PREFACE
TO THE ENGLISH EDITION.

The lectures which are published here in English were given by me at the University of Madrid in 1925; they were first published in 1927 by the Faculty of Science of the University, from a text prepared by Professor LUIGI FANTAPPIÈ, and translated into Spanish by Professor OCTAVIO DE TOLEDO.

The book has been improved and corrected and some passages have been completely rewritten. Many additions have been made, due mainly to the large number of papers and memoirs on the subject published in the last few years, and the bibliography has as far as possible been brought up to date and completed by the addition of references to everything published up to the present. These new references (about a hundred in number) and the text have been carefully collated and harmonised. The list of authors contains the names of persons almost all of whom have worked on functionals or closely related subjects.

All this shows that during its forty years of existence the subject has aroused much interest and has penetrated deeply into the various branches of mathematics and their applications. Everything concerning integral, integro-differential, and functional equations, research on functional spaces, the calculus of variations in its broadest sense, questions involving effects of the type described as "hereditary" — all these, in effect, are to-day within the domain of the theory of functionals and make use of general and systematic rules and processes belonging to that theory.

Among the new matter added to the present edition, I wish to mention the results obtained by G. C. EVANS relating to integral equations with a singular kernel and to integro-differential equations, and those obtained by his students, and the references to their applications of the theory to questions of political economy. I also wish to point out that I have discussed much more fully the work of Professor FANTAPPIÈ and the applications he has made of his own results. I have also spoken of the researches of MICHAL on transformations and integral invariants, of MOISIL on the dynamics of continuous media, and of DELSARTE on kinematics, all of which are closely connected with the theory of substitution groups. Lastly, I have summarised the results of my own recent researches into the energy equations of hereditary phenomena, and have touched on the studies that have been made of biological fluctuations under conditions involving hereditary effects.

This volume is not the first devoted to the theory of functionals. As long ago as 1913 a course of lectures on the subject given by me at the Sorbonne in 1912 was published in the "Collection Borel"; this series has also contained my lectures on integral and integro-differential equations and on composition and permutable functions, and on the last of these subjects I have also printed the lectures I gave at Princeton University and at the Rice Institute. But it is a pleasure to me to record that the first comprehensive treatise with the title "Functionals" was due to G. C. EVANS, who in 1918 published the volume *Functionals and their Applications: Selected Topics, including Integral Equations*. This consists of five chapters, dealing with functionals, differential operations on functionals, complex functionals and relations of isogeneity, functional equations, and integro-differential equations. Among the last-mentioned EVANS introduces integro-differential equations of the BÔCHER type, i.e. those in which the integrations are along curves or over fields which are arbitrary and variable. The book concludes with the generalisation of the theory of integral equations, which leads Professor EVANS, on the one hand, to MOORE's considerations on general analysis (a subject which has recently been studied by FRÉCHET), and on the other to the theory of permutable functions, which owes much subsequent progress to PÉRÈS. Almost all branches of the whole of the theory are thus contemplated in EVANS's interesting pages.

PAUL LÉVY's treatise, entitled *Leçons d'analyse fonctionnelle*, was published four years later. Here, after a thorough examination of the bases of the functional calculus, the fundamental concepts of functional operations are introduced and subjected to a searching criticism, and his original study of functional derivative equations is then developed in great detail. Professor Lévy gives an account of Gateaux's new ideas on multiple integration with an infinite number of variables, on potentials in functional fields of an infinite number of dimensions, and on functional and integro-differential equations; this material was derived from the ideas and rough notes collected and edited by Lévy himself with devoted care from the papers left by the young mathematician, who died during the war. But a considerable part of the book is justly assigned to the interesting researches carried out by Hadamard, to whom the theory of functionals owes such important contributions. Hadamard, in fact (as Tonelli also did later on), places this theory as foundation of the calculus of variations; he discovered the general expression for linear functionals, and, among other applications, gave the functional law of variation of Green's function.

The spread and the development of the theory of functionals owe much to Hadamard's work. It was he who introduced the happily chosen term "functionals", using this single word to denote what I had introduced and defined many years before, in my first memoir of 1887, by the name of "functions depending on other functions", a term I afterwards shortened into the expression "functions of lines", which is now reserved rather for a more limited class of entities.

Now from the very first lines of my 1887 memoir onwards, I have always warned mathematicians against confusing what have long been and still are called "functions of functions" with the new entities introduced by me, which are entirely distinct from them. Recently, however, to my great surprise, while examining the literature on the subject I came upon a recent treatise(1), which is intended for the use of students, in which the authors use the term "function functions" (*Funktionenfunktionen*, which resembles the ordinary *Funktionen von Funktionen* of the usual treatises on differential and integral calculus far too closely), so giving rise to an ambiguity that may well be misleading. I do not understand why the only reference to functionals in this treatise is to be found in a single footnote, which merely says that this term is used in French mathematical works, so ignoring the vast literature published in England, America, Germany, Italy, and other countries, as shown in the

(1) R. Courant and D. Hilbert: *Methoden der Mathematischen Physik*.

present volume. Still less do I understand why there is complete silence about the names of the numerous writers who have worked on the subject. And I am all the more amazed because the methods introduced and used by me from 1887 onwards (e.g. that of passing from the finite to the infinite) were put to good use later on by one of the authors of the treatise in question, and are moreover continually used in the treatise itself, while the second author has used the notion of derivation of functionals. And in German literature, including recent works on the modern theories of physics, we find not merely the ideas and methods of the theory of functionals, but even the noun "functional" itself, as readers of this volume can see.

It is my earnest wish that these lectures on functionals may encourage students of mathematics to new lines of research and new applications, and may give origin to a systematic and more extensive exposition of all the results here indicated, on the same lines as those followed in the present volume.

The hope of a more intense development and a wider extension of these studies has been revived by the founding of the review *Studia Mathematica*, which has recently been started by Professors Stefan Banach and Hugo Steinhaus. The new review, which is published at Lvov, in Poland, is to be devoted to research in functional analysis.

I cannot end this preface without expressing my warm gratitude to Professors G. C. Evans, Elena Freda, J. Pérès, and G. Vacca, for the valuable suggestions, advice, and help they have given me, and to Miss M. Long for her care and interest in the by no means easy task of translation.

Rome, June 1930.

Vito Volterra.

PREFACE
TO THE FIRST (SPANISH) EDITION

In April 1925 I had the honour of being invited by the Faculty of Science of the University of Madrid to deliver a course of lectures on "functionals" and the related theories. I wish here to express my sincere thanks to the Faculty, and in particular to its distinguished Dean, Professor OCTAVIO DE TOLEDO.

The present volume consists of the six Madrid lectures, in which I gave a rapid outline of these theories; the arrangement of the subject-matter is indicated by the chapter headings. The lectures were edited and prepared for publication by Dr. LUIGI FANTAPPIÈ, on the basis of my notes and other material, and were translated into Castilian by Professor OCTAVIO DE TOLEDO. To these two expert collaborators, the distinguished professor of the University of Madrid and the younger mathematician from Rome, I have to express my warm gratitude.

From 1887 onwards, the period when I published my earliest researches under the titles "Functions depending on other functions" and "Functions of lines", many articles and several valuable books have been published on this subject; in addition, certain chapters of analysis have undergone important changes in modern treatises as a result of the new concepts introduced along with "functionals". The theories which developed in this way and their applications have taken various directions and differ widely in their nature and scope; while the fundamental principles of functional analysis, exposed as they have been to profound and acute criticism, have gone on gradually developing and acquiring a steadily growing extension.

But a systematic treatise, collecting and co-ordinating the results obtained in the different fields and showing the underlying ideas connecting them, free alike from omissions in the analytical development and from excessive detail — such a work, giving a clear synthetic survey of all that has been done in the domain of "functionals", in my opinion does not yet exist, though many people have asked for it. These lectures certainly do not satisfy this wish, but they indicate the subjects to be discussed and show how these can be divided up into chapters, and may thus provide a framework on which the desired treatise may perhaps in time be constructed. They are accompanied by an extensive bibliography and numerous references to the more important works more or less closely connected with "functionals". I hope, therefore, that students of analysis will find them useful.

VITO VOLTERRA.

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CHAPTER I

FUNCTIONALS

Section I. Generalities and Definitions.

§ 1.

1. Before defining the quantities which we propose to study further on, we shall examine a simple problem of maxima and minima which shows how we pass naturally from the consideration of functions of a finite number of variables to quantities which depend on an infinite number of variables, i.e. on the infinite number of values assumed by an arbitrary function $x(t)$ corresponding to the values of t within an interval (a, b).

2. Let us then consider the product of two numbers x, y, whose sum is constant; it is known that this product is a *maximum* when the two numbers are equal. Translating this fact into geometrical language, we can say that among all the rectangles which have the sum of two consecutive sides constant, and therefore have a *constant perimeter*, the square is the one of *maximum area*. Passing to the more general problem of determining, among all plane polygons of n sides whose perimeter is constant, the one of maximum area, it will be seen that the solution is given by the regular polygon of n sides. If we denote the $2n$ coordinates of the n vertices of the polygon by $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$, the quantity to be made a maximum is the area A, given by the expression

$$A = \frac{1}{2} \sum_{1}^{n}{}_{i} \begin{vmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{vmatrix} = \frac{1}{2} \sum_{1}^{n}{}_{i} (x_i \Delta y_i - y_i \Delta x_i), \qquad (1)$$

a function, in this case, of the $2n$ variables x_i, y_i, with

$$\sum_{1}^{n}{}_i \sqrt{\Delta x_i^2 + \Delta y_i^2} = \text{constant}, \tag{1'}$$

and in which

$$x_{n+1} = x_1, \qquad y_{n+1} = y_1,$$
$$\Delta x_i = x_{i+1} - x_i, \quad \Delta y_i = y_{i+1} - y_i.$$

But if we pass from the case of a polygon of constant perimeter to the much more general problem of determining, among all the closed curves C of a given length l, the one that includes the maximum area A (the circle), the quantity A that we have to consider is given by the formula

$$A = \frac{1}{2}\int_C (x\,dy - y\,dx), \tag{2}$$

with

$$\int_C \sqrt{dx^2 + dy^2} = l = \text{constant}, \tag{2'}$$

and thus no longer depends on *a finite number of variables, but on the values of the coordinates of all the infinite number of points of the line C*. Instead of a problem of the ordinary differential calculus on maxima and minima, in which we have to determine a certain *finite* number of unknown quantities, we have a problem of the calculus of variations, in which the unknown is a *function* (its values for all the points of the inter. al considered) or a *curve* (the coordinates of all its points).

3. In the case of the polygon of n sides it was sufficient to find the quantities (x_1, y_1), (x_2, y_2), ... (x_n, y_n), depending on the index i, discontinuous and variable for *integral* values from 1 to n, in order to determine the polygon itself. In the case of any closed curve C, however, it will be necessary, in order to determine it, to give the coordinates x and y of ali its points

$$x = x(t), \qquad y = y(t),$$
$$x(b) = x(a), \quad y(b) = y(a),$$

in which now the quantities x, y depend on the parameter t which *varies continuously* in a certain interval (a, b); this *continuous parameter* t takes the place of the *discontinuous index* i. Corresponding to the formulae (1) and (1′), containing *sums with respect to the index* i, we get instead the formulae (2) and

(2′), containing *integrals* in which we may *take the parameter* t *as the variable of integration.*

The path followed in this particular example to pass from a problem with a finite number of unknowns to a problem in which the unknown is a function (a problem with an infinite number of unknowns) has the character of a general method. In many other cases in which this eventuality arises, it will in fact be sufficient to *replace a discontinuous index* i *by a continuous index, or parameter,* t*, and the sum with respect to this index* i *by the integral with respect to the variable of integration* t.

4. There are however problems in which the unknown is a function, but which have a different character. Thus, for instance, if we wish to determine a curve, $y = y(x)$, whose subtangent is constant ($= a$), we shall get the differential equation

$$a \frac{dy}{dx} = y,$$

and in this equation there is no integral with respect to x; all the infinite number of values of the unknown y corresponding to the infinite number of values of x do not simultaneously enter into consideration; we can say rather that this equation establishes a relation solely between *a value of* y *and the value it has at an infinitely near point,* determined by $\frac{dy}{dx}$. More generally, all the problems that give rise to an ordinary differential equation

$$f(x, y, y', \ldots y^{(n)}) = 0$$

which establishes a relation only between the values taken by y at a point x and at n infinitely near points, are of this type.

5. A substantially different problem, and related to those considered earlier, is that of determining a plane curve (of equation $y = y(x)$) passing through two given points in the plane (of coordinates x_0, $y_0 = y(x_0)$; x_1, $y_1 = y(x_1)$), such that when it rotates round a line fixed in the plane, e.g. the x-axis, it generates the surface of minimum area. In this case, the area A, the quantity to be made a minimum, is given by the expression

$$A = 2\pi \int_{x_0}^{x_1} y \sqrt{1 + y'^2}\, dx,$$

and therefore depends on all the values of y in the interval (x_0, x_1). In general, the problems treated in the calculus of variations all have this character.

§ 2.

6. From these particular examples we can reach a general definition which shall include the various cases so far considered. We shall therefore say that a quantity z is *a functional of the function* $x(t)$ *in the interval* (a, b) *when it depends on all the values taken by* $x(t)$ *when* t *varies in the interval* (a, b); or, alternatively, *when a law is given by which to every function* $x(t)$ *defined within* (a, b) *(the independent variable within a certain functional field) there can be made to correspond one and only one quantity* z, *perfectly determined, and we shall write* [1]

$$z = \mathrm{F}\left|\left[x\overset{b}{\underset{a}{(t)}}\right]\right|.$$

This definition of a *functional* recalls especially the ordinary general definition of a *function* given by Dirichlet.

If other variables $\alpha, \beta, \ldots$ also occur in the function $x(t)$ (the independent variable), we shall write

$$z = \mathrm{F}\left[x\overset{b}{\underset{a}{(t,}}\, \alpha, \beta, \ldots)\right] = z(\alpha, \beta, \ldots)$$

to indicate that the functional operator F, which when applied to the variable function x gives the quantity z, is to be applied considering x as a function of t alone, i.e. supposing that in x the quantities $\alpha, \beta, \ldots$ are constant; z will then be an ordinary

(1) The notation constantly used from my first paper of 1887 has always been

$$\mathrm{F}\left|\left[x\overset{b}{\underset{a}{(t)}}\right]\right|$$

but for the sake of simplicity, when there is no ambiguity, we can use one of the following:

$$\mathrm{F}\,|[x(t)]| \quad \text{or} \quad \mathrm{F}\left[x\overset{b}{\underset{a}{(t)}}\right]$$

or also

$$\mathrm{F}[x(t)].$$

This last notation has been used in this edition.

function of $\alpha, \beta, \ldots$, and it is to be noted that it will *not* be a function of t as x was.

The functional z may also contain certain parameters $\lambda, \mu, \ldots$

$$z = \mathrm{F}\left[x\left(\overset{b}{\underset{a}{t}}\right); \lambda, \mu, \ldots\right];$$

in this case the quantity z, for every system of values of the parameters, will be *a functional of* $x(t)$, in the sense stated above, while it will be *an ordinary function* $z(\lambda, \mu, \ldots)$ of the parameters $\lambda, \mu, \ldots$ when $x(t)$ is fixed. We can also say that in this case, corresponding to every function $x(t)$, defined within a certain field of variation, the functional operator F determines another function

$$z(\lambda, \mu, \ldots) = \mathrm{F}\left[x\left(\overset{b}{\underset{a}{t}}\right); \lambda, \mu, \ldots\right].\text{(1)}$$

7. This notion of a *functional* can be extended immediately to the quantities z which depend on all the values taken not by one but by several functions, e.g. by two, $x = x(t)$, $y = y(u)$, in the intervals (a, b) and (c, d) respectively:

$$z = \mathrm{F}\left[x\left(\overset{b}{\underset{a}{t}}\right), y\left(\overset{d}{\underset{c}{u}}\right)\right].$$

The consideration of these functionals, however, presents very little interest, as they can be immediately reduced to those *depending on a single variable function*. In fact, let $a'b' = a'c' + c'b'$ be an interval of variation of a parameter v, of amplitude equal to the sum of the two intervals (a, b) and (c, d); and let c' be a point within $(a'b')$ such that $c' - a' = b - a$, $b' - c' = d - c$. For every pair of functions $x(t)$, $y(u)$ defined in (a, b) and (c, d) respectively, let us construct a new function $X(v)$ defined as follows in the interval (a', b'): for $v = a'$, or within the interval (a', c'), $X(v) = x(a + v - a')$; for $v = b'$, or within the interval (c', b'), $X(v) = y(c + v - c')$; for $v = c'$ we shall give $X(c')$ one of the two values $x(b)$ or $y(c)$. Since

(1) We can omit the letters a, b and simply write

$$\mathrm{F}[x(t, \alpha, \beta \ldots)] \quad \text{and} \quad \mathrm{F}[x(t); \lambda, \mu \ldots]$$

when there is no ambiguity about the rôle of the variable t and of the other variables $\alpha, \beta, \ldots; \lambda, \mu, \ldots$.

the pair of functions $x(t)$, $y(u)$ completely determine the function $\bar{X}(v)$ (with the sole exception of $v = c'$), and *vice versa* $\bar{X}(v)$ determines the two functions $x(t)$, $y(u)$ (with the exception of $t = b$ for x, and $u = c$ for y), we can say, so long as the indeterminateness of the variable function at an isolated point does not affect the value of the functional, that the quantity $z = \mathrm{F}\left[\overset{b}{\underset{a}{x(t)}}, \overset{d}{\underset{c}{y(u)}}\right]$ depends on all the values taken by the *single function* $\bar{X}(v)$ in the interval (a', b'), i. e. that

$$z = \mathrm{F}\left[\overset{b}{\underset{a}{x(t)}}, \overset{d}{\underset{c}{y(u)}}\right] = \mathrm{G}\left[\overset{b'}{\underset{a'}{\bar{X}(v)}}\right].$$

8. Another generalisation of the concept of a *functional* is obtained by considering those quantities z *which depend on all the values taken by one or more functions* $\varphi_1(x_1, x_2, \ldots x_n)$, $\varphi_2(y_1, y_2, \ldots y_m), \ldots$ *of several variables defined within certain fields* $C_1, C_2, \ldots$ *respectively* (1):

$$z = \mathrm{F}[\varphi_1(x_1, x_2, \ldots x_n), \varphi_2(y_1, y_2, \ldots y_m), \ldots].$$

9. This concept of a functional also includes *functions of lines*, and, more generally, *of hyperspaces* (2). We say, in fact, that a quantity z is a function of a variable hyperspace S_r, contained in another S_n $(n > r)$ and parametrically defined by the equations

$$x_i = \varphi_i(t_1, t_2, \ldots t_r), \quad i = 1, 2, \ldots n,$$

when to every such S_r there corresponds one definite value of the quantity $z = \mathrm{F}[S_r]$, which thus comes to depend on all the values taken by the n functions φ_i at the points of the field C (of r dimensions) for which they are defined. It is to be noted, however, that this function z, a function of the hyperspace S_r, *is not* a general functional of the n functions φ_i; in fact, if the parameters t_k are changed into others u_l by means of the transformation

$$t_k = t_k(u_1, u_2, \ldots u_r), \quad k = 1, 2, \ldots r,$$

$$\frac{d(t_1, t_2, \ldots t_r)}{d(u_1, u_2, \ldots u_r)} \neq 0,$$

(1) Cf. FABRI, (19) and (20). (The numbers in parentheses in the footnotes refer to the bibliography at the end of each chapter.)

(2) Cf. VOLTERRA, (83) and (84).

the coordinates x_i will no longer be given by the functions φ of the t_k's but by other functions ψ of the u_i's:

$$x_i = \psi_i(u_1, u_2, \ldots u_r) = \varphi_i(t_1, t_2, \ldots t_r),$$

but the quantity z which depends *only on the form* of the hyper-space S_r (or manifold, as it is often called), and not on the mode of representation, will not have changed; i.e. z is a functional of the functions φ_i which does not vary when the φ_i's are replaced by other functions ψ_i obtained from them by a change of parameters.

§ 3. *Functional Fields and Abstract Aggregates.*

10. A functional $\mathrm{F}[x(t)]$ of the function $x(t)$ will be defined in general only when $x(t)$ varies within a certain determinate field of functions. Thus, for instance, the functional

$$\mathrm{F}[x(t)] = \int_a^b x(t)\,dt$$

is defined only for the field of the functions $x(t)$ that are *integrable* in the interval (a, b); the functional

$$\mathrm{F}[x(t)] = \int_a^b f\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \ldots \frac{d^nx}{dt^n}\right) dt$$

is defined only for those functions $x(t)$ for which derivatives exist up to the nth order, and for which

$$\varphi(t) = f(t, x(t), x'(t) \ldots x^{(n)}(t))$$

is integrable in the interval (a, b); and so on.

The study of *functional fields*, i.e. of those aggregates whose elements are *functions*, is therefore of the greatest interest for an accurate understanding of the concept of a functional. The widest possible functional field, that consisting of all possible functions defined in an interval (a, b), obviously constitutes a space of an infinite non-enumerable number of dimensions (i.e. the number of its dimensions is the same as the power of the continuum), since each of its elements $x(t)$ is defined only when the ∞ values taken by x corresponding to the ∞ values of t in the interval (a, b) are known. Very few functionals, however, are defined for the whole of this field. A much more restricted field, but of notable interest, is that of the *analytical*