy^\alpha\}$ of the cotangent bundle of $\mathbb{R}^m$ (or dually to a sub-bundle of the tangent bundle; i.e., a distribution). (Note: In general on a manifold M the differential ideal generated by the sections of a sub-bundle $W^* \subset T^*(M)$ will be called a *Pfaffian differential system*. In this text essentially all differential ideals will be of this type. However, they will usually be defined on manifolds lying over the one of interest.) Another special case of (2) is given by the canonical embeddings

$$J^k(\mathbb{R},\mathbb{R}^\ell) \subset J^1(\mathbb{R},\mathbb{R}^{k(\ell)})$$

of higher jet-manifolds into 1-jets.[7]

Returning to the general situation, on X we assume given a differential form φ and consider the *functional*

$$\Phi(N) = \int_N \varphi \quad , \qquad N \in V(I,\omega) . \tag{3}$$

Eventually we will restrict Φ to N's satisfying suitable boundary or endpoint conditions, but this is a somewhat subtle matter involving the structure theory of (I,ω). In particular, at first glance it appears to involve the derived flag of I, which roughly speaking tells how many derivatives are implicit in I. Endpoint conditions will be discussed in Chapter IV; in Chapter I we simply finesse the matter and argue formally.

By the *variational problem* $(I,\omega;\varphi)$ will be meant the analysis of the functional (3). For $X \subset J^1(\mathbb{R},\mathbb{R}^m)$ given by (2) above, if we take

$$\varphi = L(x,y,\dot{y})dx$$

then $(I,\omega;\varphi)$ is a classical variational problem with constraints (Lagrange problem). Another example is given by the functional (1). In general, understanding a variational problem $(I,\omega;\varphi)$ clearly will involve at least some of the structure of (I,ω) and how $d\varphi$ relates to this structure.

The first order of business is to derive the *Euler-Lagrange equations* expressing the condition that $N \in V(I,\omega)$ be an *extremal* of $(I,\omega;\varphi)$; i.e., the "differential" of Φ should vanish at N, written

$$\delta\Phi(N) = 0 \quad . \tag{4}$$

For the Lagrange problem this is to some extent accomplished in the classic treatise [13] and is discussed in many other sources (e.g. [5] and [40]). The traditional method is to use Lagrange multipliers. In general, the obvious difficulty in deriving the Euler-Lagrange equations is that only certain normal vector fields to $N \subset X$ represent infinitesimal variations of N *as an integral manifold of* (I,ω), (e.g., see page 344 of [13]). In particular there may be no compactly supported such infinitesimal variations of $N \in V(I,\omega)$, and consequently at first glance it would seem that already at this stage we must worry about endpoint conditions, thereby dragging in the structure theory of (I,ω) and running the considerable risk of becoming bogged down at the very outset.

This difficulty may be avoided by proceeding indirectly. If we simply assume the existence *in complete generality* of Euler-Lagrange equations having certain functoriality properties, then heuristic reasoning leads to a unique and remarkably symmetrical system of equations that "should" be the equations (4).[8] Rather than try to justify the heuristic reasoning at this point, we simply *define* these to be the Euler-Lagrange equations associated to the variational problem $(I,\omega;\varphi)$ and proceed to investigate these equations in their own right.

In somewhat more detail, after setting up the variational problem in Chapter I, Section a), in Chapter I, Section b) the "tangent space" $T_N(V(I,\omega))$ is defined to be the kernel of a certain linear differential operator on normal vector fields (cf. (I.b.16); in doing this the reference [38] has been helpful). In order to better understand $T_N(V(I,\omega))$, we turn in Chapter I, Section c) to some of the structure theory of Pfaffian differential systems. In particular, *Cauchy characteristics*, the *derived flag*, the important concepts of a *Pfaffian system in good form* and its basic invariant the *Cartan integer* s_1, and finally the *prolongation* of an exterior differential system are discussed. In this setting we establish an "infinitesimal Cartan-Kähler theorem," which states that a general $v \in T_N(V(I,\omega))$ depends on s_1 arbitrary functions of one variable (plus a certain number of constants).

In Chapter I, Section d) the Euler-Lagrange equations are defined (cf. (I.d.14)), and a number of examples are computed to show that in classical cases they give the right answer. We also analyze the Euler-Lagrange equations associated to the functional

$$\Phi(\gamma) = \frac{1}{2} \int_{\gamma} \kappa^2 \, ds \qquad (5)$$

where γ is a curve on a surface S and κ is its geodesic curvature. When S has constant Gaussian curvature it is found that these equations may be explicitly integrated by elliptic functions whose modulus depends on the curvature of S and on an "energy level."

In Chapter I, Section e) the basic step in this presentation of the theory is taken by writing the Euler-Lagrange equations as a Pfaffian differential system (J,ω) on an associated manifold Y that we call the *momentum space* (cf. Theorem (I.e.9)). (Note: Although we give the construction of (J,ω) on Y explicitly, from the viewpoint of the general theory of exterior differential systems it may be explained very simply: The Euler-Lagrange equations (I.d.14) contain mysterious "functions λ_α to be determined." We adjoin the λ_α as new variables, write the resulting equations as a differential system, and then (J,ω) is simply the involutive prolongation of this system.) For unconstrained and non-degenerate classical variational problems, Y is the usual momentum space $\mathbb{R} \times T^*(M)$ where $X = \mathbb{R} \times T(M)$, but in general even the dimension of Y will depend on the numerical invariants of $(I,\omega;\varphi)$, especially the Cartan integer. We call (J,ω) the *Euler-Lagrange system* and note the remarkable fact that, despite the apparent generality of the variational problem $(I,\omega;\varphi)$, (J,ω) is a very simple standard Pfaffian system: On Y there is canonically given a 1-form ψ_Y with exterior derivative $\Psi_Y = d\psi_Y$, and J is the Cauchy characteristic system (i.e., $\Psi_Y^\perp$) of Ψ_Y. In particular this leads naturally to the definition of a *non-degenerate variational problem* $(I,\omega;\varphi)$ to be one where $\dim Y = 2m+1$ (for some m) and where

$$\psi_Y \wedge (\Psi_Y)^m \neq 0 \quad . \qquad (6)$$

Thus far, all "natural" examples have turned out to be non-degenerate in this sense. By the theorem of Pfaff-Darboux the Euler-Lagrange system of a non-degenerate variational problem has a standard local normal form; frequently, this normal form is even global.

Next, also in Chapter I, Section e), we associate to a variational problem $(I,\omega;\varphi)$ satisfying a mild internal structural condition (one that is satisfied in almost all our examples, and is also satisfied on any prolongation) a quadratic form $\mathcal{Q}$. If I is the Pfaffian differential ideal generated by a sub-bundle $W^*\subset T^*(X)$ and if the sub-bundle $W_1^*\subset W^*$ generates the 1st derived system, then intrinsically $\mathcal{Q}$ is a quadratic form on the rank-s_1 bundle $(W^*/W_1^*)^*$. The variational problem is defined to be *strongly non-degenerate* in case $\mathcal{Q}$ is pointwise non-degenerate, and it is shown that "strong non-degeneracy ⇒ non-degeneracy." Strongly non-degenerate problems turn out to have the important property that only the 1st derived system and not the whole derived flag intervenes in their basic structural properties.

Before continuing this introduction we should like to emphasize our feeling that the numerous examples scattered throughout the text are of equal importance to the general theory (they may even be more important). Moreover, these examples as well as the general theory show that exterior differential systems constitute a computationally effective and theoretically natural setting for the calculus of variations. This latter philosophy is by no means original[9], and in this regard we should like to point out the sources [16], [31], and [38] as being especially helpful to us. Although by and large they deal with unconstrained problems (however, see [40]), they contain intrinsic formulations, using jet bundles and differential forms, of the classical theory and our debt to them is apparent.

As previously mentioned, one of the main concerns of this monograph is in variational problems $(I,\omega;\varphi)$ whose Euler-Lagrange differential systems (J,ω) are explicitly solvable in the old-fashioned sense of being "suitably" integrable by quadratures. Here a most useful tool is (a suitable generalization of) Emmy Noether's theorem [55], which associates a 1st integral of (J,ω) to each infinitesimal symmetry of $(I,\omega;\varphi)$. In Chapter II, Section b) Noether's theorem is combined with the general formalism to show that several natural differen ial-geometric variational problems are quasi-integrable by quadratures. One of these is the functional, motivated by physical considerations,

$$\Phi(\gamma) = \frac{1}{2}\int_\gamma \kappa^2\, ds \qquad (\kappa = \text{curvature}) \tag{7}$$

defined for curves $\gamma\subset\mathbb{E}^3$. A partial result here is due to Radon [57] and is discussed in Blaschke [3]. Using the formalism developed thus far it is essentially known *a priori* that the Euler-Lagrange equations associated to (7) are quasi-integrable by quadratures, and it is a simple matter to carry out the integration. The same is also true of the variational problem associated to the functional (7) with the integral constraint

$$\int_\gamma ds = \ell = \text{constant}$$

(this one turns out to be equivalent to the unconstrained problem (7) on a surface of constant curvature), and of the *Delauney problem*

$$\begin{cases} \Phi(\gamma) = \int_\gamma ds \\ \text{with the constraint } \kappa = \text{constant} \end{cases} \tag{8}$$

which was much discussed classically (a treatment may be found in [14]). The Euler-Lagrange equations associated to (7) and (8) have "phase portraits" given respectively by elliptic and rational algebraic curves.

These examples begin to make clear the general point that once one accepts the basic construction

$$(I,\omega;\varphi) \quad \text{on} \quad X \rightsquigarrow (J,\omega) \quad \text{on} \quad Y \text{ ,}$$

the computation of examples, 1^{st} integrals, and later on Jacobi equations, has an algorithmic character. Carrying out this algorithm in practice has as its essential step the computation of the structure equations of the differential system (I,ω) and the relation of $d\varphi$ to these equations. Once this is done the determination of (J,ω), 1^{st} integrals, Jacobi equations, etc. is reduced to formal algebraic manipulations that seem to always have the same flavor.

As stated earlier, in this monograph we have restricted attention to the case of one independent variable. Now it seems likely that even more interesting problems will arise in higher dimensions, and in this regard we should like to call attention to the functional[10]

$$\Phi(M) = \frac{1}{k} \int_M \|II\|^k dA \text{ ,} \tag{9}$$

defined on submanifolds $M^n \subset \mathbb{E}^{n+r}$, where $\|II\|$ is the length of the

2^{nd} fundamental form. The case $n=1$, $k=2$ is the functional (7). In general we might think of (9) as standing in the same relation to (7) as does the minimal surface functional

$$V(M) = \int_M dA \tag{10}$$

to geodesics. To reduce higher dimensional problems to one independent variable it is natural to look at surfaces of revolution. In the minimal surface case it is well-known that the Euler-Lagrange equations associated to (10) are integrable by quadratures (catenary). Using Noether's theorem we are almost, but not completely, able to integrate the very interesting Euler-Lagrange equations associated to (9) in the case $n=k=2$ of a surface of revolution (cf. the end of Chapter II, Section a)).

As suggested at the beginning of the Introduction, one of the goals of this text is to treat variational problems for functionals (3) where I is an invariant exterior differential system and φ is an invariant form on a Lie group. Of course a principle motivation is the aforementioned fact that general curves γ in many homogeneous spaces G/H have canonical liftings to integral manifolds of an invariant differential system I on G. Now it is a well-known and beautiful fact that a left invariant positive definite quadratic Lagrangian system on any Lie group G has associated *Euler equations* that describe the motion along integral curves of the Euler-Lagrange equations of the momentum vector λ in the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G. In particular λ moves on a coadjoint orbit (Kostant-Souriau; loc. cit., and an Appendix to [2]). In Chapter III, Section a) these results are generalized to the setting of an invariant variational problem $(I,\omega;\varphi)$, and then the resulting Euler equations and coadjoint orbit description are used to integrate several problems, including the Euler-Lagrange equations associated to the functional (7) defined on curves in a Riemannian manifold of constant curvature.

With regard to the Lagrange problem on Lie groups we should like to call attention to the papers [8], [9], which are related to the discussion in Chapter III, Section a).

Thus far the Euler-Lagrange equations have only been arrived at by heuristic reasoning; in particular, they have not yet been shown to yield critical values of the functional (3). In fact, without specifying endpoint conditions this doesn't even make sense. Examination of special cases leads in Chapter IV, Section a) to a natural class of variational problems $(I,\omega;\varphi)$, that are said to be *well-posed*, whose endpoint conditions are given by the leaves of a canonical foliation on the momentum space Y. The situation may be summarized by the diagram

$$\begin{array}{ccc} & Y & \\ {\scriptstyle\tilde{\omega}}\swarrow & & \searrow{\scriptstyle\pi} \\ Q & & X \end{array} \qquad (11)$$

where Q is the quotient (assumed to exist) by the endpoint foliation. We call Q the *reduced momentum space*, and the understanding of the geometry of the basic diagram (11) turns out to be the key to the deeper aspects of the theory.[12]

For well-posed variational problems it is shown in Chapter IV, Section a) that the solutions to the Euler-Lagrange equations do in fact give critical points of the restriction of the functional (3) to subsets $V(I,\omega;[A,B])\subset V(I,\omega)$ consisting of integral manifolds satisfying endpoint conditions. Following this the endpoint conditions are interpreted in a number of examples,[13] and it is proved that a strongly non-degenerate variational problem is well-posed.

In Chapter IV, Section b) the important concepts of *Jacobi vector fields* and *conjugate points* are defined for well-posed variational problems. (Actually the definition of the Jacobi equations makes sense in general.) The point here is to work on the momentum space Y and not down on X. Not only is this natural theoretically, but as shown by examples the definition "upstairs" leads to effective computation of some examples.[14]

In the classical calculus of variations there is an intimate and very beautiful connection between the Jacobi equations and the 2nd variation. In the general setting when one tries to compute the 2nd variation down on X this causes considerable difficulty, but when lifted to Y the situation becomes quite simple and elegant. Again, as in the classical unconstrained case it is possible to define the *index form* as a quadratic form on the space of Jacobi vector fields and establish a simple connection between the index and 2nd variation.

Following this it is proved in Chapter IV, Section c) that, if $(I,\omega;\varphi)$ is a strongly non-degenerate variational problem whose quadratic form Q is positive definite, then for sufficiently close endpoint conditions a solution to the Euler-Lagrange equations yields a local minimum for the functional (3). This is an extension of a standard classical result, but it is not just a direct generalization since the point turns out to be to show that in the diagram (11) there is a (unique) exterior differential system (G,ω) on the reduced momentum space Q such that

$$\tilde{\omega}^* G = \pi^* I_1$$

where $I_1 \subset I$ is the 1^{st} derived system (recall that Q is a quadratic form on $(W^*/W_1^*)^*$). However, once one insists on setting up the calculus of variations in the general framework of exterior differential systems and introducing only those concepts intrinsic to this theory, results such as the one just mentioned become quite natural and the proofs not more difficult than in the classical case.

Next, in Chapter IV, Section d) the analogues of the classical concept of a *field* (sometimes called geodesic field) and the *Hamilton-Jacobi equation* are defined. Some examples are computed, and then these concepts are used, as in the classical case, to show that if N is a solution to the Euler-Lagrange equations of a positive-definite strongly non-degenerate variational problem, and if moreover N contains no pair of conjugate points, then N gives a local minimum for the functional (3). Again the essential ingredient beyond the classical case is the relation of the 1^{st} derived system of I to the diagram (11).

In summary, it would appear that the concept of a *positive strongly non-degenerate variational problem* is a good notion that includes the classical cases together with the Lagrange problem in sufficient generality to be useful, while at the same time allowing an extension of the main points of the classical theory.

Finally, in Chapter IV, Section e) we specifically discuss the classical Lagrange problem. It is shown that, with a minor modification of the previous endpoint conditions, this problem fits into the general theory. It is interesting that our theory does *not* reduce to the classical method of Lagrange multipliers; the exact relation together with several examples are also discussed.