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Edited by A. Dold and B. Eckmann

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Adviser: Roberto Conti

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Nonlinear Diffusion Problems

Montecatini Terme 1985

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Lectures given at the 2nd 1985 Session of
the Centro Internazionale Matematico Estivo
(C.I.M.E.) held at Montecatini Terme, Italy
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Edited by A. Fasano and M. Primicerio



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PREFACE

This volume contains the texts of the three series of lectures given at the C.I.M.E. Session on "Some Problems in Nonlinear Diffusion" held at "La Querceta", Montecatini, from June 10 to June 18, 1985.

The general theme of the session was the study of the effects of nonlinearity in diffusion problems. Two main topics were considered: diffusion problems with degeneracy (such as in the porous media equation), and reaction-diffusion problems.

The first topic has been treated in the lectures by prof. Donald G. Aronson (University of Minnesota, Minneapolis). He considered a variety of aspects, ranging from physical background to regularity and asymptotic behaviour of solutions, also including peculiar subjects like waiting times and Hamilton-Jacobi equation.

Mathematical modelling of reaction-diffusion problem with reference to the chemical engineering applications has been illustrated by prof. Ivar Stakgold (University of Delaware, Newark). Various types of approximations have been discussed and the corresponding mathematical aspects have been investigated devoting special attention to the possible formation of dead cores.

Steady state processes in reaction-diffusion have been the main subject of the lectures by prof. Jesus Hernandez (Universidad Autonoma, Madrid). His large overview of qualitative methods covers in particular comparison arguments, the stability of solutions, and the use of topological degree theory.

The volume is complemented by a seminar on "rearrangements of functions and partial differential equations" which was presented by prof. Giorgio Talenti (Università di Firenze).

We wish to thank the lecturers and the participants, as well as the CIME scientific committee. We feel that the Session was quite successful for the interest shown by the audience and the extremely high quality of the lectures delivered.

A. Fasano

M. Primicerio

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THE POROUS MEDIUM EQUATION

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Introduction

These lectures are intended as a brief introduction to the mathematical study of non-linear diffusion mechanisms. Rather than attempt a comprehensive survey, I have elected to focus my attention on the so called porous medium equation. This allows me to convey the broad outlines of the theory without too many technicalities. Even with this restriction, it is impossible in eight lectures to cover everything of importance that has been done in the past twelve or so years. The choices I have made reflect my personal taste and should not be taken as a value judgement on the omitted material. Because of the volume of material covered, the level of proof is quite variable. Some results are presented without proof, some with fairly detailed sketches of proof, and still others with only cryptic hints. In any event, references to the main sources are always provided. Various time constraints have made it impossible for me to make an extensive revision of my original lecture notes. Thus the text presented here, though lacking in polish, is very close to what was actually said in the lectures.

Outline

- Lecture 1. Physical background. Selfsimilar solutions. Basic existence and uniqueness theory.
- Lecture 2. Basic estimates. Regularity results for one dimensional flows.
- Lecture 3. The interface in one dimensional flow: waiting times, local smoothness, corner points, asymptotic behavior, ultimate smoothness.
- Lecture 4. The porous medium equation as a finite speed approximation to a Hamilton-Jacobi equation.
- Lecture 5. Regularity in d-dimensional flow: global Holder continuity, ultimate Lipschitz continuity, local counterexamples.
- Lecture 6. Initial trace theory.
- Lecture 7. Asymptotic behavior of solutions to initial value and boundary value problems in $\mathbb{R}^d$.
- Lecture 8. Stabilization theory.

Lecture 1

PHYSICAL BACKGROUND. Consider an ideal gas flowing isentropically in a homogeneous porous medium. The flow is governed by the following three laws [M].

Equation of state: $p = p_0 \rho^\alpha$,

where $p = p(x, t)$ is the pressure, $\rho = \rho(x, t)$ is the density, and $\alpha \in [1, \infty)$ and $p_0 \in \mathbb{R}^+$ are constants. Here $x \in \mathbb{R}^d$ for some $d \geq 1$.

Conservation of mass: $\kappa \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0$,

where $\vec{v} = \vec{v}(x, t)$ is the velocity vector and $\kappa \in \mathbb{R}^+$ is the porosity of the medium (i.e., the volume fraction available to the gas).

Darcy's Law: $\kappa \vec{v} = -\mu \nabla p$,

where $\nu \in \mathbb{R}^+$ is the viscosity of the gas and $\mu \in \mathbb{R}^+$ is the permeability of the medium.

Note that Darcy's law is an empirically derived law [D] which replaces the usual conservation of momentum in the standard (Navier-Stokes) description of gas flow.

If we eliminate p and $\vec{v}$ from the equations and scale away all of the resulting constants we obtain the porous medium equation

$$\frac{\partial u}{\partial t} = \Delta(u^m), \quad (1)$$

where $m = 1 + \alpha \geq 2$. The quantity u represents a scaled density and so it is natural to assume that $u \geq 0$.

Equation (1) arises in many other applications, e.g., in the theory of ionized gases at high temperature [ZR] for values of $m > 1$, and in various models in plasma physics [BH] for values of $m < 1$. Of course, for $m = 1$ equation (1) is the classical equation of heat conduction. In these lectures I will focus on the case $m > 1$.

Other models (e.g., ground water flow) lead to equations similar to (1), but with u^m replaced by a more general nonlinear term $\varphi(u)$. I will concentrate on the porous medium case since it serves as a paradigm for the more general theory and its theory is much more complete. Finally, many problems lead to porous medium type equations with source or drift terms:

$$\frac{\partial u}{\partial t} = \Delta \varphi(u) + A \cdot \nabla \psi(u) + \sigma(u).$$

Examples occur in ground water problems and in population dynamics problems. References can be found in [Ar4], [BP] and [P]. I will discuss some aspects of the population case in the last two lectures.

If we compute the Laplacian in (1) the result is

$$\Delta(u^m) = \operatorname{div}(m u^{m-1} \operatorname{grad} u).$$

Thus equation (1) is uniformly parabolic in any region where u is bounded away from zero, but is degenerate in the neighborhood of any point where $u = 0$. In terms of standard Fickian diffusion theory, the diffusivity $m u^{m-1}$ vanishes with u . The most striking manifestation of this nonlinear degeneracy is that in porous medium flow there is a finite speed of propagation of disturbances from rest. This is in stark contrast to the linear heat equation ($m = 1$) where there is an infinite speed of propagation.

SELF-SIMILAR SOLUTIONS. There are several explicit self-similar solutions of the porous medium equation. It is useful to look briefly at some of them since they provide a preview of much of the theory.

An important class of self-similar solutions can be found by assuming that u has the form

$$u(x, t) = (t_0 \pm t)^{-\alpha} f(\xi)$$

with

$$\xi \equiv x(t_0 \pm t)^{-\beta}$$

where $t_0 \in \mathbb{R}$ is arbitrary and $t_0 \pm t > 0$. The numbers α and β as well as the function f must be determined. Substituting in (1) yields

$$(t_0 \pm t)^{-m\alpha-2\beta} \Delta(f^m) = \pm (t_0 \pm t)^{-\alpha-1} (-\beta \xi \cdot \nabla f - \alpha f).$$

Thus, if

$$(m-1)\alpha + 2\beta = 1 \tag{2}$$

we obtain a partial differential equation for $f = f(\xi)$:

$$\Delta(f^m) \pm (\beta \xi \cdot \nabla f + \alpha f) = 0. \tag{3}$$

Some further restrictions are needed in order to fix α and β .

Barenblatt Solution [Bl]. The Barenblatt solution of (1) is a radially symmetric self-similar solution of the form

$$U(x, t; M) = t^{-\alpha} \{ (A-B|x|^2 t^{-2\beta})_+ \}^{1/(m-1)} \equiv t^{-\alpha} F(|x| t^{-\beta})$$

which satisfies

$$\int_{\mathbb{R}^d} U(x, t; M) dx = M \quad \text{for all } t \in \mathbb{R}^+ \tag{4}$$

and arbitrary $M \in \mathbb{R}^+$. Here $(\cdot)_+ = \max(\cdot, 0)$. In order that (2) and (4) hold we must have $\alpha = \beta d$ with

$$\beta = \frac{1}{2 + (m-1)d},$$

and

$$\omega_d \int_0^\infty F(\zeta) \zeta^{d-1} d\zeta = M, \tag{5}$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. If

$$B = (m-1)\beta/2m$$

then $f(\xi) = F(|\xi|)$ is a solution to (3) for $|\xi| \neq (A/B)^{1/2}$ with arbitrary $A \in \mathbb{R}^+$. Finally, the value of A is determined by the condition (5). Specifically, A must satisfy

$$\omega_d A^{1/2\beta(m-1)} B^{-d/2} \int_0^{\pi/2} (\cos \theta)^{\frac{m+1}{m-1}} (\sin \theta)^{d-1} d\theta = M.$$

The Barenblatt solution U is a classical solution of (1) and, indeed, a C^∞ function on the set

$$\theta[U] \equiv \{(x, t) \in \mathbb{R}^d \times \mathbb{R}^+ : U(x, t) > 0\}.$$

Set

$$r(t) \equiv A^{1/2} t^\beta / B^{1/2}.$$

Then, clearly,

$$\theta[U] = \{(x, t) \in \mathbb{R}^d \times \mathbb{R}^+ : |x| < r(t)\}.$$

The set

$$I[U] \equiv \{(x, t) \in \mathbb{R}^d \times [0, \infty) : |x| = r(t)\}$$

is called the interface (or free boundary) since it is the boundary of

$$\text{supp } U = \text{Cl } \theta[U].$$

Actually, U is a classical solution of (1) in $(\mathbb{R}^d \times \mathbb{R}^+) \setminus I$, but it is not a classical solution in all of $\mathbb{R}^d \times \mathbb{R}^+$ since $\nabla(U^{m-1})$ has jump discontinuities across I . As we shall see later on, U is a solution of (1) in $\mathbb{R}^d \times \mathbb{R}^+$ in the appropriate generalized or weak sense and is uniquely determined by the initial values on $t = 0$.

Note that $\text{supp } U(\cdot, t; M) = \text{Cl } B_{r(t)}(0)$ increases monotonically with t at the finite rate $\dot{r}(t)$ for $t > 0$. As $t \downarrow 0$, $\text{supp } U(\cdot, t)$ shrinks to the origin and it follows from (4) that $U(\cdot, t; M) dx \rightarrow M \delta_0(dx)$, i.e., initially the Barenblatt solution is a multiple of the Dirac measure concentrated at $x = 0$. It is a worthwhile exercise to show that $U(x, t; 1)$ approaches the fundamental solution of the heat conduction equation as $m \downarrow 1$.

The Barenblatt solution can be embedded in a two parameter family of selfsimilar solutions by using the scale invariance properties of the porous medium equation. If $u(x, t)$ is a solution of (1), then for any positive constants p and q

$$w(x, t) \equiv \left(\frac{q}{2}\right)^{\frac{1}{m-1}} \frac{1}{p^2} u(px, qt)$$

is also a solution. Set

$$W(x, t; M, p, q) \equiv \left(\frac{q}{2}\right)^{\frac{1}{m-1}} \frac{1}{p^2} U(px, qt; M).$$

Then W is a selfsimilar solution of (1) with

$$\int_{\mathbb{R}^d} W(x, t; M, p, q) dx = M \left(\frac{q}{2^{d(m-1)}}\right)^{\frac{1}{m-1}},$$

i.e., with

$$W(\cdot, t; M, p, q) \rightarrow M \left(\frac{q}{2^{d(m-1)}}\right)^{\frac{1}{m-1}} \delta_0(dx)$$

as $t \downarrow 0$.

It follows from the equation of state that the pressure corresponding to the scaled density u is proportional to u^{m-1} . Since pressure and velocity are related by Darcy's law, the scaled pressure

$$v \equiv \frac{m}{m-1} u^{m-1} \quad (6)$$

will play a very important role in the development of the theory. Formally, the equation for the pressure is

$$\frac{\partial v}{\partial t} = (m-1)v \Delta v + |\nabla v|^2. \quad (7)$$

The pressure corresponding to the Barenblatt solution is given by

$$V(x, t; M) \equiv \frac{\beta}{2t} \{r^2(t) - |x|^2\}_+.$$

Observe that V is continuous, while V_t and ∇V are bounded but have jump discontinuities across I .

Quadratic Pressure Solution. Set $\alpha = 1/(m-1)$, $\beta = 0$, and

$$t_0 = \frac{m-1}{2m[2+d(m-1)]}.$$

We seek a selfsimilar solution of (1) of the form

$$(t_0 - t)^{-\frac{1}{m-1}} f(x) = (t_0 - t)^{-\frac{1}{m-1}} F(|x|).$$

It is not difficult to verify that $f(x) = (t_0 |x|^2)^{\frac{1}{m-1}}$ is a solution of (3) so that

$$\hat{U}(x, t) \equiv \left(\frac{t_0 |x|^2}{t_0 - t} \right)^{\frac{1}{m-1}}$$

is a solution of the required form. Note that $\hat{U}$ is a classical solution of (1) in $\mathbb{R}^d \times (0, t_0)$ with initial values

$$\hat{U}(x, 0) = |x|^{\frac{2}{m-1}}.$$

However, $\hat{U}(\cdot, t) \rightarrow +\infty$ as $t \uparrow t_0$ in $\mathbb{R}^d \setminus \{0\}$. The pressure corresponding to $\hat{U}$ is given by

$$\hat{V}(x, t) \equiv \frac{mt_0 |x|^2}{(m-1)(t_0 - t)}.$$

Using scale invariance, we can embed $\hat{U}$ in a one parameter family of solutions. In particular, for any $q > 0$

$$\hat{W}(x, t; q) \equiv q^{\frac{1}{m-1}} \hat{U}(x, q, t) = \left(\frac{t_0 |x|^2}{\frac{t_0}{q} - t} \right)^{\frac{1}{m-1}}$$

is a selfsimilar solution with quadratic pressure.

For $d = 1$, define

$$\mathcal{U}(x, t) = \begin{cases} \hat{U}(x, t) & \text{in } (-\infty, 0] \times [0, t_0) \\ 0 & \text{in } \mathbb{R}^+ \times [0, t_0). \end{cases}$$

Then, as we shall see later, $\hat{u}$ is the weak solution of (1) in $\mathbb{R} \times [0, t_0]$ with initial values

$$\hat{u}(x, 0) = \begin{cases} |x|^{\frac{2}{m-1}} & \text{in } (-\infty, 0] \\ 0 & \text{in } \mathbb{R}^+. \end{cases}$$

In this case $\text{supp } \hat{u} = (-\infty, 0] \times [0, t_0]$ and the interface is the line segment $x = 0$ for $t \in [0, t_0]$. This example shows that the solution of (1) is not necessarily global in time, and that the support of a solution may not expand for some positive time.

Linear Pressure Solution. Take $d = 1$, $\alpha = -1/(m-1)$, $\beta = 1$ and $t_0 = 0$. We look for a solution of (1) in the form

$$t^{\frac{1}{m-1}} f\left(\frac{x}{t}\right) \text{ in } \mathbb{R} \times \mathbb{R}^+.$$

It is easy to verify that for arbitrary $\gamma > 0$

$$\tilde{u}(x, t) \equiv \left\{ \frac{m-1}{m} \gamma t \left(\gamma \pm \frac{x}{t} \right) \right\}^{\frac{1}{m-1}}$$

is such a solution. The corresponding pressure

$$\tilde{v}(x, t) = \gamma \left(\gamma t \pm x \right)_+$$

is a linear wave.

Additional information about selfsimilar solutions and further references can be found in [ZR], [B2], and [PG].

BASIC THEORY As is indicated by the various selfsimilar solutions we have discussed, we cannot expect to find a classical solution to the initial value problem for the porous medium equation, at least if we allow $u(x, 0)$ to be zero at some points. Thus we need some notion of generalized solution.

To be definite, consider the 1-dimensional initial value problem

$$\begin{aligned} u_t &\equiv (u^m)_{xx} \quad \text{for } (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(\cdot, 0) &= u_0 \quad \text{for } x \in \mathbb{R}, \end{aligned} \tag{8}$$

where u_0 is a given nonnegative function. A continuous nonnegative bounded function $u = u(x, t): \mathbb{R} \times \mathbb{R}^+ \rightarrow [0, \infty)$ is said to be a generalized solution of (7) if $(u^m)_x$ exists and is bounded in the sense of distributions, and if for every $T \in \mathbb{R}^+$

$$\int_{\mathbb{R} \times (0, T)} u \psi_t - \psi_x (u^m)_x + \int_{\mathbb{R}} \psi(x, 0) u_0 = 0$$