Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1054

Vidar Thomée

Galerkin Finite Element Methods for Parabolic Problems

Springer-Verlag
Berlin Heidelberg New York Tokyo

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1054

Vidar Thomée

Galerkin Finite Element Methods for Parabolic Problems

Springer-Verlag
Berlin Heidelberg New York Tokyo 1984

Author

Vidar Thomée
Department of Mathematics
Chalmers University of Technology and the University of Göteborg
S-412 96 – Göteborg, Sweden

AMS Subject Classifications (1980): 65 N 30

ISBN 3-540-12911-1 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-12911-1 Springer-Verlag New York Heidelberg Berlin Tokyo

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

Lecture Notes in Mathematics

For information about Vols. 1–844, please contact your bookseller or Springer-Verlag.

Vol. 845: A. Tannenbaum, Invariance and System Theory: Algebraic and Geometric Aspects. X, 161 pages. 1981.

Vol. 846: Ordinary and Partial Differential Equations, Proceedings. Edited by W. N. Everitt and B. D. Sleeman. XIV, 384 pages. 1981.

Vol. 847: U. Koschorke, Vector Fields and Other Vector Bundle Morphisms – A Singularity Approach. IV, 304 pages. 1981.

Vol. 848: Algebra, Carbondale 1980. Proceedings. Ed. by R. K. Amayo. VI, 298 pages. 1981.

Vol. 849: P. Major, Multiple Wiener-Itô Integrals. VII, 127 pages. 1981.

Vol. 850: Séminaire de Probabilités XV. 1979/80. Avec table générale des exposés de 1966/67 à 1978/79. Edited by J. Azéma and M. Yor. IV, 704 pages. 1981.

Vol. 851: Stochastic Integrals. Proceedings, 1980. Edited by D. Williams. IX, 540 pages. 1981.

Vol. 852: L. Schwartz, Geometry and Probability in Banach Spaces. X, 101 pages. 1981.

Vol. 853: N. Boboc, G. Bucur, A. Cornea, Order and Convexity in Potential Theory: H-Cones. IV, 286 pages. 1981.

Vol. 854: Algebraic K-Theory. Evanston 1980. Proceedings. Edited by E. M. Friedlander and M. R. Stein. V, 517 pages. 1981.

Vol. 855: Semigroups. Proceedings 1978. Edited by H. Jürgensen, M. Petrich and H. J. Weinert. V, 221 pages. 1981.

Vol. 856: R. Lascar, Propagation des Singularités des Solutions d'Equations Pseudo-Différentielles à Caractéristiques de Multiplicités Variables. VIII, 237 pages. 1981.

Vol. 857: M. Miyanishi. Non-complete Algebraic Surfaces. XVIII, 244 pages. 1981.

Vol. 858: E. A. Coddington, H. S. V. de Snoo: Regular Boundary Value Problems Associated with Pairs of Ordinary Differential Expressions. V, 225 pages. 1981.

Vol. 859: Logic Year 1979–80. Proceedings. Edited by M. Lerman, J. Schmerl and R. Soare. VIII, 326 pages. 1981.

Vol. 860: Probability in Banach Spaces III. Proceedings, 1980. Edited by A. Beck. VI, 329 pages. 1981.

Vol. 861: Analytical Methods in Probability Theory. Proceedings 1980. Edited by D. Dugué, E. Lukacs, V. K. [illegible], 183 pages. 1981.

Vol. 862: Algebraic Geometry. Proceedings 1980. Edited by A. Libgober and P. Wagreich. V, 281 pages. 1981.

Vol. 863: Processus Aléatoires à Deux Indices. Proceedings, 1980. Edited by H. Korezlioglu, G. Mazziotto and J. Szpirglas. V, 274 pages. 1981.

Vol. 864: Complex Analysis and Spectral Theory. Proceedings, 1979/80. Edited by V. P. Havin and N. K. Nikol'skii, VI, 480 pages. 1981.

Vol. 865: R. W. Bruggeman, Fourier Coefficients of Automorphic Forms. III, 201 pages. 1981.

Vol. 866: J.-M. Bismut, Mécanique Aléatoire. XVI, 563 pages. 1981.

Vol. 867: Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin. Proceedings, 1980. Edited by M.-P. Malliavin. V, 476 pages. 1981.

Vol. 868: Surfaces Algébriques. Proceedings 1976–78. Edited by J. Giraud, L. Illusie et M. Raynaud. V, 314 pages. 1981.

Vol. 869: A. V. Zelevinsky, Representations of Finite Classical Groups. IV, 184 pages. 1981.

Vol. 870: Shape Theory and Geometric Topology. Proceedings, 1981. Edited by S. Mardešić and J. Segal. V, 265 pages. 1981.

Vol. 871: Continuous Lattices. Proceedings, 1979. Edited by B. Banaschewski and R.-E. Hoffmann. X, 413 pages. 1981.

Vol. 872: Set Theory and Model Theory. Proceedings, 1979. Edited by R. B. Jensen and A. Prestel. V, 174 pages. 1981.

Vol. 873: Constructive Mathematics, Proceedings, 1980. Edited by F. Richman. VII, 347 pages. 1981.

Vol. 874: Abelian Group Theory. Proceedings, 1981. Edited by R. Göbel and E. Walker. XXI, 447 pages. 1981.

Vol. 875: H. Zieschang, Finite Groups of Mapping Classes of Surfaces. VIII, 340 pages. 1981.

Vol. 876: J. P. Bickel, N. El Karoui and M. Yor. Ecole d'Eté de Probabilités de Saint-Flour IX – 1979. Edited by P. L. Hennequin. XI, 280 pages. 1981.

Vol. 877: J. Erven, B.-J. Falkowski, Low Order Cohomology and Applications. VI, 126 pages. 1981.

Vol. 878: Numerical Solution of Nonlinear Equations. Proceedings, 1980. Edited by E. L. Allgower, K. Glashoff, and H.-O. Peitgen. XIV, 440 pages. 1981.

Vol. 879: V. V. Sazonov, Normal Approximation – Some Recent Advances. VII, 105 pages. 1981.

Vol. 880: Non Commutative Harmonic Analysis and Lie Groups. Proceedings, 1980. Edited by J. Carmona and M. Vergne. IV, 553 pages. 1981.

Vol. 881: R. Lutz, M. Goze, Nonstandard Analysis. XIV, 261 pages. 1981.

Vol. 882: Integral Representations and Applications. Proceedings, 1980. Edited by K. Roggenkamp. XII, 479 pages. 1981.

Vol. 883: Cylindric Set Algebras. By L. Henkin, J. D. Monk, A. Tarski, H. Andréka, and I. Németi. VII, 323 pages. 1981.

Vol. 884: Combinatorial Mathematics VIII. Proceedings, 1980. Edited by K. L. McAvaney. XIII, 359 pages. 1981.

Vol. 885: Combinatorics and Graph Theory. Edited by S. B. Rao. Proceedings, 1980. VII, 500 pages. 1981.

Vol. 886: Fixed Point Theory. Proceedings, 1980. Edited by E. Fadell and G. Fournier. XII, 511 pages. 1981.

Vol. 887: F. van Oystaeyen, A. Verschoren, Non-commutative Algebraic Geometry, VI, 404 pages. 1981.

Vol. 888: Padé Approximation and its Applications. Proceedings, 1980. Edited by M. G. de Bruin and H. van Rossum. VI, 383 pages. 1981.

Vol. 889: J. Bourgain, New Classes of $\mathcal{L}^p$-Spaces. V, 143 pages. 1981.

Vol. 890: Model Theory and Arithmetic. Proceedings, 1979/80. Edited by C. Berline, K. McAloon, and J.-P. Ressayre. VI, 306 pages. 1981.

Vol. 891: Logic Symposia, Hakone, 1979, 1980. Proceedings, 1979, 1980. Edited by G. H. Müller, G. Takeuti, and T. Tugué. XI, 394 pages. 1981.

Vol. 892: H. Cajar, Billingsley Dimension in Probability Spaces. III, 106 pages. 1981.

Vol. 893: Geometries and Groups. Proceedings. Edited by M. Aigner and D. Jungnickel. X, 250 pages. 1981.

Vol. 894: Geometry Symposium. Utrecht 1980, Proceedings. Edited by E. Looijenga, D. Siersma, and F. Takens. V, 153 pages. 1981.

Vol. 895: J.A. Hillman, Alexander Ideals of Links. V, 178 pages. 1981.

Vol. 896: B. Angéniol, Familles de Cycles Algébriques – Schéma de Chow. VI, 140 pages. 1981.

Vol. 897: W. Buchholz, S. Feferman, W. Pohlers, W. Sieg, Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies. V, 383 pages. 1981.

Vol. 898: Dynamical Systems and Turbulence, Warwick, 1980. Proceedings. Edited by D. Rand and L.-S. Young. VI, 390 pages. 1981.

Vol. 899: Analytic Number Theory. Proceedings, 1980. Edited by M.I. Knopp. X, 478 pages. 1981.

continuation on page 239

PREFACE

The purpose of this work is to present, in an essentially self-contained form, a survey of the mathematics of Galerkin finite element methods as applied to parabolic problems. The selection of topics is not meant to be exhaustive, but rather reflects the author's involvement in the field over the past ten years. The goal has been mainly pedagogical, with emphasis on collecting ideas and methods of analysis in simple model situations, rather than on pursuing each approach to its limits. The notes thus summarize recent developments, and the reader is often referred to the literature for more complete results on a given topic. Because the formulation and analysis of Galerkin methods for parabolic problems are generally based on facts concerning the corresponding stationary elliptic problems, the necessary elliptic results are included in the text for completeness.

The following is an outline of the contents of the notes:

In the introductory Chapter 1 we consider the simplest Galerkin finite element method for the standard initial boundary value problem with homogeneous Dirichlet boundary conditions on a bounded domain for the heat equation, using the standard associated weak formulation of the problem and employing first piecewise linear and then more general piecewise polynomial approximating functions vanishing on the boundary of the domain. For this model problem we demonstrate the basic error estimates in energy and mean square norms, first for the semidiscrete problem resulting from discretization in the space variables only and then also for the most commonly used completely discrete schemes obtained by discretizing the semidiscrete equation with respect to the time variable.

In the following five chapters we consider several extensions and generalizations of these results in the case of the semidiscrete approximation, and show error estimates in a variety of norms. First, in Chapter 2, we express the semidiscrete problem by means of an approximate solution operator for the elliptic problem in a manner which does not require the approximating functions to satisfy the homogeneous boundary conditions. A discrete method of Nitsche based on a non-standard weak formulation of the elliptic problem is used as an example. In Chapter 3 more precise results are shown in the case of the homogeneous heat equation. These require an accurate description of the smoothness of the solution for given initial data, expressed in terms of certain function spaces $\dot{H}^s(\Omega)$ which will be used repeatedly in these notes and which take into account both the smoothness and the boundary behavior of its elements. We also demonstrate that the smoothing property of the solution operator for positive time has an analogue in the semidiscrete situation and that, as a consequence, the finite element solution then converges to full order even when the initial data are non-smooth. The results of Chapters 2 and 3 are extended to more general parabolic equations in Chapter 4. In Chapter 5 some a priori bounds and error estimates with respect to the maximum-norm are derived in a simple situation, and in Chapter 6 negative norm estimates are shown, in certain cases together with related results concerning the convergence at specific points (superconvergence).

In the next three chapters we consider the discretization in time of the semidiscrete problem. First, in Chapter 7, we study the homogeneous heat equation and give analogues of our previous results both for smooth and for non-smooth data. The methods used for time discretizations are of one-step type and rely on rational approximations to the exponential, allowing the standard Euler and Crank-Nicolson procedures as special cases. In Chapter 8 we study completely discrete one-step methods for the inhomogeneous heat equation in which the forcing term is evaluated at a fixed finite number of points per time step. In Chapter 9 we apply Galerkin's method for the time discretization and seek discrete solutions as piecewise polynomials in the time variable which may be discontinuous at the nodes of the now not necessarily uniform partition of the time axis. In this procedure the forcing term enters in integrated form rather than at a finite number of points.

In Chapter 10 we discuss the application of the standard Galerkin method to a nonlinear parabolic equation. We show error estimates for the semidiscrete problem and then pay special attention to the formulation and analysis of time stepping procedures which are linear in the unknown functions.

In the following three chapters we consider various modifications of the standard Galerkin method. In Chapter 11 we analyze the so called lumped mass method for which in certain cases a maximum principle is valid. In Chapter 12 we discuss the H^1 and H^{-1} methods. In the first of these, the Galerkin method is based on a weak formulation with respect to an inner product in H^1 and for the second, the method uses trial and test functions from different finite dimensional spaces. In Chapter 13, the approximation scheme is based on a mixed formulation of the initial boundary value problem in which the solution and its gradient are sought independently in different spaces.

In the final Chapter 14 we consider the singular problem obtained by introducing polar coordinates in a spherically symmetric problem in a ball in R^3 and discuss two Galerkin methods based on two different weak formulations defined by two different inner products.

References to the literature are given at the end on each chapter. The numbering of theorems, lemmas and formulas is made for each chapter separately, and when a reference is made to a different chapter this is explicitly stated.

These notes have developed from courses that I have given at the University of Queensland, Australia, in 1979, Université Pierre et Marie Curie (Paris VI) in 1980, and Jilin University, China, in 1982, and also, of course, from my teaching over the years in my own university, Chalmers University of Technology, Göteborg, Sweden. I wish to thank all my students and colleagues in these institutions for the inspiration they have provided. Most of my own work in this field has been intimately connected with my association during more than a decade with J.H. Bramble, A.H. Schatz and L. Wahlbin of Cornell University and I wish to express my gratitude to them for their congenial collaboration and to the U.S. National Science Foundation for supporting this collaboration during 12 summers.

Finally, I wish to thank Stig Larsson and Nie Yi Yong, who have read the whole manuscript in detail and are responsible for many improvements, and Boel Engebrand who so expertly typed these notes.

Göteborg in December 1983

Vidar Thomée

TABLE OF CONTENTS.

1. THE STANDARD GALERKIN METHOD.

In this introductory chapter we shall consider the standard Galerkin method for the approximate solution of the initial-boundary value problem for the heat equation.

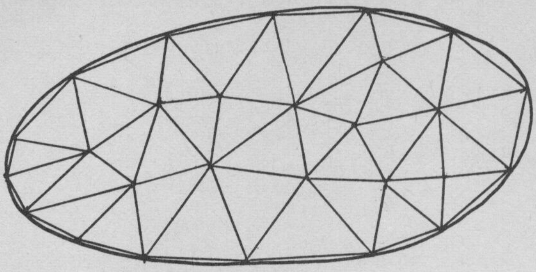
Let Ω be a domain in R^d with smooth boundary $\partial\Omega$ and consider the initial-boundary value problem

$$
\begin{aligned}
(1) \qquad u_t - \Delta u &= f && \text{in } \Omega \times [0,\infty), \\
u &= 0 && \text{on } \partial\Omega \times [0,\infty), \\
u(x,0) &= v(x) && \text{in } \Omega,
\end{aligned}
$$

where u_t denotes $\partial u/\partial t$ and $\Delta = \Sigma_1^d \partial^2/\partial x_j^2$ the Laplacian. In the first step we shall want to approximate $u(x,t)$ by means of a function $u_h(x,t)$ which, for each fixed t, belongs to a finite-dimensional linear space S_h of functions of x with certain approximation properties. This function will be a solution of a finite system of ordinary differential equations and is referred to as a semidiscrete solution. We shall then proceed to discretize (1) also in the time variable so as to produce a completely discrete scheme for the approximate solution of our problem.

Before we turn to the differential equation, we consider briefly the approximation of smooth functions in Ω which vanish on $\partial\Omega$. For concreteness, we shall exemplify by piecewise linear functions in a convex plane domain.

Thus let $\mathcal{T}_h$ denote a partition of Ω into disjoint triangles τ such that no vertex of any triangle lies on the interior of a side of another triangle and such that the union of the triangles determine a polygonal domain $\Omega_h \subset \Omega$ whose boundary vertices lie on $\partial\Omega$ (cf. fig.).



Let h denote the maximal length of a side of the triangulation $\mathcal{T}_h$. Thus h is a parameter which decreases as the triangulation is made finer. We shall assume that the angles of the triangulations are bounded below, independently of h, and often also that the triangulations are quasi-uniform in the sense that the triangles of $\mathcal{T}_h$ are of essentially the same size, which may be expressed by demanding that the area of τ in $\mathcal{T}_h$ is bounded below by ch^2 with $c > 0$ independent of h.

Let now S_h denote the continuous functions on the closure $\bar{\Omega}$ of Ω which are linear in each triangle of $\mathcal{T}_h$ and which vanish outside Ω_h. Let $\{P_j\}_1^{N_h}$ be the interior vertices of $\mathcal{T}_h$. A function in S_h is then uniquely determined by its values at the points P_j and thus depends on N_h parameters. Let φ_j be the "pyramid function" in S_h which takes the value 1 at P_j but vanishes at the other vertices. Then $\{\varphi_j\}_1^{N_h}$ forms a basis for S_h, and every χ in S_h admits the representation

$$\chi(x) = \sum_{j=1}^{N_h} \alpha_j \varphi_j(x) , \quad \text{with} \quad \alpha_j = \chi(P_j) .$$

Given a smooth function v on Ω which vanishes on $\partial\Omega$, we can now, for instance, approximate it by its interpolant $I_h v$ in S_h, which we define by requiring that it agrees with v at the interior vertices, i.e. $I_h v(P_j) = v(P_j)$ for $j = 1,\ldots,N_h$. We shall need some results concerning the error in this interpolation.

We shall denote below by $\|\cdot\|$ the L_2 or mean square norm over Ω and by $\|\cdot\|_r$ that in the Sobolev space $H^r(\Omega) = W_2^r(\Omega)$. Thus, for real-valued functions v,

$$\|v\| = (\int_\Omega v^2 dx)^{1/2} ,$$

and for r a positive integer,

$$\|v\|_r = \Big(\sum_{|\alpha|\le r} \|D^\alpha v\|^2\Big)^{1/2},$$

where with $\alpha = (\alpha_1,\ldots,\alpha_d)$, $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_d)^{\alpha_d}$ denotes an arbitrary derivative with respect to x of order $|\alpha| = \sum_{j=1}^{d} \alpha_j$ so that the sum contains all such derivatives of order at most r. We recall that for functions in $H_0^1(\Omega)$, i.e. the functions v with $\nabla v = \operatorname{grad} v$ in $L_2(\Omega)$ and which vanish on $\partial\Omega$, $\|\nabla v\|$ and $\|v\|_1$ are equivalent norms.

The following error estimates for the interpolant just defined are well-known, namely

$$\|I_h v - v\| \le Ch^2 \|v\|_2,$$

and

$$\|\nabla I_h v - \nabla v\| \le Ch \|v\|_2,$$

where, as will always be the case in the sequel, the statements of the inequalities assume that v is sufficiently regular for the norms on the right to be finite.

We shall now return to the general case of a domain Ω in R^d and assume that we are given a family $\{S_h\}$ of finite-dimensional subspaces of $H_0^1(\Omega)$ such that for some integer $r \ge 2$ and small h,

(2) $$\inf_{\chi\in S_h} \{\|v-\chi\| + h\|\nabla(v-\chi)\|\} \le Ch^s \|v\|_s, \quad 1 \le s \le r, \quad \text{for} \quad v \in H^s(\Omega) \cap H_0^1(\Omega).$$

The above example of piecewise linear functions corresponds to $d = r = 2$. Also in the general situation estimates such as (2) may often be obtained by exhibiting an interpolation operator I_h into S_h such that

(3) $$\|I_h v - v\| + h\|\nabla(I_h v - v)\| \le Ch^s \|v\|_s, \quad 1 \le s \le r.$$

For the case that $\partial\Omega$ is curved and $r > 2$ there are difficulties near the boundary, but the above situation may be accomplished, in principle, by mapping a curved triangle onto a straight-edged one (isoparametric elements). We shall not dwell on this.

The optimal orders to which functions and their gradients may be approximated under our assumption (2) are $O(h^r)$ and $O(h^{r-1})$, respectively, and we shall attempt below to obtain approximations of these orders for the solution of the heat equation.

For the purpose of defining thus an approximate solution to the initial boundary value problem (1) we first write this problem in weak form: We multiply the heat equation by a smooth function φ which vanishes on $\partial\Omega$, integrate over Ω, and apply Green's formula to the second term, to obtain, for all such φ, with (v,w) denoting the inner product $\int_\Omega vw\,dx$ in $L_2(\Omega)$,

$$(u_t,\varphi) + (\nabla u,\nabla\varphi) = (f,\varphi) \quad \text{for} \quad t \geq 0 .$$

We may then pose the approximate problem to find $u_h(t)$, belonging to S_h for each t, such that

$$(4) \qquad (u_{h,t},\chi) + (\nabla u_h,\nabla\chi) = (f,\chi) \quad \text{for all} \quad \chi \quad \text{in} \quad S_h, \quad t \geq 0 ,$$

together with the initial condition

$$u_h(0) = v_h ,$$

where v_h is some approximation of v in S_h. Since we have only discretized in the space variables, this is referred to as a semidiscrete problem. Later, we shall discretize also in the time variables to produce completely discrete schemes.

In terms of a basis $\{\varphi_j\}_1^{N_h}$ for S_h our semidiscrete problem may be stated: Find the coefficients $\alpha_j(t)$ in

$$u_h(x,t) = \sum_{j=1}^{N_h} \alpha_j(t)\varphi_j(x)$$

such that

$$\sum_{j=1}^{N_h} \alpha_j'(t)(\varphi_j,\varphi_k) + \sum_{j=1}^{N_h} \alpha_j(t)(\nabla\varphi_j,\nabla\varphi_k) = (f,\varphi_k), \quad k = 1,\ldots,N_h ,$$

and, with γ_j the components of the given initial approximation v_h,

$$\alpha_j(0) = \gamma_j , \quad j = 1,\ldots,N_h .$$

In matrix notation this may be expressed as

$$A\alpha'(t) + B\alpha(t) = \tilde{f} \quad \text{for} \quad t \geq 0 , \quad \text{with} \quad \alpha(0) = \gamma ,$$

where $A = (a_{jk})$ is the mass matrix with elements $a_{jk} = (\varphi_j,\varphi_k)$, $B = (b_{jk})$ the stiffness matrix with $b_{jk} = (\nabla\varphi_j, \nabla\varphi_k)$, $\tilde{f} = (f_k)$ the vector with entries $f_k = (f, \varphi_k)$, $\alpha(t)$ the vector of unknowns $\alpha_j(t)$ and $\gamma = (\gamma_k)$. The dimension of all these items equals N_h, the dimension of S_h.

Since the mass matrix A is a Gram matrix, and thus in particular positive definite and invertible, the above system of ordinary differential equations may be written

$$\alpha'(t) + A^{-1}B\alpha(t) = A^{-1}\tilde{f} \quad \text{for} \quad t \geq 0, \quad \text{with} \quad \alpha(0) = \gamma,$$

and hence obviously has a unique solution for positive t.

We shall prove the following estimate for the error between the solutions of the semidiscrete and continuous problems.

<u>Theorem 1</u>. Let u_h and u be the solutions of (4) and (1), respectively. Then

$$\|u_h(t) - u(t)\| \leq \|v_h - v\| + Ch^r\{\|v\|_r + \int_0^t \|u_t\|_r ds\} \quad \text{for} \quad t \geq 0.$$

Here we require, of course, that the solution of the continuous problem has the regularity implicitly assumed by the presence of the norms on the right and that v vanishes on $\partial\Omega$. Note also that if (3) holds and $v_h = I_h v$, then the first term on the right is dominated by the second. The same holds true if $v_h = P_0 v$, where P_0 denotes the L_2-projection of v onto S_h, since this choice is the best approximation of v in S_h with respect to the L_2 norm. Another such optimal order choice of v_h is the projection to be defined next.

For the purpose of the proof of Theorem 1 we introduce the so called elliptic or Ritz projection P_1 onto S_h as the orthogonal projection with respect to the inner product $(\nabla v, \nabla w)$, so that

$$(5) \qquad (\nabla P_1 u, \nabla\chi) = (\nabla u, \nabla\chi) \quad \text{for} \quad \chi \quad \text{in} \quad S_h.$$

In fact, $P_1 u$ is the finite element approximation of the solution of the corresponding elliptic problem whose exact solution is u. From the well established error analysis for the elliptic problem we quote the following error estimate.

Lemma 1. With P_1 defined by (5) we have

$$\|P_1v-v\|+h\|\nabla(P_1v-v)\| \le Ch^s\|v\|_s \quad \text{for} \quad 1\le s\le r\,, \quad v\in H^s(\Omega)\cap H_0^1(\Omega)\,.$$

Proof. We start with the gradient. We have using (5)

$$\|\nabla(P_1v-v)\|^2 = (\nabla(P_1v-v),\nabla(P_1v-v)) = (\nabla(P_1v-v),\nabla(\chi-v))$$
$$\le \|\nabla(P_1v-v)\|\ \|\nabla(\chi-v)\|\,,$$

and hence by (2),

$$\|\nabla(P_1v-v)\| \le \inf_{\chi\in S_h}\|\nabla(\chi-v)\| \le Ch^{s-1}\|v\|_s\,.$$

For the L_2 norm we proceed by duality. Let φ be arbitrary in $L_2(\Omega)$, take $\psi\in H^2(\Omega)\cap H_0^1(\Omega)$ as the solution of

$$-\Delta\psi = \varphi \quad \text{in} \quad \Omega\,, \quad \psi = 0 \quad \text{on} \quad \partial\Omega\,,$$

and recall the a priori inequality

$$\|\psi\|_2 \le C\|\Delta\psi\| = C\|\varphi\|\,.$$

Then

$$(P_1v-v,\varphi) = -(P_1v-v,\Delta\psi) = (\nabla(P_1v-v),\nabla\psi) = (\nabla(P_1v-v),\nabla(\psi-P_1\psi))$$
$$\le \|\nabla(P_1v-v)\|\ \|\nabla(\psi-P_1\psi)\| \le Ch^{s-1}\|v\|_s\ h\|\psi\|_2 \le Ch^s\|v\|_s\|\varphi\|\,,$$

which completes the proof if we choose $\varphi = P_1v-v$.

We now turn to the proof of Theorem 1. In the main step we shall compare the solution of the semidiscrete problem to the elliptic projection of the exact solution. We write

$$(6) \qquad u_h-u = (u_h-P_1u)+(P_1u-u) = \theta+\rho\,.$$

The second term is easily bounded by Lemma 1 and obvious estimates:

$$\|\rho(t)\| \le Ch^r\|u(t)\|_r = Ch^r\|v+\int_0^t u_t\,ds\|_r \le Ch^r\{\|v\|_r+\int_0^t\|u_t\|_r\,ds\}\,.$$

In order to estimate θ, we note that

$$(7) \qquad (\theta_t,\chi) + (\nabla\theta,\nabla\chi) = (u_{h,t},\chi) + (\nabla u_h,\nabla\chi) - (P_1 u_t,\chi) - (\nabla P_1 u,\nabla\chi)$$
$$= (f,\chi) - (P_1 u_t,\chi) - (\nabla u,\nabla\chi) = (u_t - P_1 u_t,\chi),$$

or

$$(8) \qquad (\theta_t,\chi) + (\nabla\theta,\nabla\chi) = -(\rho_t,\chi) \quad \text{for } \chi \text{ in } S_h .$$

In this derivation we have used the definition of P_1 and the easily established fact that this operator commutes with time differentiation. Since θ belongs to S_h we may choose $\chi = \theta$ in (8) and conclude

$$(9) \qquad (\theta_t,\theta) + \|\nabla\theta\|^2 = -(\rho_t,\theta) ,$$

or, since the first term equals $\frac{1}{2}\frac{d}{dt}\|\theta\|^2$ and the second is non-negative,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 \leq \|\rho_t\|\,\|\theta\| .$$

This yields

$$\frac{d}{dt}\|\theta\| \leq \|\rho_t\| ,$$

or, after integration,

$$\|\theta(t)\| \leq \|\theta(0)\| + \int_0^t \|\rho_t\|ds .$$

Here

$$\|\theta(0)\| = \|v_h - P_1 v\| \leq \|v_h - v\| + \|P_1 v - v\| \leq \|v_h - v\| + Ch^r\|v\|_r ,$$

and further

$$\|\rho_t\| = \|P_1 u_t - u_t\| \leq Ch^r\|u_t\|_r .$$

Together these estimates show the theorem.

In the above proof we made use in (9) of the fact that $\|\nabla\theta\|^2$ is non-negative. By a somewhat less wasteful treatment of this term one may demonstrate that the effect of the initial data upon the error tends to zero exponentially as t grows. In fact, with λ_1 the smallest eigenvalue of $-\Delta$ with Dirichlet boundary data, we have

$$\|\nabla\chi\|^2 \geq \lambda_1\|\chi\|^2 \qquad \text{for} \quad \chi \in S_h ,$$

and hence (9) yields

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \lambda_1\|\theta\|^2 \leq \|\rho_t\| \ \|\theta\| ,$$

or

$$\frac{d}{dt}\|\theta\| + \lambda_1\|\theta\| \leq \|\rho_t\| ,$$

and

$$\|\theta(t)\| \leq e^{-\lambda_1 t}\|\theta(0)\| + \int_0^t e^{-\lambda_1(t-s)}\|\rho_t(s)\|ds$$

$$\leq e^{-\lambda_1 t}\|v_h - v\| + Ch^2\{e^{-\lambda_1 t}\|v\|_2 + \int_0^t e^{-\lambda_1(t-s)}\|u_t(s)\|_2 ds\} .$$

Since

$$\|\rho(t)\| \leq Ch^2\|u(t)\|_2 ,$$

we conclude that

$$\|u_h(t) - u(t)\|$$

$$\leq e^{-\lambda_1 t}\|v_h - v\| + Ch^2\{e^{-\lambda_1 t}\|v\|_2 + \|u(t)\|_2 + \int_0^t e^{-\lambda_1(t-s)}\|u_t(s)\|_2 ds\} .$$

We shall not pursue this analysis for large t below.

We shall briefly look at another approach to the proof of Theorem 1 which consists in working with the equation for θ in operator form. For this purpose we introduce a "discrete Laplacian" Δ_h, which we think of as an operator from S_h into itself, by

$$(\Delta_h\psi,\chi) = -(\nabla\psi,\nabla\chi) \quad \text{for} \quad \psi,\ \chi \quad \text{in} \quad S_h;$$

this analogue of Green's formula clearly defines $\Delta_h\psi = \sum_{j=1}^{N_h} d_j\varphi_j$ from

$$\sum_{j=1}^{N_h} d_j(\varphi_j,\varphi_k) = -(\nabla\psi,\nabla\varphi_k), \quad k = 1,\ldots,N_h ,$$

since the matrix of this system is the positive definite mass matrix encountered above. The operator Δ_h is easily seen to be selfadjoint and $-\Delta_h$ is positive definite. Note that Δ_h is related to our other operators by

(10) $$\Delta_h P_1 = P_0 \Delta .$$

For, with $\chi \in S_h$,

$$(\Delta_h P_1 v, \chi) = -(\nabla P_1 v, \nabla \chi) = -(\nabla v, \nabla \chi) = (\Delta v, \chi) = (P_0 \Delta v, \chi) .$$

The semidiscrete equation now takes the form

$$(u_{h,t}, \chi) - (\Delta_h u_h, \chi) = (P_0 f, \chi) \quad \text{for } \chi \text{ in } S_h ,$$

or, since the factors on the left are all in S_h,

$$u_{h,t} - \Delta_h u_h = P_0 f .$$

Using (10) we hence obtain for θ

$$\theta_t - \Delta_h \theta = (u_{h,t} - \Delta_h u_h) - (P_1 u_t - \Delta_h P_1 u)$$

$$= P_0 f + (P_0 - P_1) u_t - P_0 (u_t - \Delta u) = P_0 (I - P_1) u_t = -P_0 \rho_t ,$$

or

(11) $$\theta_t - \Delta_h \theta = -P_0 \rho_t .$$

Let us denote by $E_h(t)$ the solution operator of the homogeneous semidiscrete equation

$$u_{h,t} - \Delta_h u_h = 0 \quad \text{for } t \geq 0 ,$$

i.e. the operator which takes the initial data $u_h(0) = v_h$ into the solution $u_h(t)$ at time t, so that $u_h(t) = E_h(t) v_h$. (This operator can also be thought of as the semigroup generated by $-\Delta_h$.) Duhamel's principle then tells us that the solution of the inhomogeneous equation (11) is

$$\theta(t) = E_h(t)\theta(0) - \int_0^t E_h(t-s) P_0 \rho_t(s) ds .$$

We now note that $E_h(t)$ is stable in L_2, or, more precisely,

$$\|E_h(t) v_h\| \leq \|v_h\| \quad \text{for } v_h \text{ in } S_h ,\ t \geq 0 .$$