Lecture Notes in Mathematics

1484

Hans Delfs

Homology of Locally Semialgebraic Spaces

Springer-Verlag

Author

Hans Delfs
Fakultät für Mathematik
Universität Regensburg
Universitätsstraße 31
W-8400 Regensburg, FRG

Mathematics Subject Classification (1991): 14G30, 14F45, 55N30, 55N35, 14C17, 54D18

ISBN 3-540-54615-4 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-54615-4 Springer-Verlag New York Berlin Heidelberg

Typesetting: Camera ready by author
Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
46/3140-543210 - Printed on acid-free paper

Editorial Policy

for the publication of monographs

In what follows all references to monographs, are applicable also to multiauthorship volumes such as seminar notes.

§ 1. Lecture Notes aim to report new developments - quickly, informally, and at a high level. Monograph manuscripts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. Furthermore, the manuscripts should provide sufficient motivation, examples and applications. This clearly distinguishes Lecture Notes manuscripts from journal articles which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this "lecture notes" character. For similar reasons it is unusual for Ph. D. theses to be accepted for the Lecture Notes series.

§ 2. Manuscripts or plans for Lecture Notes volumes should be submitted (preferably in duplicate) either to one of the series editors or to Springer- Verlag, Heidelberg . These proposals are then refereed. A final decision concerning publication can only be made on the basis of the complete manuscript, but a preliminary decision can often be based on partial information: a fairly detailed outline describing the planned contents of each chapter, and an indication of the estimated length, a bibliography, and one or two sample chapters - or a first draft of the manuscript. The editors will try to make the preliminary decision as definite as they can on the basis of the available information.

§ 3. Final manuscripts should be in English. They should contain at least 100 pages of scientific text and should include

- a table of contents;
- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

Further remarks and relevant addresses at the back of this book.

Lecture Notes in Mathematics 1484

Introduction

The basic task in real algebraic (or better semialgebraic) geometry is to study the set of solutions of a finite system of polynomial inequalities over a real closed field R. Such a set is called a semialgebraic set over R. The semialgebraic subsets of R^n are obtained from the basic open sets

$$U(P) = \{x \in R^n \mid P(x) > 0\},$$

where $P \in R[X_1, \ldots, X_n]$ is a polynomial, by finitely many applications of the set theoretic operations of uniting, intersecting and complementing. The interval topology of R induces a topology on every semialgebraic set over R called strong topology. Unfortunately the arising topological spaces are totally disconnected, except in the case $R = \mathbb{R}$. This pathology can be remedied. Strong topology is replaced by semialgebraic topology, a topology in the sense of Grothendieck (cf. [A]): Only open semialgebraic subsets are admitted as „open sets", and essentially only coverings by finitely many open semialgebraic subsets are admitted as „open coverings". These restricted topological spaces are the basic objects studied in semialgebraic topology. (For details concerning this and other notions from semialgebraic geometry we refer to [Br], [BCR], [DK], [DK_3], [DK_4]).
It is easier to study not only semialgebraic sets which are embedded in some algebraic variety over R but to study more generally spaces which, locally, look like a semialgebraic set. This observation leads to the notion of semialgebraic spaces.
An affine semialgebraic space over R is a ringed space which is isomorphic to a semialgebraic subset N of some affine R-variety V equipped with its sheaf $\mathcal{O}_N$ of semialgebraic functions (cf. [DK]). N is considered in its semialgebraic topology. A locally semialgebraic space $(M, \mathcal{O}_M)$ over a real closed field R is a ringed space ($\mathcal{O}_M$ the structure sheaf) which posesses an admissible open covering $(U_i \mid i \in I)$ such that $(U_i, \mathcal{O}_M \mid U_i)$ is an affine semialgebraic space over R for every $i \in I$ (cf. [DK_3], [DK_4]).
The category of locally semialgebraic spaces seems to be the appropriate framework for topological considerations over an arbitrary real closed field. (For the study of many questions especially in homotoy theory it turns out to be very useful to work in the even larger category of weakly semialgebraic spaces, cf. [K_2]).

This book is a contribution to the fundamentals of semialgebraic topology and consists of two main parts. The first is primarily concerned with the study of sheaves and their cohomology on locally semialgebraic spaces. In the second part we develop a homology theory for locally complete locally semialgebraic spaces over a real closed field R. It is the semialgebraic analogue of the homology for locally compact topological spaces introduced by Borel and Moore ([BM]). Finally we apply the sheaf and homology theory to varieties over an algebraically closed field of characteristic zero and develop a semialgebraic („topological") approach to intersection theory on these varieties.
A first observation is that a sheaf $\mathcal{F}$ on a locally semialgebraic space M over R is not determined by its stalks $\mathcal{F}_x, x \in M$, (Example I.1.7). But fortunately we may embed M into a space $\tilde{M}$ which

- is a topological space in the usual sense
- has the same sheaf theory as M.

If M is a semialgebraic subset of an affine R-variety V, then $\tilde{M}$ is just the associated constructible subset of the real spectrum Sper $R[V]$ of the coordinate ring of V (cf. I, §1). The space $\tilde{M}$ can be endowed with a structure sheaf $\mathcal{O}_{\tilde{M}}$. In this way it becomes a locally ringed space which locally looks like a constructible subset K of some real spectrum Sper A equipped with the sheaf of abstract semialgebraic functions on K defined by Brumfiel ([Br_1]) and Schwartz ([S], [S_1]) (see also [D_2, §1]). Such a space is called an abstract locally semialgebraic space. In Chapter II we investigate sheaves on these spaces. This has the great advantage that abstract locally semialgebraic spaces are topological spaces in the classical sense and therefore classical topological sheaf theory applies to them. (But notice that these spaces are almost never Hausdorff). Since the sheaf theories on M and $\tilde{M}$ coincide, our results are also valid for locally semialgebraic spaces over a real closed field.

In Chapter I we present a short introduction to the theory of abstract locally semialgebraic spaces. We give the definition and explain the connection with the geometric case (§1). Important classes of subsets of a given space are endowed with a subspace structure in §2. Some basic notions which turn out to be useful in sheaf theory are defined and discussed in §3. The subspace X^{max} of closed points of an abstract locally semialgebraic space X often is a „nice" topologcial space. In §4 we explain, for example, under what conditions X^{max} is a locally compact space. The most important class of locally semialgebraic spaces consisting of the regular and paracompact spaces is investigated in §5.

Slightly more generally N. Schwartz considers in [S_1] spaces which locally are proconstructible subsets of a variety. He calls these spaces „real closed spaces". Most of the results in chapter II may be easily extended to this more general case. In chapters III and IV we study only the geometric case, locally semialgebraic spaces over a real closed field, and for this purpose the notion of abstract locally semialgebraic spaces as introduced in chapter I is sufficient.

Chapter II is devoted to the study of sheaves on abstract locally semialgebraic spaces. In particular we deal with the cohomology groups of a space X with coefficients in a sheaf $\mathcal{F}$ and support in a family Φ of closed subsets. Usually we are only interested in families Φ of supports which are generated by their locally semialgebraic members. We define paracompactifying support families Φ (this notion is different from the corresponding notion in topological sheaf theory!) and discuss some relations between the properties of Φ and $\Phi \cap X^{max}$ (§1). The homomorphisms induced in cohomology by locally semialgebraic maps are described in §2. There we also prove that homotopic maps induce the same homomorphism in cohomology. Applying this result we see that, in the geometric case, it is sometimes possible to replace certain families of supports by paracompactifying ones.
Since regular and paracompact locally semialgebraic spaces in many ways show a similar behaviour as paracompact topological spaces, sections of a sheaf over a partially quasicompact subset can be extended to a neighbourhood (§3). This is one of the main reasons why soft sheaves are acyclic as is shown in §4. There is another important class of acyclic sheaves consisting of those sheaves which are flabby in the semialgebraic sense. The results on acyclic sheaves are applied in §5 to describe the cohomology of certain subsets. In particular we learn how the cohomology groups of X and X^{max} and how the cohomology groups of M and M_{top} are related (where M is a space over the field $\mathbb{R}$ of real numbers and M_{top} is the set M considered in its strong topology).

One of the central results of the book is proven in §6: The cohomology groups of a locally semialgebraic space over a real closed field R with coefficients in a sheaf and arbitrary supports do not change when the base field R is extended. We need this result in §7 to derive the semialgebraic proper base change theorem. This base change theorem is generalized to the case of non proper locally semialgebraic maps in §8. In §9 and §10 we state some facts about the cohomological dimension of geometric spaces and hypercohomology which are needed later on.

Chapter III deals with semialgebraic Borel-Moore-homology. This is a homology theory designed for locally semialgebraic spaces over a real closed field R which are locally complete. Recall that topological Borel-Moore-homology is defined for locally compact spaces and the notion „locally complete" is the semialgebraic substitute for „locally compact". Every affine semialgebraic space M (and more generally every paracompact regular locally semialgebraic space) can be triangulated, i.e. it is isomorphic to a simplicial complex X over R (cf. [DK_1, §2], [DK_3, II, §4], introduction of chap. III). But in general the simplicial complex X is not closed, i.e. there may be open simplicies σ in X whose faces do not belong to X. Nevertheless these simplicies should contribute to the homology of M. So it seems to be quite natural to take the open simplices as building blocks of a homology theory (and not the closed simplices as in classical simplicial homology). Following this idea we define Borel-Moore-homology groups with constant coefficients and closed supports by use of triangulations and open simplices (§1 and §2). The basic properties of these groups are derived by easy „simplicial arguments". No sheaf theory is needed for this elementary introduction to Borel-Moore-homology. We already sketched this approach in [D_1]. These elementary methods suffice to prove the substantial result that every algebraic variety over R posesses a fundamental class (§3).
But of course there are also problems which are difficult to handle by more or less combinatorial considerations. A typical example is Poincaré duality for arbitrary families of supports. It is possible to give an elementary proof but it is long and difficult because the combinatorics involved is rather complicated. Here the use of sheaves turns out to simplify the problem considerably.
Therefore we generalize our definitions in §2 and introduce Borel-Moore-homology groups with arbitrary supports and coefficients in an arbitrary sheaf in §5. In particular the important case of locally constant sheaves is included. The groups are defined by means of a complex $(\Delta_k \mid k \in \mathbf{Z})$ of sheaves of simplical chains (see §4). The sheaves Δ_k are similar to the sheaves of PL-chains in PL-topology. The main differences to PL-theory are that we work with open instead of closed simplices and that a simplicial chain is identified with all its semialgebraic and not only with its linear subdivisions. Semialgebraic subdivision is discussed in §4.
Using weakly semialgebraic spaces and a lot of homotopy theory M. Knebusch was able to prove the really nontrivial result that the semialgebraic homology with compact support may be calculated by singular chains, even if the base field R is non archimedian (cf. [K_2]). So the sheaf theoretical approach to homology might also be based, as in classical topology [B], on the sheaves of locally finite singular chains. The semialgebraic triangulation theorem implies that the sheaves of simplicial chains have very nice properties. Therefore it is really an advantage to work with them. For example, the sheaves Δ_k are flabby in the semialgebraic sense and the basic properties of Borel-Moore-homology can be easily deduced from the sheaf theory developed in Chapter II.

In §6 we study the functorial behaviour and in §7 the homotopy invariance of Borel-Moore-homology. A cap product between cohomology and homology is introduced in §8. The proof of a general Poincaré duality theorem is given in §9. In §10 we investigate the behaviour of the Borel-Moore-homology when the real closed groundfield R is enlarged. The relation between the semialgebraic homology of a locally semialgebraic space M over R and the topological Borel-Moore-homology of $\tilde{M}^{\text{max}}$ (resp. M_{top} if $R = \mathbb{R}$) is discussed in §11. Finally we show in §12 that the semialgebraic Borel-Moore-homology groups could also be defined by use of injective resolutions and duals of complexes of sheaves. Such a definition would be analogous to the definition given by Borel and Moore in the topological case ([BM]). (Dualising complexes were also used to define étale Borel-Moore-homology for varieties over arbitrary algebraically closed fields, cf. [DV, exp. VIII]).

In Chapter IV we discuss some aspects of intersection theory. We take an algebraically closed field C of characteristic zero and choose a real closed field $R \subset C$ with $C = R(\sqrt{-1})$. In §1 we prove that every locally isoalgebraic space over C (cf. [Hu] or Appendix, §1, for the definition) posesses a fundamental class. Using Poincaré duality we establish an intersection product in the homology of locally semialgebraic manifolds (§2). This intersection product enables us to define the intersection of locally isoalgebraic cycles on a locally isoalgebraic manifold (§3). In particular we are able to describe the algebraic intersection multiplicities of subvarieties of a smooth algebraic variety V over C in a purely „topologcial" (i.e. semialgebraic) way, and we obtain a multiplicative homomorphism $A_*(V) \to H_*(V)$ from the Chow ring of V to the semialgebraic Borel-Moore-homology.

Throughout this book R is a real closed field. All rings are assumed to be commutative with 1. The real spectrum of a ring A is denoted by $\operatorname{Sper} A$. (The basic theory concerning real spectra is contained in [BCR],[CR] or [L]). The sections of a sheaf $\mathcal{F}$ over an open set U are denoted by $\Gamma(U, \mathcal{F})$. If objects B and C are canonically isomorphic, then we often simply write $B = C$. A list of symbols and an index may be found at the end of this book.

I wish to thank M. Knebusch, R. Huber, R. Robson and N. Schwartz for many helpful discussions on the subject. Finally special thanks are due to M. Richter and R. Bonn for their patience and excellence in typing the manuscript.

Regensburg, May 1991

Hans Delfs

TABLE OF CONTENTS

CHAPTER I: Abstract locally semialgebraic spaces.

This chapter contains a short introduction to the theory of abstract locally semialgebraic spaces. We present the basic definitions and some observations needed in the subsequent parts of these notes.

§1 - Abstract and geometric spaces

We refer the reader to [DK_3] for an extensive treatment of locally semialgebraic spaces over the real closed field R. A survey about the basic theory over R is also given in [DK_4]. As in algebraic geometry it is sometimes useful to study not only these „geometric spaces" but also a more general class of „abstract spaces". This was made possible by the introduction of abstract semialgebraic functions (cf. [Br_1, §3], [S], [S_1], [D_2, §1]) on the real spectrum (cf. [BCR], [K_1]) of an arbitrary ring A (commutative, with 1).

Definition 1. a) Let $\mathrm{Sper}\, A$ be the real spectrum of a ring A. A pair $(K, \mathcal{O}_K)$ consisting of a constructible subset K of $\mathrm{Sper}\, A$ and the sheaf $\mathcal{O}_K$ of abstract semialgebraic functions on K ([D_2, §1]) is called a *semialgebraic subspace* of $\mathrm{Sper}\, A$.
b) An *abstract affine semialgebraic space* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic, as a locally ringed space, to a semialgebraic subspace $(K, \mathcal{O}_K)$ of an affine real spectrum $\mathrm{Sper}\, A$.
c) An *abstract locally semialgebraic space* is a locally ringed space $(X, \mathcal{O}_X)$ which has an open covering $(X_i \mid i \in I)$ such that $(X_i, \mathcal{O}_X \mid X_i)$ is an abstract affine semialgebraic space for every $i \in I$. Such a covering $(X_i \mid i \in I)$ is called an open affine covering of X. If X is in addition quasicompact, then $(X, \mathcal{O}_X)$ is called an *abstract semialgebraic space*. (Note that „quasicompact" means that X admits a *finite* open affine covering).
d) A *locally semialgebraic map* between abstract locally semialgebraic spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a morphism (f, δ) in the category of locally ringed spaces, i.e. $f : X \to Y$ is a continuous map and $\delta : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism of sheaves of local rings. We often omit δ in our notation and simply write $f : X \to Y$.

Abstract locally semialgebraic spaces were first defined and studied by N. Schwartz ([S], [S_1]). Slightly more generally he considers in [S_1] spaces which locally are proconstructible subsets of a variety. He calls these spaces „real closed spaces". We will use many of his definitions and results.

We make the following *general assumption*: All abstract locally semialgebraic spaces $(X, \mathcal{O}_X)$ are assumed to be *quasiseparated* ([S_1], II.4.14). This means that the intersection $U \cap V$ of any two open quasicompact subsets U, V of X is also quasicompact, or, equivalently, that $X_i \cap X_j$ is quasicompact for any two members X_i, X_j of an open affine cover $(X_i \mid i \in I)$ of X ([S_1, II.4.16]).

In the category of abstract locally semialgebraic spaces arbitrary fibre products exist ([S_1, II.3.1]). The residue class field $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ of a space X in a point $x \in X \{\mathcal{O}_{X,x}$ the stalk of $\mathcal{O}_X$ in x, $\mathfrak{m}_{X,x}$ the maximal ideal of the local ring $\mathcal{O}_{X,x}\}$ is real closed and will always be denoted by $k(x)$.

If X is a semialgebraic subspace of $\mathrm{Sper}\, A$, then $k(x)$ is the real closure of the residue class field $A(\mathfrak{p}(x))$ of A in $\mathfrak{p}(x)$, the support of x, with respect to the ordering of $A(\mathfrak{p}(x))$ which is induced by x. The image of an element $f \in \Gamma(X, \mathcal{O}_X)$ in $k(x)$ under the natural map

$\Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x} \to k(x)$ is denoted by $f(x)$. The elements $f \in \Gamma(X, \mathcal{O}_X)$ are called *locally semialgebraic functions* on X.

Definition 2. Let $(X, \mathcal{O}_X)$ be an affine abstract semialgebraic space. A subset Y of X is called *semialgebraic* (or *constructible*) if Y is a finite union of sets of the form

$$\{x \in X \mid f(x) = 0,\, g_j(x) > 0,\, j = 1, \ldots, s\},$$

$f \in \Gamma(X, \mathcal{O}_X), g_j \in \Gamma(X, \mathcal{O}_X)(1 \le j \le r)$. The set of semialgebraic subsets of X is denoted by $\gamma(X)$.

Of course, if $(X, \mathcal{O}_X)$ is a semialgebraic subspace of Sper A, then $\Gamma(X)$ consists of those constructible subsets of Sper A that are contained in X.

Definition 3. Let $(X, \mathcal{O}_X)$ be an abstract locally semialgebraic space and $(X_i \mid i \in I)$ be an open affine covering of X. A subset Y of X is called *locally semialgebraic* (or *locally constructible*) if $Y \cap X_i$ is a semialgebraic subset of X_i for every $i \in I$. If Y is also quasicompact, then it is called *semialgebraic* (or *constructible*). Since all spaces are assumed to be quasi-separated, the intersection $X_i \cap X_j$ is a semialgebraic subset of X_i, and it is easy to see that our Definition 3 does not depend on the choice of the open affine covering of X. The family of locally semialgebraic (semialgebraic) subsets of X is denoted by $\mathcal{T}(X)$ (by $\gamma(X)$). By $\mathring{\mathcal{T}}(X)\,\{\mathring{\gamma}(X)\}$ we denote the family of those sets in $\mathcal{T}(X)\,\{$ in $\gamma(X)\}$ which are open in X, and by $\bar{\mathcal{T}}(X)\,\{\bar{\gamma}(X)\}$ the family of those sets which are closed in X.

From now on we usually omit the structure sheaves in our notation and simply write X instead of $(X, \mathcal{O}_X)$.

Definition 4. A locally semialgebraic map $f: X \to Y$ is called *semialgebraic* (or *quasicompact*) if $f^{-1}(A)$ is a semialgebraic (or, equivalently, a quasicompact) subset of X for every semialgebraic subset $A \in \gamma(Y)$ of Y.

Every locally semialgebraic map $f: X \to Y$ whose domain X is a semialgebraic space is semialgebraic.

The concept of an abstract locally semialgebraic space is the natural generalization of the notion of a locally semialgebraic space over a real closed field R as we will explain now.

Example 1.1. Let V be an affine algebraic variety over R. We consider a semialgebraic subspace $(M, \mathcal{O}_M)$ of V (cf. [DK, §7]). By definition, M is a semialgebraic subset of the set $V(R)$ of R-rational points of V, and $\mathcal{O}_M$ is the sheaf of semialgebraic functions on M. Here the set M is considered in its semialgebraic topology (loc. cit.), and a function $f: M \to R$ is called semialgebraic if it has a semialgebraic graph and is continuous with respect to the strong topologies. Now let Sper $R[V]$ be the real spectrum of the coordinate ring $R[V]$ of V. The set $V(R)$ is contained in Sper $R[V]$. We may associate a constructible subset $\tilde{A}$ of Sper $R[V]$ to every semialgebraic subset A of $V(R)$ (cf. [CR, §5]). $\tilde{A}$ is defined by the same equalities and inequalities as A. It is the unique constructible subset $\tilde{A}$ of Sper $R[V]$ with $\tilde{A} \cap V(R) = A$. The important point is that the sheaf theories on the semialgebraic space M over R and the topological space $\tilde{M}$ coincide (loc. cit.). More precisely, if $\mathcal{F}$ is a sheaf on M, then we obtain a sheaf $\tilde{\mathcal{F}}$ on $\tilde{M}$ by defining

$$\Gamma(\tilde{U}, \tilde{\mathcal{F}}) := \Gamma(U, \mathcal{F})$$

for every open semialgebraic subset U of M. (Note that the sets $\tilde{U}$ form a basis of the topology of $\tilde{M}$). On the other hand, if $\mathcal{G}$ is a sheaf on $\tilde{M}$, then we get a sheaf $\mathcal{F}$ on M by restriction:

$$\Gamma(U,\mathcal{F}) := \Gamma(\tilde{U},\mathcal{G}).$$

Obviously we have $\tilde{\mathcal{F}} = \mathcal{G}$. The sheaf $\mathcal{O}_M$ of semialgebraic functions on M corresponds to the sheaf $\mathcal{O}_{\tilde{M}}$ of abstract semialgebraic functions on $\tilde{M}$. i.e. $\tilde{\mathcal{O}}_M = \mathcal{O}_{\tilde{M}}$ (cf. [D_2, 1.9]). We see that it is quite natural to assign the abstract affine semialgebraic space $(\tilde{M},\mathcal{O}_{\tilde{M}})$ to $(M,\mathcal{O}_M)$.

The topological space $\tilde{M}$ has the following useful description. We denote the family of semialgebraic subsets of M by $\gamma(M)$ and the family of open semialgebraic subsets by $\mathring{\gamma}(M)$. Let $Y(M)$ be the set of ultrafilters of the Boolean lattice $\gamma(M)$. The sets

$$Y(U)^M := \{F \in Y(M) \mid U \in F\}, \quad U \in \mathring{\gamma}(M),$$

are the basis of a topology on $Y(M)$. The set $Y(M)$ endowed with this topology is canonically homeomorphic to $\tilde{M}$ ([Brö, p. 260], [CC, §1]).

Now suppose W is another affine R-variety, $(N,\mathcal{O}_N)$ is a semialgebraic subspace of W and $f: M \to N$ is a semialgebraic map (i.e. f is continuous and has a semialgebraic graph). The map f induces a continuous map $\tilde{f}: \tilde{M} \to \tilde{N}$ which may be described as follows: The image $\tilde{f}(F)$ of an ultrafilter $F \in Y(M)$ is the ultrafilter of $\gamma(N)$ generated by the sets $f(A), A \in F$.

Composition with f yields a map of sheaves

$$\delta: \mathcal{O}_N \to f_*\mathcal{O}_M$$

(cf. [DK, §7]). Now observe that $(f_*\mathcal{O}_M)^\sim = \tilde{f}_*\tilde{\mathcal{O}}_M = \tilde{f}_*\mathcal{O}_{\tilde{M}}$. Therefore δ gives a map $\tilde{\delta}: \mathcal{O}_{\tilde{N}} \to \tilde{f}_*\mathcal{O}_{\tilde{M}}$. This map $\tilde{\delta}$ is a map of sheaves of local rings. So our given semialgebraic map $f: M \to N$ yields a morphism $(\tilde{f},\tilde{\delta}): (\tilde{M},\mathcal{O}_{\tilde{M}}) \to (\tilde{N},\mathcal{O}_{\tilde{N}})$ in the category of abstract semialgebraic spaces.

Using the definitions of locally semialgebraic spaces and locally semialgebraic maps over R in [DK_3, I, §1] and Definition 1 above, it is now easy to see that the assignments

$$(M,\mathcal{O}_M) \mapsto (\tilde{M},\mathcal{O}_{\tilde{M}})$$
$$f \mapsto (\tilde{f},\tilde{\delta})$$

extend to a functor from the category of locally semialgebraic spaces over R to the category of abstract locally semialgebraic spaces over R. This functor will always be denoted by $^\sim$. By use of $^\sim$ we may consider the category of locally semialgebraic spaces over R as a *full* subcategory of the category of abstract locally semialgebraic spaces over Sper R. The functor $^\sim$ also preserves fibre products ([S_1, III.2.1]). We may and will consider a locally semialgebraic space M over R as a subset of $\tilde{M}$. The set M is dense in $\tilde{M}$ and consists of the points $x \in \tilde{M}$ with $k(x) = R$.

Let M be a locally semialgebraic space over R. As in [DK_3] we denote the family of locally semialgebraic subsets (open, closed locally semialgebraic subsets) of M by $\mathcal{T}(M)(\mathring{\mathcal{T}}(M), \bar{\mathcal{T}}(M))$ and the family of semialgebraic subsets (open, closed semialgebraic subsets) by $\gamma(M)(\mathring{\gamma}(M), \bar{\gamma}(M))$.

Proposition 1.2. Intersection with $M, Y \mapsto Y \cap M$, yields canonical bijections

$$\mathcal{T}(\tilde{M}) \xrightarrow{\sim} \mathcal{T}(M), \mathring{\mathcal{T}}(\tilde{M}) \xrightarrow{\sim} \mathring{\mathcal{T}}(M), \bar{\mathcal{T}}(\tilde{M}) \xrightarrow{\sim} \bar{\mathcal{T}}(M),$$
$$\gamma(\tilde{M}) \xrightarrow{\sim} \gamma(M), \mathring{\gamma}(\tilde{M}) \xrightarrow{\sim} \mathring{\gamma}(M), \bar{\gamma}(\tilde{M}) \xrightarrow{\sim} \bar{\gamma}(M).$$

Proof. It suffices to prove this for a semialgebraic subspace M of an affine variety V over R. In this case the result is well known ([CR, §5]).

If $A \in \mathcal{T}(M)$, then $\tilde{A}$ always denotes the (unique) locally semialgebraic subset of $\tilde{M}$ with $\tilde{A} \cap M = A$.

Remark 1.3. Let $f: M \to N$ be a locally semialgebraic map between locally semialgebraic spaces M, N over R. The explicit description of the map $\tilde{f}: \tilde{M} \to \tilde{N}$ in the affine case (cf. Example 1.1) shows that $\tilde{f}^{-1}(\tilde{A}) = \widetilde{f^{-1}(A)}$ for every $A \in \mathcal{T}(N)$. If f is semialgebraic then $f(B) \in \mathcal{T}(N)$ for every $B \in \mathcal{T}(M)$ ([DK$_3$, I.5.3]) and we have $\tilde{f}(\tilde{B}) = \widetilde{f(B)}$.

Again let M be a locally semialgebraic space over R. As in the affine case (cf. Example 1.1) we assign a sheaf $\tilde{\mathcal{F}}$ on $\tilde{M}$ to every sheaf $\mathcal{F}$ on M:

$$\Gamma(Y, \tilde{\mathcal{F}}) := \Gamma(Y \cap M, \mathcal{F}) \quad (Y \in \mathring{\mathcal{T}}(\tilde{M})).$$

Conversely, if $\mathcal{G}$ is a sheaf on $\tilde{M}$ then we obtain a sheaf $\mathcal{F}$ on M by defining

$$\Gamma(U, \mathcal{F}) := \Gamma(\tilde{U}, \mathcal{G}) \quad (U \in \mathring{\mathcal{T}}(M)).$$

Obviously these assignements are inverse to each other. Thus we have

Proposition 1.4. The categories of sheaves on $\tilde{M}$ and M are canonically isomorphic.

So, for all questions concerning sheaves, we may equally well work on $\tilde{M}$ instead on M. For example, since $\Gamma(M, \mathcal{F}) = \Gamma(\tilde{M}, \tilde{\mathcal{F}})$ for every sheaf $\mathcal{F}$ on M the sheaf cohomology theories of M and $\tilde{M}$ coincide.

Corollary 1.5. $H^q(M, \mathcal{F}) = H^q(\tilde{M}, \tilde{\mathcal{F}})$ for every (abelian) sheaf $\mathcal{F}$ on M and every $q \geq 0$.

More generally, we may consider the direct image $f_*\mathcal{F}$ of a sheaf $\mathcal{F}$ on M under a locally semialgebraic map $f: M \to N$ between locally semialgebraic spaces over R. Since $\widetilde{f^{-1}(U)} = \tilde{f}^{-1}(\tilde{U})$ for every $U \in \mathring{\mathcal{T}}(N)$ (Remark 1.3), we have $\widetilde{f_*\mathcal{F}} = \tilde{f}_*\tilde{\mathcal{F}}$. Together with Prop. 1.4 this implies that the right derived functors $R^q f_*$ and $R^q \tilde{f}_* (q \geq 0)$ of f_* and $\tilde{f}_*$ also „coincide".

Corollary 1.6. $(R^q f_* \mathcal{F})^\sim = R^q \tilde{f}_* \tilde{\mathcal{F}}$ for every (abelian) sheaf $\mathcal{F}$ on M and every $q \geq 0$.

In the following we often do not distinguish between sheaves on M and $\tilde{M}$, i.e. sheaves on M are also regarded as sheaves on $\tilde{M}$ (and vice versa). This turns out to be very useful since $\tilde{M}$ is a topological space in the usual sense. So a sheaf $\mathcal{G}$ on M is determined by its stalks $\mathcal{G}_x (:= \tilde{\mathcal{G}}_x), x \in \tilde{M}$. But, in general, it is not determined by the family $(\mathcal{G}_x)_{x \in M}$ of stalks in the points of M.

Example 1.7. Let $M =]0,1[$ be the unit interval over $\mathbf{R}$, considered as a semialgebraic space over $\mathbf{R}$. Then $\tilde{M}$ is the set of ultrafilters of the Boolean lattice γ (]0,1[). Let $\mathcal{G}$ be the associated sheaf on M of the presheaf

$$]a,b[\mapsto \begin{cases} \mathbf{Z} & \text{if} \quad 0 = a < b \\ 0 & \text{else} \end{cases}$$

Let x_0 be the ultrafilter generated by the intervals $]0,\varepsilon[, \varepsilon > 0$. Then $\mathcal{G}_{x_0} = \mathbf{Z}$, but $\mathcal{G}_x = 0$ for every $x \in M$.

We close this section with an example of a semialgebraic map.

Example 1.8. A homomorphism $\varphi: A \rightarrow B$ induces a continuous map $f = \mathrm{Sper}\,\varphi : \mathrm{Sper}\,B \rightarrow \mathrm{Sper}\,A$ (cf. [CR]). Let $\mathcal{O}_{\mathrm{Sper}\,A}$ and $\mathcal{O}_{\mathrm{Sper}\,B}$ be the sheaves of abstract semialgebraic functions on Sper A and Sper B. Composing with f yields a homomorphism $\mathcal{O}_{\mathrm{Sper}\,A} \rightarrow f_*\mathcal{O}_{\mathrm{Sper}\,B}$ of sheaves of local rings (cf. [D_2, Prop. 1.8]). In this way f becomes a semialgebraic map between the abstract semialgebraic spaces Sper B and Sper A.

Henceforth an abstract locally semialgebraic space $(X, \mathcal{O}_X)$ will be simply called *space* or *abstract space*. „ *Affine space* “ means „affine (abstract) semialgebraic space“. A locally semialgebraic space M over R is called a *geometric space* over R. The structure sheaves will usually be omitted in our notation. Unless otherwise stated all maps between spaces are locally semialgebraic maps.

§2 - Subspaces

In this section we explain how certain subsets Y of an abstract space X may be canonically endowed with a subspace structure.

If $Y \subset X$ is open, then obviously $(Y, \mathcal{O}_X \mid Y)$ is an abstract space, called an *open* subspace of X ([S_1, II.2.3]). Note that Y is also quasiseparated.

Our next goal is to define a subspace structure on an arbitrary locally semialgebraic subset Y of X.

Let $U \in \mathring{\gamma}(X)$ be an open affine subspace of X and $f:(U, \mathcal{O}_X \mid U) \xrightarrow{\sim} (U', \mathcal{O}_{U'})$ be an isomorphism onto some semialgebraic subspace $(U', \mathcal{O}_{U'})$ of some real spectrum Sper A. Let $K := U \cap Y$ and $K' := f(K)$. Then K' is a constructible subset of Sper A and we have the sheaf $\mathcal{O}_{K'}$ of abstract semialgebraic functions on $K' \subset$ Sper A. Let $f_1 : K \to K'$ be the restriction of f. We define $\mathcal{O}_K := f_1^* \mathcal{O}_{K'}$. Then $(K, \mathcal{O}_K)$ is an affine (abstract) semialgebraic space, and we have endowed Y with a subspace structure on the open affine part U of X. Now suppose $V \in \mathring{\gamma}(U)$ and $g:(V, \mathcal{O}_X \mid V) \to (V', \mathcal{O}_{V'})$ is an isomorphism onto some semialgebraic subspace $(V', \mathcal{O}_{V'})$ of some real spectrum Sper B. Let $L := V \cap Y, L' := g(L)$ and $g_1 : L \to L'$ be the restriction of g. Let $\mathcal{O}_L$ be the inverse image $g_1^* \mathcal{O}_{L'}$ of the sheaf $\mathcal{O}_{L'}$ of abstract semialgebraic functions on $L' \subset$ Sper B.

Lemma 2.1. $\mathcal{O}_L = \mathcal{O}_K \mid L$.

Proof. cf. [S_1, II.2].

Lemma 2.1. says that the subspace structures $(U \cap Y, \mathcal{O}_{U \cap Y})$ on the affine parts $U \cap Y$, $U \in \mathring{\gamma}(X)$ affine, glue together to form a subspace structure $(Y, \mathcal{O}_Y)$ on Y. Obviously $(Y, \mathcal{O}_Y)$ is an abstract locally semialgebraic space and the inclusion map $Y \hookrightarrow X$ is a locally semialgebraic map. These spaces $(Y, \mathcal{O}_Y)$ are called the *locally semialgebraic subspaces* of $(X, \mathcal{O}_X)$.

Example 2.2. Let $(M, \mathcal{O}_M)$ be a geometric space over R and $N \in \mathcal{T}(M)$. Then we equipped N with a subspace structure $(N, \mathcal{O}_N)$ in [DK_3, §3]. Via the functor $\sim$ the geometric space $(N, \mathcal{O}_N)$ over R corresponds to the subspace $\tilde{N}$ of the abstract space $(\tilde{M}, \mathcal{O}_{\tilde{M}})$ we defined here. This is a trivial consequence of the definitions.

Proposition 2.3. Let $f : X \to Y$ be a locally semialgebraic map between abstract spaces and $Z \in \mathcal{T}(Y)$. Assume $f(X) \subset Z$. Then the map $g : X \to Z$ obtained from f by restriction of the image space is a locally semialgebraic map from X to the subspace Z of Y.

Proof. cf. [S_1, II.2.15].

Finally we consider fibres of locally semialgebraic maps. So let $f : X \to Y$ be a map between abstract spaces and let $y \in Y$. There is a natural locally semialgebraic map

$$\text{Sper}\, k(y) = \text{Spec}\, k(y) \to Y$$

mapping the unique point of Spec $k(y)$ to y. We consider the fibre product $X \times_Y$ Sper $k(y)$. Let $p : X \times_Y \text{Sper}\, k(y) \to X$ be the projection.

Proposition 2.4. p induces a homeomorphism

$$p_1 : X \times_Y \operatorname{Sper} k(y) \to f^{-1}(y).$$

Proof. cf. [S_1, II.3.2].

We shift the structure sheaf of $X \times_Y \operatorname{Sper} k(y)$ to $f^{-1}(y)$ by p_1. Then $f^{-1}(y)$ becomes an abstract locally semialgebraic space. If $f^{-1}(y)$ is a locally semialgebraic subset of X, then the space structure of $f^{-1}(y)$ defined here coincides with the subspace structure on $f^{-1}(y)$ we considered before ([S_1, II.3.2].

Notation. For an abstract space X we denote the affine space $X \times_{\operatorname{Sper} \mathbb{Z}} \operatorname{Sper} \mathbb{Z}[T_1, \ldots, T_n]$ over X by $\mathbb{A}^n_X$.

Definition 1 (cf. [S_1, II.7.1]). A map $f : X \to Y$ between abstract spaces X and Y is *locally of finite type* if every $x \in X$ has an open neighbourhood $U \in \mathring{\gamma}(X)$ such that the restriction $f \mid U : U \to Y$ admits a factorization

$$\begin{array}{ccc} U & \xrightarrow[\sim]{h} & K \\ & f \searrow \quad \swarrow p & \\ & Y & \end{array}$$

where h is an isomorphism from U onto some semialgebraic subspace K of some affine space $\mathbb{A}^n_y$ and p is induced by the natural projection $\mathbb{A}^n_y \to Y$.

The following result is rather obvious (cf. [S_1, III.1.3]).

Proposition 2.5. Let $f : X \to Y$ be a map between abstract spaces which is locally of finite type. Then the fibre $f^{-1}(y)$ over a point $y \in Y$ is a geometric space over $k(y)$. (Of course this means: There is a geometric space M over $k(y)$ with $\tilde{M} \cong f^{-1}(y)$).

§3 - Some basic notions

We carry over some of the notions introduced in [DK_3, Chap. I] to the abstract case.

Definition 1 ([S_1, II.4.1]). A map $f: X \to Y$ between spaces is called *separated* if the diagonal map $\Delta_f = (\mathrm{id}, \mathrm{id}): X \to X \times_Y X$ is closed.

A geometric space M over R is called separated if it is Hausdorff in its strong topology.

Proposition 3.1. Let M be a geometric space over R. Then M is separated if and only if the map $f: \tilde{M} \to \mathrm{Sper}\, R$ from $\tilde{M}$ to the one-point-space $\mathrm{Sper}\, R$ is separated.

This is Theorem III.3.1 in [S_1].

From now on all geometric spaces are assumed to be separated.

Definition 2 (cf. [DK_3, I, §4, Def. 2]). An abstract space X is called *paracompact* if it possesses a locally finite open affine covering $(X_i \mid i \in I)$.
Here „locally finite" has its usual meaning in topology: Every $x \in X$ has a neighbourhood U which meets only finitely many sets X_i.

Remark 3.2. Since semialgebraic subsets of X are quasicompact, an open covering $(X_i \mid i \in I)$ of X is locally finite if and only if, for every $U \in \mathring{\gamma}(X)$, all but finitely many sets X_i have empty intersection with U.

An immediate consequence of the definition is

Proposition 3.3. Every locally semialgebraic subspace of a paracompact space is paracompact.

Proposition 3.4. An abstract space X is paracompact if and only if every open covering $(X_i \mid i \in I)$ posesses a locally finite refinement $(Y_j \mid j \in J)$ with $Y_j \in \mathring{\gamma}(X)$.

Proof. See [DK_3, I, Prop. 4.5].

N.B. Despite Prop. 3.4 our notation of paracompactness differs from the usual one in topology since abstract spaces are almost never Hausdorff.

Definition 3 (cf. [DK_3, I, §4, Def. 3]). An abstract space X is called *Lindelöf* if it posesses an open covering $(X_i \mid i \in \mathbb{N})$ by countably many semialgebraic subsets $X_i \in \mathring{\gamma}(X)$.

Proposition 3.5. A space X is Lindelöf if and only if every open covering $(X_i \mid i \in I)$ of X has a countable refinement $(Y_j \mid j \in \mathbb{N})$.

Proof. See [DK_3, I, Prop. 4.16].

Proposition 3.6. Let X be a paracompact connected abstract space. Then X is Lindelöf.

Proof. See [DK_3, I, Prop. 4.17].