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Equivariant Stable Homotopy Theory

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Preface

Our primary purpose in this volume is to establish the foundations of equivariant stable homotopy theory. To this end, we shall construct a stable homotopy category of G-spectra enjoying all of the good properties one might reasonably expect, where G is a compact Lie group. We shall use this category to study equivariant duality, equivariant transfer, the Burnside ring, and related topics in equivariant homology and cohomology theory.

This volume originated as a sequel to the volume "H_∞ ring spectra and their applications" in this series [20]. However, our goals changed as work progressed, and most of this volume is now wholly independent of [20]. In fact, we have two essentially disjoint motives for undertaking this study. On the one hand, we are interested in equivariant homotopy theory, the algebraic topology of spaces with group actions, as a fascinating subject of study in its own right. On the other hand, we are interested in equivariant homotopy theory as a tool for obtaining useful information in classical nonequivariant homotopy theory. This division of motivation is reflected in a division of material into two halves. The first half, chapters I-V, is primarily addressed to the reader interested in equivariant theory. The second half, chapters VI-X, is primarily addressed to the reader interested in nonequivariant applications. It gives the construction and analysis of extended powers of spectra that served as the starting point for [20]. It also gives a systematic study of generalized Thom spectra. With a very few minor and peripheral exceptions, the second half depends only on chapter I and the first four sections of chapter II from the first half. The reader is referred to [105] for a very brief guided tour of some of the high spots of the second half.

Chapter I gives the more elementary features of the equivariant stable category, such as the theory of G-CW spectra and a desuspension theorem allowing for desuspension of G-spectra by all representations of G in the given ambient "indexing universe". Chapter II gives the construction of smash products and function G-spectra. It also gives various change of universe and change of groups theorems. Chapter III gives a reasonably comprehensive treatment of equivariant duality theory, including Spanier-Whitehead, Atiyah, and Poincaré duality. Chapter IV studies transfer maps associated to equivariant bundles, with emphasis on their calculational behavior in cohomology. Chapter V studies the Burnside ring and its role in equivariant stable homotopy theory. It includes various related splitting theorems in equivariant homology and cohomology theory.

Although we have encountered quite a few new phenomena, our main goals in the first half have been the equivariant generalization of known nonequivariant results and the generalization and sharpening of known equivariant results. We therefore owe ideas and material to numerous other mathematicians. Our general debt to the work of Boardman [13,14] and Adams [1] in nonequivariant stable homotopy theory will be apparent throughout. The idea for a key proof in chapter I is due to Hauschild. The main change of groups theorems in chapter II are generalizations of results of Wirthmuller [144] and Adams [3], and the study of subquotient cohomology theories in II§9 is based on ideas of Costenoble.

Our debts are particularly large in chapters III, IV, and V. Our treatment of duality is largely based on ideas in the lovely paper [47] of Dold and Puppe and on (nonequivariant) details in the papers [63,64,65] of their students Henn and Hommel; equivariant duality was first studied by Wirthmuller [145]. Our treatment of transfer naturally owes much to the basic work of Becker and Gottlieb [10,11] and Dold [46], and transfer was first studied equivariantly by Nishida [117] and Waner [141]. Our IV§6 is a reexposition and equivariant generalization of Feshbach's work [53,54] on the double coset formula, and he cleared away our confusion on several points. While our initial definitions are a bit different, a good deal of chapter V is a reexposition in our context of tom Dieck's pioneering work [38-44] on the Burnside ring of a compact Lie group and the splitting of equivariant stable homotopy. This chapter also includes new proofs and generalizations of results of Araki [4].

A word about our level of generality is in order. We don't restrict to finite groups since, for the most part, relatively little simplification would result. We don't generalize beyond compact Lie groups because we believe that only the most formal and elementary portions of equivariant stable homotopy theory would then be available. The point is that, in all of our work, the depth and interest lies in the interplay between homotopy theory and representation theory. Technically, part of the point is that the cohomology theories represented by our G-spectra are RO(G)-graded and not just Z-graded. This implies huge amounts of algebraic structure which would be invisible in more formal and less specific homotopical contexts.

While a great deal of our work concerns equivariant cohomology theory, we have not given a systematic study here. Lewis, McClure, and I have used the equivariant stable category to invent "ordinary RO(G)-graded cohomology theories" [88], and the three of us and Waner are preparing a more thorough account [90]. (Hauschild, Waner, and I are also preparing an account of equivariant infinite loop space theory, which is less directly impinged upon by this volume.)

Chapters VI-VIII establish rigorous foundations for the earlier volume [20], which we shall refer to as $[H_\infty]$ here. That volume presupposed extended powers $D_jE = E\Sigma_j \ltimes_{\Sigma_j} E^{(j)}$ of spectra with various good properties. There E was a nonequivariant spectrum, but our construction will apply equally well to G-spectra E for any compact Lie group G.

In fact, extended powers result by specialization of what is probably the most fundamental construction in equivariant stable homotopy theory, namely the twisted half-smash product $X \ltimes E$ of a G-space X and a G-spectrum E. (The "twisting" is encoded by changes of universe continuously parametrized by X.) This construction is presented in chapter VI, although various special cases will have been encountered earlier.

We develop a theory of "operad ring G-spectra" and in particular construct free operad ring G-spectra in chapter VII. When G is finite, special cases give approximations of iterated loop G-spaces $\Omega^V \Sigma^V X$, and we obtain equivariant generalizations of Snaith's stable splittings of spaces $\Omega^n \Sigma^n X$.

We prove some homological properties of nonequivariant extended powers that were used in [H_∞] in chapter VIII.

Chapters IX and X give a careful treatment of the Thom spectra associated to maps into stable classifying spaces. These have been used extensively in recent years, and many people have felt a need for a detailed foundational study. In chapter IX, we work nonequivariantly and concentrate on technical problems arising in the context of spherical fibrations (as opposed to vector bundles). In chapter X, we work equivariantly but restrict ourselves to the context of G-vector bundles. There result two specializations to the context of nonequivariant vector bundles, the second of which is the more useful since it deals naturally with elements of $KO(X)$ of arbitrary virtual dimension.

We must again acknowledge our debts to other mathematicians. We owe various details to Bruner, Elmendorf, and McClure. The paper of Tsuchiya [138] gave an early first approximation of our definitions of extended powers and H_∞ ring spectra. As explained at the end of VII§2, Robinson's A_∞ ring spectra [124] fit naturally into our context. The proof of the splitting theorem in VII§5 is that taught us by Ralph Cohen [34]. We owe the formulations of some of our results on Thom spectra to Boardman [12] and of others to Mahowald [93], whose work led to our detailed study of these objects.

Each chapter of this book has an introduction summarizing its main ideas and results. There is a preamble comparing our approach to the nonequivariant stable category with earlier ones, and there is an appendix giving some of the more esoteric proofs. References are generally by name (Lemma 5.4) when to results in the same chapter and by number (II.5.4) when to results in other chapters.

Finally, I should say a word about the genesis and authorship of this volume. Chapter VIII and part of chapter VI are based on Steinberger's thesis [133], and chapter VII started from unpublished 1978 notes of his. Chapter IX and the Appendix are based on Lewis' thesis [83], and the definition and axiomatization of the transfer in chapter IV are simplifications of his work in [85]. Chapter V incorporates material from unpublished 1980 notes of McClure. All of the rest of the equivariant material is later joint work of Lewis and myself.

The authorship of the several chapters is as follows.

Chapters I through IV: Lewis and May

Chapter V: Lewis, May, and McClure

Chapters VI and VII: Lewis, May, and Steinberger

Chapter VIII: May and Steinberger

Chapter IX: Lewis

Chapter X: Lewis and May

The Appendix and the indices were prepared by Lewis.

J. Peter May
June 20, 1985

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Contents

Preamble: A polemical introduction to the stable category

by J. P. May

Nonequivariantly, the virtues of having a good stable category are by now well understood. In such a category, the basic formal properties of homology and cohomology theories become trivialities. Many arguments that could be carried out ad hoc without a stable category became much cleaner with one. More important, many common arguments simply cannot be made rigorous without use of such a category.

Equivariantly, it is even more important to have a good stable category. Much basic equivariant algebra only arises in a fully stable context. For example, one already has that $[S^n, S^n]$ is Z for $n \geq 1$ and that $[S^n, X]$ is a Z-module for $n \geq 2$. The equivariant analog of Z is the Burnside ring $A(G)$, and, unless G is finite, there need be no representation V of G large enough that $[S^V, S^V]_G$ is $A(G)$ or that $[S^V, X]_G$ is an $A(G)$-module. Even when G is finite, and regardless of connectivity hypotheses, none of the ordinary homotopy groups $[S^n, X]_G$ of a G-space X need be $A(G)$-modules, whereas all of the homotopy groups of G-spectra are $A(G)$-modules. Much more evidence will appear as we proceed.

Our construction of the equivariant stable category is a generalization of my construction of the nonequivariant stable category. Since the latter is less familiar than the earlier constructions of Boardman and Adams, a comparison of the various approaches may be helpful to the informed reader. I can't resist quoting from Boardman's 1969 Historical Introduction [13, p.1]. "This introduction... is addressed without compromise to the experts. (The novice has the advantage of not having been misled by previous theories.)"

Boardman continues "In this advertisement we compare our category $\underline{S}$ of CW-spectra, <u>or rather its homotopy category</u> $\underline{S}_h$, with various competing products. We find the comparison quite conclusive, because the more good properties the competitors have, the closer they are to $\underline{S}_h$". All experts now accept this absolutely. Boardman's category $\underline{S}_h$ is definitively the right one, and any good stable category must be equivalent to it. It does not follow, however, that his category $\underline{S}$, <u>before passage to homotopy</u>, is the right one, and we are convinced that it is not.

Boardman's construction of his category $\underline{S}$ proceeds as follows. He begins with the category $\mathcal{F}$ of finite CW complexes. He constructs the category $\mathcal{F}_s$ of finite CW spectra by a purely categorical procedure of stabilization with respect to the suspension functor. He then constructs $\underline{S}$ from $\mathcal{F}_s$ by a much deeper purely categorical procedure of adjoining colimits of all directed diagrams of finite CW spectra and inclusions. The intuition is that, however CW spectra are defined, they ought to be the colimits of their finite subspectra, and finite CW spectra ought to be desuspensions of finite CW complexes. An advantage of this approach is that one can obtain conceptual proofs of theorems about $\underline{S}_h$ almost automatically by feeding information about finite CW complexes into a categorical black box. A disadvantage, to paraphrase Adams [1, p.123], is that the construction is inaccessible to those without a specialized knowledge of category theory.

In fact, in his Historical Introduction [13, p.4], Boardman pointed out an alternative description of a category equivalent to his, and he gave details of the comparison in [14,§10]. Define a CW prespectrum D to be a sequence of CW complexes D_n and cellular inclusions $\Sigma D_n \to D_{n+1}$. Define a map $f: D \to D'$ to be a family of based maps $f_n: D_n \to D'_n$ strictly compatible with the given inclusions. Let $\mathcal{P}(D,D')$ denote the set of maps $D \to D'$. Say that a subprespectrum C is dense (or cofinal) in D if for any finite subcomplex X of D_n, $\Sigma^k X$ is contained in C_{n+k} for some k. Then [14,10.3] implies that $\underline{S}$ is equivalent to the category of CW-prespectra D and morphisms

$$\underline{S}(D,D') = \coprod_{(C,C')} \mathcal{P}(C,C')/(\approx) = \coprod_{C} \mathcal{P}(C,D')/(\approx),$$

where C and C' run through the dense subprespectra of D and D' and where $f: C \to C'$ is equivalent to $\bar{f}: \bar{C} \to \bar{C}'$ if and only if the composites

$$C \cap \bar{C} \xrightarrow{f} C' \subset D' \quad \text{and} \quad C \cap \bar{C} \xrightarrow{\bar{f}} \bar{C}' \subset D'$$

are equal.

Adams [1] turned this result into a definition and proceeded from there. (He called a map $D \to D'$ a "function", an element of $\underline{S}(D,D')$ a "map", and a homotopy class of "maps" a "morphism"; he also called a CW prespectrum a CW spectrum.) A similarly explicit starting point was taken by Puppe [122]. An advantage of this approach (to some people!) is that it is blessedly free of category theory. A disadvantage is that many proofs, for example in the theory of smash products, become unpleasantly ad hoc. To quote Boardman again [14,p.52], "The complication will show why we do not adopt this as definition".

It seems reasonable to seek an alternative construction with all of the advantages and none of the disadvantages. Staring at the definition, we see that $\underline{S}$ is constructed from the category of CW prespectra and maps by applying a kind of limit procedure to morphisms while leaving the objects strictly alone. This is the meaning of Adams' slogan [1, p.142] "cells now - maps later".

From our point of view, this is precisely analogous to developing sheaf theory without ever introducing sheaves or sheafification. There is a perfectly sensible way to "spectrify" so as to force elements of $\underline{S}(D,D')$ to be on the same concrete level as maps $D \to D'$. Define a spectrum E to be a sequence of based spaces E_n

and based homeomorphisms $E_n \to \Omega E_{n+1}$. Define a map $f: E \to E'$ to be a sequence of based maps $f_n: E_n \to E'_n$ strictly compatible with the homeomorphisms. Let $\mathcal{S}(E,E')$ denote the set of maps $E \to E'$. Define the spectrum LD associated to a CW prespectrum D by

$$(LD)_n = \operatorname*{colim}_{k \geq 0} \Omega^k D_{n+k} ,$$

where the colimit is taken with respect to iterated loops on adjoint inclusions $D_i \to \Omega D_{i+1}$. Since Ω commutes with colimits, there are evident homeomorphisms $\Omega(LD)_{n+1} \cong (LD)_n$. One finds by a laborious inspection of definitions that

$$\underline{S}(D,D') \cong \mathcal{S}(LD,LD').$$

Of course, only the expert seeking concordance with earlier definitions need worry about the verification: we shall take the category $\mathcal{S}$ as our starting point.

Obviously the spaces $(LD)_n$ are no longer CW complexes (although they do have the homotopy types of CW complexes), and we have imposed no CW requirement in our definition of spectra. It should now be apparent that, despite their rigid structure, spectra are considerably more general objects than CW prespectra. Working in a stable world in which the only spectra are those coming from CW prespectra is precisely analogous to working in an unstable world in which the only spaces are the CW complexes. Just as any space has the weak homotopy type of a CW complex, so any spectrum has the weak homotopy type of one coming from a CW prespectrum. (Verification of the last assertion requires only elementary constructions with space level CW approximations and mapping cylinders and was already implicit in my 1969 paper [95].)

The extra generality allowed by our definition of spectra is vital to our theory. Throughout this volume, we shall be making concrete spectrum-level constructions which simply don't exist in the world of CW prespectra.

Dropping CW conditions in our definition of spectra clearly entails dropping CW conditions in our definition of prespectra. For us, a prespectrum is a sequence of spaces D_n and maps $\Sigma D_n \to D_{n+1}$. Maps of prespectra are defined as above. By our analogy with sheaf theory, we are morally bound to extend the construction L above to a spectrification functor $L: \mathcal{P} \to \mathcal{S}$ left adjoint to the obvious forgetful functor from spectra to prespectra. When the adjoints $D_i \to \Omega D_{i+1}$ are not inclusions, LD is slightly mysterious and its construction is due to Lewis [83], who will give details in the Appendix. Starting from D one constructs a prespectrum D' and map $D \to D'$ by letting D'_i be the image of D_i in ΩD_{i+1}. The resulting maps $D'_i \to \Omega D'_{i+1}$ are a bit closer to being inclusions. Iterating this construction (transfinitely many times!) one arrives at a prespectrum $\bar{D}$ and map $D \to \bar{D}$ such that the maps $\bar{D}_i \to \Omega\bar{D}_{i+1}$ are inclusions. One defines LD by applying the elementary colimit construction above to $\bar{D}$; one has a composite natural map $D \to LD$ which is the unit of the adjunction. Actually, the explicit construction is of little importance. The essential point is that, by standard and elementary category theory, L obviously exists and is obviously unique.

We now see that our category of spectra has arbitrary limits and colimits. Indeed, the category of prespectra obviously has all limits and colimits since these can be constructed spacewise. All such limit constructions preserve spectra. Colimit constructions do not, and colimits of spectra are obtained by applying L to prespectrum level colimits. Thus, and this will take some getting used to by the experts, limits for us are simpler constructions than colimits. In fact, right adjoints in general are simpler constructions than left adjoints. For example, it is trivial for us to write down explicit products and pullbacks of spectra and explicit function spectra. These don't exist in the world of CW-prespectra.

Moreover, we shall often prove non-obvious facts about left adjoints simply by quoting obvious facts about right adjoints. This might seem altogether too categorical, but in fact the opposite procedure has long been standard practice. Function spectra in $\underline{S}h$ (not $\underline{S}$!) are usually obtained by quoting Brown's representability theorem - something at least as sophisticated as any category theory we use - and then proving things about these right adjoints by quoting known facts about the left adjoint smash product functors.

Of course, one does want a theory of CW spectra, but there is no longer the slightest reason to retreat to the space or prespectrum level to develop it. We have a good category of spectra, with cones, pushouts, and colimits. To define CW spectra, we need only define sphere spectra and proceed exactly as on the space level, using spectrum level attaching maps. The resulting CW spectra are all homotopy equivalent to spectra coming from CW prespectra; conversely, any spectrum coming from a CW-prespectrum is homotopy equivalent to a CW-spectrum. A CW-spectrum is the colimit of its finite subspectra, and a finite CW spectrum is a desuspension of a finite CW complex (that is, of its associated suspension spectrum). Our stable category is constructed from the homotopy category of spectra by adjoining formal inverses to the weak equivalences. It is equivalent to the homotopy category of CW spectra. By the discussion above, it is also equivalent to Boardman's category $\underline{S}h$. Without exception, everything in the literature done in Boardman's category can just as well be interpreted as having been done in our category.

In one respect I have lied a bit above. We don't usually index prespectra and spectra on integers but rather on finite dimensional inner product spaces. When one thinks of D_n, one thinks of S^n and thus of $\mathbb{R}^n$. Implicitly, one is thinking of $\mathbf{R}^\infty$ with its standard basis. Even nonequivariantly, a coordinate free approach has considerable advantages. For example, it leads to an extremely simple conceptual treatment of smash products and, as Quinn, Ray, and I realized in 1973 [99], it is vital to the theory of structured ring spectra. In the equivariant context, one must deal with all representations, and coordinate-free indexing is obviously called for, as tom Dieck realized even earlier.

Modulo the appropriate indexing, virtually everything said above about my approach to the nonequivariant stable category applies verbatim in the equivariant context. The few exceptions are relevant to possible generalizations of the earlier constructions. Our G-CW spectra are built up from "G-sphere spectra" $G/H^+ \wedge S^n$, and any G-CW spectrum is the colimit of its finite subspectra. However, it is not true that a finite G-CW spectrum is isomorphic (as opposed to homotopy equivalent) to a desuspension of a finite G-CW complex unless one redefines the latter by allowing G-spheres $G/H^+ \wedge S^V$ associated to G-representations V as domains of attaching maps. This loses the cellular approximation theorem and would presumably cause difficulties in a Boardman style approach to the G-stable category. Since G-spheres S^V are not known to have canonical G-CW structures, the appropriate notion of a G-CW prespectrum is not immediately apparent. We shall give a definition which is related to our notions exactly as described above in the nonequivariant case. However, a full treatment, including smash and twisted half-smash products, would be inordinately lengthy and complicated. In any case, right adjoints, such as fixed point functors, are even more important equivariantly than nonequivariantly, and a treatment lacking such constructions on the spectrum level would be most unnatural.

I should say that there is also a semisimplicial construction of the stable category due to Kan (and Whitehead) [69,70] and elaborated by Burghelea and Deleanu [21]. Except perhaps when G is finite, it is ill-adapted to equivariant generalization, and it is also inconvenient for the study of structured ring spectra.

One last point addressed to the experts. We shall not introduce graded morphisms here. Regardless of what approach one takes, graded morphisms are really nothing more than a notational device. The device can aid in keeping track of the signs which arise in the study of cohomology theories, but it can in principle add nothing substantive to the mathematics. In the equivariant context, the grading would have to be over $RO(G)$ and its introduction would serve only to obscure the exposition.

I. THE EQUIVARIANT STABLE CATEGORY

by L. G. Lewis, Jr. and J. P. May

We gave a preliminary definition of spectra in [H_∞,I§1] as sequences of spaces E_i and homeomorphisms $E_i \cong \Omega E_{i+1}$ This "coordinatized" notion is wholly inadequate for the study of either structured ring spectra or equivariant stable homotopy theory. While our main concern in [H_∞] was with the first of these subjects, we are here most interested in the second. Because of the role played by permutation groups in the construction of extended powers, we need a fair amount of equivariant stable homotopy theory to make rigorous the constructions used in [H_∞] in any case. While this motivates us only as far as the study of G-spectra for finite groups G, it turns out that a complete treatment of the foundations of equivariant stable homotopy theory in the proper generality of compact Lie groups is obtainable with very little extra effort.

Thus, throughout the first five chapters, G will be a compact Lie group. Considerations special to permutation groups will not appear until late in Chapter VI. We shall construct a good "stable category" of G-spectra, where "stability" is to be interpreted as allowing for desuspensions by arbitrary finite dimensional real representations of G.

After some recollections about equivariant homotopy theory in section 1, we begin work in section 2 by setting up categories of G-prespectra and G-spectra and discussing various adjoint functors relating them to each other and to G-spaces. We give both coordinate-free and coordinatized notions of G-spectra and show that these give rise to equivalent categories. The freedom to pass back and forth between the two is vital to the theory.

In section 3, we introduce the smash products of G-spaces and G-spectra and the associated adjoint function G-spectra. The analogous constructions between G-spectra and G-spectra are deeper and will be presented in the next chapter. The simpler constructions suffice for development of most of the basic machinery of homotopy theory. We also introduce orbit spectra and fixed point spectra.

In section 4, we introduce left adjoints "$\Lambda^Z\Sigma^\infty$" to the $Z^{\underline{th}}$ space functors from G-spectra to G-spaces, where Z runs through the relevant indexing representations. These functors play a basic role in the passage back and forth between space level and spectrum level information. In particular, we use instances of these functors to define sphere G-spectra $S^n_H = G/H^+ \wedge S^n$ for integers n and closed subgroups H of G. (The term subgroup shall mean closed subgroup henceforward.) We then define homotopy groups in terms of these sphere spectra and define weak equivalences in terms of the resulting homotopy groups.

In section 5, we introduce G-CW spectra. We follow a general approach, developed in more detail in [107], in which such basic results as the cellular approximation theorem, Whitehead's theorem, and the Brown representability theorem are almost formal trivialities. With these results, we see that arbitrary G-spectra are weakly equivalent to G-CW spectra. This allows us to construct the equivariant stable category by formally inverting the weak equivalences in the homotopy category of G-spectra. The result is equivalent to the homotopy category of G-CW spectra, both points of view being essential to a fully satisfactory theory.

In section 6, we summarize the basic properties of the stable category, the most important being the equivariant desuspension theorem. This asserts that Ω^V and Σ^V are adjoint self equivalences of the stable category for any G-representation V. We then indicate briefly how to define represented equivariant cohomology theories. The natural representing objects for cohomology theories on G-spaces are cruder than our G-spectra, and we make use of an elementary iterated mapping cylinder construction on the G-prespectrum level to obtain a precise comparison. This cylinder construction has various other applications. On the G-spectrum level, it turns out to admit a simple description as a telescope, and this leads to a $\lim^1$ exact sequence for the calculation of the cohomology of G-spectra in terms of the cohomologies of their component G-spaces.

In section 7, we give a number of deferred proofs based on use of a shift desuspension functor Λ^Z (in terms of which the earlier functor $\Lambda^Z\Sigma^\infty$ is a composite). In particular, we prove the equivariant desuspension theorem. This depends on the assertion that a map of G-spectra is a weak equivalence if and only if each of its component maps of G-spaces is a weak equivalence. This is the only place in the chapter where equivariance plays a really major role in a proof, the corresponding nonequivariant assertion being utterly trivial. We learned the basic line of argument from Henning Hauschild, although the full strength of the result depends on our definitional framework.

In section 8, we give various results concerning special kinds of G-prespectra and G-spectra. In particular, we show that, up to homotopy, our G-CW spectra come from G-CW prespectra of a suitably naive sort.

We shall defer some details of proof to the Appendix, on the grounds that the arguments in question would unduly interrupt the exposition.

We remind the reader that G is always a compact Lie group and that everything in sight is G-equivariant. Once the definitions are in place, we generally omit the G from the notations, writing spectra for G-spectra, etc.

§1. Recollections about equivariant homotopy theory

Since the basic definitions of equivariant homotopy theory are not as widely known as they ought to be, we give a brief summary before turning to G-spectra.

Let $G\mathcal{U}$ denote the category of compactly generated weak Hausdorff left G-spaces. (The weak Hausdorff condition asserts that the diagonal is closed in the compactly generated product; it is the most natural separation axiom to adopt for compactly generated spaces; see [111,83].) Let $G\mathcal{J}$ denote the category of based left G-spaces, with G acting trivially on basepoints. These categories are closed under such standard constructions as (compactly generated) products and function spaces, G acting diagonally on products and by conjugation on function spaces. The usual adjunction homeomorphisms hold and are G-equivariant. For unbased G-spaces X,Y, and Z we have

$$Z^{X \times Y} \cong (Z^Y)^X.$$

We write F(X,Y) for the function space of based maps $X \to Y$; for based G-spaces X,Y, and Z we have

$$F(X \wedge Y, Z) \cong F(X, F(Y,Z)).$$

The usual machinery of homotopy theory is available in the categories $G\mathcal{U}$ and $G\mathcal{J}$, homotopies being maps $X \times I \to Y$ in $G\mathcal{U}$ or $X \wedge I^+ \to Y$ in $G\mathcal{J}$. Cofibrations are defined in either category by the homotopy extension property (and are automatically closed inclusions). Similarly, fibrations are defined by the covering homotopy property. We shall use standard results without further comment; see e.g. [17,143, or 107]. Write $hG\mathcal{U}$ and $hG\mathcal{J}$ for the respective homotopy categories and write $\pi(X,Y)_G$ for the set of homotopy classes of based maps $X \to Y$.

Turning to homotopy groups, let $S^n = I^n/\partial I^n$ with trivial G-action and with $S^0 = \{0,1\}$. For $H \subset G$, define a G-space S^n_H by

$$S^n_H = (G/H)^+ \wedge S^n = (G/H) \times S^n/(G/H) \times \{*\}.$$

We think of S^n_H as a generalized sphere. It is well understood that the homotopy groups of a based G-space X should be taken to be the collection of homotopy groups

$$\pi^H_n X = \pi_n X^H \cong \pi(S^n, X)_H \cong \pi(S^n_H, X)_G .$$

Here the last isomorphism comes from the adjunction