Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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F. Ghione C. Peskine E. Sernesi (Eds.)

Space Curves

Proceedings, Rocca di Papa 1985

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INTRODUCTION

The conference on "Curves in Projective' Space" held in Rocca di Papa during the week 3-8 June 1985 was organized by the "II Università di Roma" and partly supported by the "C.N.R.".

The main subjects studied were:

a) Good and bad families of projective curves.

b) Postulation of projective space curves.

c) Classical problems in enumerative geometry.

During this meeting many formal and informal lectures were held and most of them, transmitted to us in a written form, are hereby published.

We take pleasure in expressing our gratitude to all partecipants for the excellent working atmosphere we enjoyed at the conference and many thanks to the secretaries who made it possible. Unfortunately one of them, Mrs. Gigliola Guzzardi, died prematurely last year and we always think affectionately of her.

We specially thank Mrs. Nicoletta Mantovani for the beautiful typing job of this volume.

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BEYOND THE MAXIMAL RANK CONJECTURE FOR CURVES IN $\mathbb{P}^3$

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INTRODUCTION

We work on an algebraically closed field of characteristic zero. A curve C in $\mathbb{P}^3$ is said to be of *maximal rank* if for any $k \geq 1$, the natural map of restriction $r_C(k): H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k)) \to H^0(C, \mathcal{O}_C(k))$ is surjective or injective. Here we prove the so called " *maximal rank conjecture for curves in* $\mathbb{P}^3$" (see |4|). Namely the following:

Theorem 1: *If* $d \geq (3g+12)/4$, $g \geq 0$, *there exists a smooth connected curve in* $\mathbb{P}^3$ *of degree* d, *genus* g, *with general moduli and of maximal rank.*

This theorem will be (almost) a by-product of a more general result (theorem 2) which we now describe. For $d \geq (g+9)/2$ we define an irreducible component, $W(d,g)$, of Hilb $\mathbb{P}^3$ whose general element is a smooth, connected curve with $h^1(C, N_C) = 0$. Hence $W(d,g)$ is generically reduced of dimension 4d. Then we prove that $W(d,g)$ has the right number of moduli. Indeed let $p: W(d,g) \to \mathcal{M}_g$ be the rational map induced by the universal curve on $W(d,g)$, we show that p is dominant if $4d-3g-12 \geq 0$ while $\dim(p(W(d,g))) = \dim W(d,g) - \dim \mathrm{Aut}(\mathbb{P}^3)$ otherwise. The notion of components with the right number of moduli has been introduced by Sernesi who proved their existence for $d \geq (g+18)/3 (d \geq 9)$ (see |10|). It seems to be the useful generalization of the notion of "general moduli". Now our main result is the following:

Theorem 2: *There exists a function* $u: \mathbb{N} \to \mathbb{R}_+$ *with* $\lim_{g \to +\infty} u(g) = 1/2$ *such that for every* $d \geq g \cdot u(g)$, *a general element of* $W(d,g)$ *has maximal rank.*

It seems likely that theorem 2 is not the best possible. We don't know what should be the best bound but we make the following:

Conjecture: *There exists a constant* K *such that for all natural numbers* d,g *with* $g \le Kd^{3/2}$ *there exists an irreducible component,* A(d,g) , *of* Hilb $\mathbb{P}^3$ *whose general element is a smooth, connected curve of degree* d , *genus* g *of maximal rank.*

Moreover it seems reasonable to expect that A(d,g) has the right number of moduli. The present paper is organized as follows. In section I we introduce W(d,g) and prove it has the right number of moduli. In section III, IV,... we make an inductive construction which proves theorem 2. The idea of the inductive procedure is taken from |5|, |7|: thank you very much! Theorem 2 proves the first theorem except for a finite number of (d,g) which are considered in X .

I) GOOD COMPONENTS OF THE HILBERT SCHEME.

In this section we describe components of the Hilbert scheme in which we would like to find curves of maximal rank.

I.1: **Definition:** *We denote by* Z(d,g) *the closure in* Hilb $\mathbb{P}^3$ *of the set of non-singular curves of degree* d *and genus* g .

Note that for $d \ge g+3$, Z(d,g) is irreducible |3|.

I.2: **Lemma:** *Let* X *be a reduced, locally complete intersection curve with* $h^1(N_X)=0$. *Let* L *be a* k*-secant line to* X *with* $1 \le k \le 3$. *If* k=3 *assume that the three tangents to* X *at the points of* X *are not coplanar. Then* $h^1(N_{X \cup L})=0$. *Futhermore if* X *is smoothable (i.e.* X *is in* Z(d,g) *for suitable* d,g*) then* $X \cup L$ *is smoothable too.*

Proof: See |10|, |6|, |1|.

I.3: **Definition:** *If* $d \ge 7$ *and* $d-3 < g \le 2d-9$ *write* d=s+t+3 *and* g=s+2t , $s \ge 3$, $t \ge 1$. *Denote by* C *a rational normal curve. Let* $D_1,\dots,D_s$ *be a disjoint* 2*-secant lines to* C . *Let* X *be the union of* $C,D_1,\dots,D_s$ *and of* t *disjoint lines* $T_1,\dots,T_t$ *such that for each* $i: T_i \cap C = \emptyset$, $T_i \cap D_j \neq \emptyset$ *if and only if* $1 \le j \le 3$. *Then we define* W(d,g) *as the unique (see* I.2*) irreducible component of* Hilb $\mathbb{P}^3$ *containing* X .

For simplicity if $d \geq g+3$, $g \geq 0$ *we put* $W(d,g)=Z(d,g)$.

Note that $W(d,g)$ does not depend on the choice of C, D_j, T_i but only of the configuration.

I.4: **Lemma:** *Let* X *be a Cohen-Macaulay equidimensional curve in* $\mathbb{P}^3$. *Set* $N_X := \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^3}}(I_X, \mathcal{O}_X)$. *Then* $\chi(N_X)=4.\deg(X)$.

Proof : |8| p.95, lemma 2.2.11.

I.5: **Lemma:** *Let* X *be a smoothable curve which is reduced and generically a complete intersection with* $h^1(N_X)=0$. *Let* P *be a smooth point of* X . *Let* D,R *be distinct lines through* P *such that* D *is a 3-secant line to* X *and* R *is a* k*-secant line to* $X, 2 \leq k \leq 3$. *If* $k=3$ *assume furthermore that* P *is sufficiently general. Set:* $D \cap X = \{P, x_1, x_2\}$, $R \cap X = \{y_i\}_{1 \leq i \leq k}$ $y_1 = P$. *Assume that:* (i) $T_P X \cup D \cup R$ *spans* $\mathbb{P}^3$ (ii) $T_{x_1} X \cap T_{x_2} X = \emptyset$ (iii) *if* $k=3$, $T_{y_i} X$, $1 \leq i \leq 3$, *are not coplanar.*
Then $Y := X \cup D \cup R$ *is smoothable with* $h^1(N_y)=0$ *and hence is a smooth point of* Hilb $\mathbb{P}^3$.

Proof: We claim that Y is smoothable since R is specialization of k-secant lines to X which do not meet D. If $k=2$ this is clear. If $k=3$ we may assume that the projection of X from P is birational $(\mathrm{ch}(K)=0)$ and as good as possible, i.e the singularities of the projection will be m multiple points with genus defect $d_1,\ldots,d_m$ with $d_1=1$ (corresponding to an ordinary node, E) and d_i as small as possible. Consider the family of projections of X from points of X near P . The numbers $m, d_1, \ldots, d_m$ will be constant and there will be an ordinary node corresponding to E . This prove the claim. Now let's prove that Y is a smooth point of Hilb $\mathbb{P}^3$. According to I.4 it is enough to show that $h^1(N_Y)=0$. Set $Z := X \cup R$. By definition N_Z and N_Y are torsion free. Moreover: $N_Y|Z \supset N_Z$ and $N_Y|D \supset N_D$. Hence $h^1(N_Y|Z)=0$ and the Mayer-Vietoris sequence:

$$0 \longrightarrow N_Y \longrightarrow N_Y|Z \oplus N_Y|D \longrightarrow N_Y|Z\cap D \longrightarrow 0 \qquad (1)$$

is exact. Put $F := N_Y|D$ and let T be its torsion subsheaf. The exact sequence: $0 \to T \to F \to F/T \to 0$ splits because $\mathrm{Ext}^1(F/T,T)=0$. Indeed $\mathcal{E}xt^1(F/T,T)=0$ because

F/T is locally free and $h^1(D, Hom_D(F/T,T))=0$ because $Hom_D(F/T,T)$ has finite support. We have: $F|D \supset N_{X\cup D}|D = \mathcal{O}_D(a_1) \oplus \mathcal{O}_D(a_2)$ with $a_i \geq 2$ by the assumption (ii) (see |1|, Prop.4). Hence $N_Y||D = T \oplus \mathcal{O}_D(b_1 \oplus \mathcal{O}_D(b_2)$ with $b_I \geq 2$. Thus the restriction map: $H^o(N_Y|D) \to H^o(N_Y|Z\cap D)$ is surjective. From (1) we get $h^1(N_Y)=0$.

I.6: **Corollary**: *Let* C *be in* $W(d,g)$ *and let* $D_1,\ldots,D_n$ *be disjoint* 2-*secant lines to* C . *Set* $Y:=C\cup D_1\cup\ldots\cup D_n$. *Let* R_i , $1\leq i\leq t$, *be disjoint* k-*secant lines to* Y , $2\leq k\leq 3$. *If* $k=3$ *assume that* R_i *intersects* C *at most at one point. Then* $Y\cup R_i\cup\ldots\cup R_t$ *is in* $W(d',g')$ *for suitable* d',g' .

Proof : This follows from I.5 noting that we can freely more the lines D_i and then also the lines R_j .

I.7: **Components with the right number of moduli.**

Let $\rho(d,g,3)=4d-3g-12$ be the Brill-Noether number. Let H be an irreducible component of $Z(d,g)$. There is a natural rational map, p , from H to $\mathcal{M}_g$. In |10| Sernesi introduced the following notion:

I.7.1: **Definition:** *The irreducible component* H *has the right number of moduli if* $\dim(\mathrm{Im}(p))=\min\{3g-3, 3g-3+\rho(d,g,3)\}$.

Components of $Z(d,g)$ with the right number of moduli exist for: $3d\geq g+18$ $(d\geq 9)$ |10| .

I.8: **Proposition:** $W(d,g)$ *has the right number of moduli.*

Proof: Denote by $W'(d,g)$ the open subset of $W(d,g)$ consisting of semistable curves and let $p(d,g)$ denote the natural map from $W'(d,g)$ to $\mathcal{M}_g$. We may assume that $d<g+3$, $d\geq 7$ and the result true for $W(d-3,g-4)$. Note that $\rho(d+3,g+4,3)= =\rho(d,g,3)$. First assume $\rho(d,g,3)\geq 0$. We want to prove that the general fibers of $p(d,g)$ and $p(d-3,g-4)$ have the same dimension. Take a general curve, C , in $W'(d-3,g-4)$ with general moduli. Let D,D' be 2-secant lines to C and let D'' be a 1-secant to C such that: $D\cap D'=\emptyset$, $D\cap D''\neq\emptyset$, $D'\cap D''\neq\emptyset$. Then $Y=C\cup D\cup D'\cup D''$

is in $W'(d,g)$ (by I.6). Now the fiber $p(d,g)^{-1}(Y)$ is isomorphic to $p(d-3,g-4)^{-1}(C)$. If $\rho=\rho(d,g,3)<0$ is even then we start from a curve Z in $W'(d+\rho/2,g+\rho)$ at which the fiber of p has dimension 15. Then we add $\rho/2$ 3-secant lines to Z obtaining a stable curve with the same property. If $\rho=-(2k-1)$, we take Z' in $W'(d-k-1,g-2k-1)$ at which the fiber of p has dimension 15. Then we add one 2-secant and k 3-secant lines obtaining a stable curve with the same property.

II) **THE CRITICAL VALUE**

II.1: **The normal decomposition.**

For integers d,g with $g\geq 0$, $d-3\leq g\leq 2d-9$ there exist unique non-negative integers s,t such that: $d=s+t+3$, $g=s+2t$. The meaning of this decomposition is that we may obtain a curve in $W(d,g)$ as the union of a twisted cubic, C; s 2-secant lines, $D_1,\ldots,D_s$ to C and t suitable 3-secant lines to $C\cup D_1\cup\ldots\cup D_s$.

II.2: **Speciality and k-secant lines.**

Let M be a curve with $H^1(\mathcal{O}_M(2))=0$. Let L be a k-secant line to M, $1\leq k\leq 3$. By a Mayer-Vietoris exact sequence it follows that $h^1(\mathcal{O}_{M\cup L}(2))=0$. Hence all the curves we will consider in this paper will have $h^1(\mathcal{O}_C(2))=0$.

II.3: **The critical value.**

For a couple (d,g) we define its critical value $m:=v(d,g)$ by: $m=\min\{k\geq 2/kd-g+1\leq\binom{k+3}{3}\}$. If C is a curve of degree d, genus h we set $v(C)=v(d,g)$. If $m=2$ then $d\leq 7$. These cases are well known so we may assume $m\geq 3$. From II.2 and |9| p.99, a suitable curve, X, in $W(d,g)$ is of maximal rank if and only if $r_X(m-1)$ is injective and $r_X(m)$ is surjective. Finally to show the existence of a curve of maximal rank in $W(d,g)$ it is enough to show the existence of one curve, Z, with $r_Z(m)$ surjective and of one curve, Z', with $r_{Z'}(m-1)$ injective. In order to construct Z,Z' we will consider a general construction $(C;d,g)$ (see III, IV). We will show in details how this construction proves the existence of $Z(V)$. In

VIII we will indicate the modifications needed for the injective case.

III) THE CONSTRUCTION (C;d,g): NUMERICAL SETTING.

Troughout this section (d,g) is a fixed couple of integers with $g \geq 0$, $d-3 < g \leq 2d-9$. Also m denotes the critical value of (d,g) (see II.3).

III.1: Associated degree and genus.

The associate degree and genus to (d,g;m), (d',g'), is defined as follows: by definition of m we have: $\binom{m+3}{3} = md-g+1+q$, $q \geq 0$.
If $q < m-2$ then $(d',g')=(d,g)$. If $q \geq m-2$ let $q=a(m-2)+b$, $0 \leq b \leq m-3$. Then: $d'=d+a$, $g'=g+2a$. Note that (d',g') is the unique couple of integers such that: $2(d'-d)=g'-g$ and $\binom{m+3}{3}-m+3 \leq md'-g'+1 \leq \binom{m+3}{3}$. Also $d'=3+s+t'$, $g'=s+2t'$ with $t' \geq t$ (where s,t is the normal decomposition of d,g, see II.1). The geometric meaning of d',g' is as follows: they are the degree and genus of a curve, C', obtained by adding $(d'-d)$ 3-secant lines to a curve of degree d, genus g in such a way that $v(C')=m$. Furthermore d' is maximal with this property.

III.2: The integers d(n), b(n).

In |2| we defined integers $d(n)$, $b(n)$ by the relations: $n,d(n)+1-(d(n)-3+b(n)= =\binom{n+3}{3}$, $0 \leq b(n) \leq n-2$; $d(0)=3$, $b(0)=0$; $d(1)=4$, $b(1)=0$. According to the value of n mod.6 we have:

$d(6k) = 6k^2+7k+3$, $b(6k)=0$; $d(6k+1)=6k^2+9k+4$, $b(6k+1)=2k$

$d(6k+2)=6k^2+11k+6$, $b(6k+2)=0$; $d(6k+3)=6k^2+13k+8$, $b(6k+3)=0$

$d(6k+4)=6k^2+15k+10$, $b(6k+4)=2k+1$; $d(6k+5)=6k^2+17k+13$, $b(6k+5)=0$

Recall that the $d(n)$'s are the critical degrees for curves in $W(d,d-3)$ (|2|).

III.3: **The integer** r .

With notations as above we derive from (d',g') a further integer, r : r is the maximal non-negative integer k such that:

$k \leq m$ and $m-k$ is even $(\bullet)$

$s-d(k)+3 \geq 2(m-k)-1 \qquad (\bullet\bullet)$

If $m \geq 12$ then r is well defined (it is enough to check $s \geq 2m-3$). We may assume $r \geq 1$. Note that $(\bullet\bullet)$ may be written as: $2(d'-d(r))-2(m-r-2)+d(r)-3 \geq g'$. We have: $s-d(r)+3=2(m-r-2)+a$ with $a \geq 3$. So we see that if we add: $d'-d(r)-2(m-r-2)$ 3-secants and $2(m-r-2)$ 2-secants to a curve of degree $d(r)$, genus $d(r)-3$, we obtain a curve of degree d' and genus greater or equal to g' . To have a curve of genus g' we have to add less 3-secants, more precisely, adding $d'-d(r)-2(m-r-2)-a$ suitable 3-secants and $2(m-r-2)+a$ suitable 2-secants to a curve of degree $d(r)$, genus $d(r)-3$, we obtain a curve of $W(d',g')$.

III.4: **The intermediate degrees and genus** $x(n)$, $g(n)$.

Now we will try, starting from $d(r)$, $d(r)-3$, to reach d',g' . For every integer n with: $r \leq n \leq m$, $m-n$ even, we define integers $x(n)$, $y(n)$ by: $x(r)=d(r)$, $y(r)=b(r)$, $g(r)=d(r)-3$. If $n \geq r+2$: $n, x(n)+1-(2x(n)-d(r)-3-a-2(n-r-2))+y(n)=\binom{n+3}{3}$, $0 \leq y(n) \leq \max(0,n-3)$. Also we set: $g(n)=2x(n)-d(r)-3-a-2(n-r-2)$.

Now for every n with $m-n$ even and $r+2 \leq n \leq m$ we want to find inductively a curve $X(n) \in W(x(n),g(n))$ with $r_{X(n)}(n)$ surjective (i.e. $\dim(\ker r_{X(n)}(n))=y(n)$) . For this we will start from a curve, Y , satisfying $H(r)$ (see IV) hence $\deg(Y)=$ $=d(r), g(Y)=d(r)-3$. At the first step we will add $x(r+2)-d(r)-a$ 3-secants and a 2-secants to Y (see $H(r) \Rightarrow R(r+2)$, VI.2,3). This will give us $X(r+2)$. Then for $n \geq r+4$, $X(n)$ will be obtained from a curve in $W(x(n-2),g(n-2))$ by adding four 2-secants and $x(n)-x(n-2)-4$ 3-secants (VI.4,5,6). Finally after $(m-r)/2$ steps we will get a curve, $X(m)$ of degree d' , genus g' with $r_{X(m)}(m)$ surjective. Then taking off (if possible, see V.2) $(d'-d)$ 3-secants lines from $X(m)$ we will get the desired curve Z' with: $Z' \in W(d,g)$ and $r_{Z'}(m)$ surjective.

IV) **THE INDUCTIVE HYPOTHESIS FOR** $(C;d,g)$.

The two inductive statements we use for $(C;d,g)$ are:

IV.1: **H(r) , $r \geq 0$.**

There exists (Y,Z,D,S) such that:

(1) Y is the union of a smooth curve Z with $Z \in W(r+3,r)$ if r is even (resp. $Z \in W(r+2,r-1)$ if r is odd) and of $d(r)-r-3$ (resp. $d(r)-r-2$) disjoint 2-secant lines to Z .

(2) D is a 2-secant line to Z , $S \subset D \setminus (Y \cap D)$ is a finite subset with $\mathrm{card}(S)=b(r)$

(3) $h^{0}(\mathcal{I}_{Y \cup S}(r))=0$.

IV.2: **R(n;d,g) , $r+2 \leq n \leq m$, $m-n$ even.**

For every integer y with $0 \leq 2y \leq y(n)$ there exists $(X(n),Z,S,S',D,D',R,R',Q)$ such that:

(1) $X(n) \in W(x(n),g(n))$, $X(n)$ is the union of $Z \in W(x(n)-2(m-n),g(n)-2(m-n))$ and of $2(m-n)$ 2-secant lines to Z . Moreover if $n=m, X(m)$ is the union of a curve Z' in $W(x(m-2),g(m-2))$ and of t 3-secants to Z' and four 2-secants to Z'.

(2) D,D',R,R' are disjoint 2-secant lines to $X(n)$; they intersect different lines of $X(n) \setminus Z$ each of then at one point; $S \subset D$, $S' \subset D'$ are finite subsets, $\mathrm{card}(S)=y(n)-y$, $\mathrm{card}(S')=y$.

(3) Q is a smooth quadric surface containing D,D',R,R' and with $\dim(X(n) \cap Q)=0$. If $p:Q \to D$ denotes the natural projection then $p(S')$ $S \cap p[Z \cap (Q \setminus (D \cup D' \cup R \cup R'))]$

(4) $h^{0}(\mathcal{I}_{X(n) \cup S \cup S'}(n))=0$.

V) **R(m;d,g) IMPLIES THE SURJECTIVITY OF $r_C(m)$ FOR GENERAL C IN W(d,g) .**

V.1: **Lemma:** *Assume* $R(m;d,g)$ *true. If* $m \geq 21$ *and if* C *is sufficiently general in* $W(d,g)$ *then the natural map* $r_C(m)$ *is surjective.*

Proof : From $R(m;d,g)$ we get a curve $X(m)$ in $W(x(m),g(m))$ with $r_{X(m)}(m)$ surjective. Taking off $(d'-d)$ 3-secant lines from $X(m)$ we get a curve Y in $W(d,g)$ with $r_Y(m)$ surjective (note that $x(m)=d'$). But for this we need: $x(m)-x(m-2)-4 \geq d'-d$. Since $v(d,g)=m$ we have $d \geq d(m-1)+1$. Hence for $m \geq 21$ the inequality above follows from V.2 below.

V.2: **Lemma:** *For* $m \geq 21$ *we have* $x(m-2) \leq d(m-1)-3$.

Proof : By definition: $(m-2)\cdot d(m-1)+4+b(m-1)=\binom{m+2}{3}$ and $(m-2)\cdot x(m-2)-g(m-2)+1+$ $+y(m-2)=\binom{m+1}{3}$. We have $g(m-2) \leq 2x(m-2)-6$. If $x(m-2) \geq d(m-1)-2$ we get $\binom{m+1}{2} \leq$ $\leq 2d(m-1)+2m-11+b(m-1)-y(m-2)$ $(\cdot)$. Since $b(m-1) \leq m-3$ and $y(m-2) \geq 0$: $\binom{m+1}{2} \leq 2d(m-1)+$ $+3m-14$ $(\cdot\cdot)$. For $m \geq 7$: $d(m-1) \leq (m+3)^2/6$ (use III.2) and $(\cdot\cdot)$ reads: $0 \geq m^2-27m+66$ which is false if $m \geq 25$. For the remaining cases use $(\cdot)$ and the explicit values of $d(m-1)$, $b(m-1)$ (III.2).

VI) PROOF OF THE INDUCTIVE HYPOTHESIS.

VI.1: **Lemma:** *For* $r \geq 0$, $H(r)$ *is true.*

Proof : Note that for $r \geq 2$, $H(r)$ is stronger than the assertion called $H(r)$ in |2|. However it is possible to modify the proof of |2| lemma 6.1 (proof that $H(r-2)$ implies $H(r)$) and prove the assertion just stated. Indeed in the proof of |2| 6.1 we take (Y,Z,S,D) satisfying $H(r-2)$ just stated; we take another 2-secant line D' to Z . At the end of the construction we find $W \supset Y \cup D \cup D'$, $\deg(W)=d(r)$, $p_a(W)=d(r)-3$, with $h^0(\mathcal{I}_W(r))=b(r)$, W union of $Z \cup D \cup D'$ and 2-secant lines to $Z \cup D \cup D'$. Then we deform $Z \cup D \cup D'$ to a smooth curve Z' , $\deg(Z')=\deg(Z)+2$. We can deform the lines of $W \setminus (Z \cup D \cup D')$ obtaining $W' \supset Z'$ with $h^0(\mathcal{I}_{W'}(r))=b(r)$. To handle the postulation of $Y \cap Q$ for general Y satisfying $H(r-2)$ we use |2| §8 and the explicit values of $d(r-2)$, $b(r-2)$.

VI.2: **Lemma:** *If* $r \geq 5$, $d(r)-r-3 \geq 2(m-r-2)$ *and* $a \geq b(r)+2$ *then* $H(r)$ *implies* $R(r+2;d,g)$.

Proof : Take (Y,Z,D,S) satisfying $H(r)$ and fix y with $0 \leq 2y \leq y(r+2)$. Let D' be a 2-secant line to Z with $D \cap D' = \emptyset$. Let Q be a smooth quadric containing D,D' . We may assume that Q does not contain any further 2-secant line to Y . As $2\deg(Z)-2 \geq r+1$ there exist $F=\{P_1,\dots,P_{r+1}\}$ with $F \subset (Z \cap (Q \setminus D))$. Since any surface of degree r containing $r+1$ points of D contains D , there exists $F' \subset F$, $\mathrm{card}(F')=b(r)$, such that $h^0(\mathcal{I}_{Y \cup p(F')}(r))=0$ ($p:Q \to D$ is the projection). For convenience let us assume that $F'=\{P_1,\dots,P_{b(r)}\}$. On Q we consider $x(r+2)-d(r)$ disjoint lines, D_i , $-1 \leq i \leq x(r+2)$ $x(r+2)-d(r)-2$, satisfying the following conditions:

(a) each D_i intersects D

(b) for $-1 \leq i \leq 0$, D_i intersects a line, L_i , of $Y \cup Z$. Furthermore $L_{-1} \neq L_0$ and $(L_{-1} \cup L_0) \cap D \cup D')=\emptyset$

(c) for $1 \leq i \leq b(r)$: $D_i=[P_i,p(P_i)]$

(d) for $1 \leq i \leq x(r+2)-d(r)-2$, $D_i \cap (Y \setminus Z)=\emptyset$ and D_i meets Z if and only if $i \leq x(r+2)-d(r)-a+b(r)$.

Then we choose $T \subset D_0$, $T' \subset D_{-1}$ with $\mathrm{card}(T)=y(r+2)-y$, $\mathrm{card}(T')=y$, $T \cup T'$ disjoint from Y and such that $p'(T') \subset T \cap p'(Z \cap (Q \setminus (D \cup D')))$, p' is the projection on D_0 . To show that these choices are possible we have to check some numerical facts. Namely: (i) $\deg(Y \setminus Z) \geq 2$, (ii) $x(r+2)-d(r)-a+b(r) \leq x(r+2)-d(r)-2$, (iii) $2\deg(Z) \geq x(r+2)-d(r)-a-b(r)$, (iv) $b(r) \leq x(r+2)-d(r)-a+b(r)$, (v) $2\deg(Z) \geq y(r+2)$. Actually instead of (i) we will need $\deg(Y \setminus Z) \geq 4$ which is true because $d(r) \geq r+7$ for $r \geq 5$ (use III.2). Now (ii) is precisely the assumption $b(r)+2 \leq a$. For (iii), (v) it is enough to have $2\deg(Z) \geq r+2$ which is clearly true. For (iv) see VII.6. Now we set: $X:=Y \cup D \cup D' \cup D_1 \cup \dots \cup D_z (\bigcup_{j=1}^{b(r)} \chi(D_j \cap D))$ where $z=x(r+2)-d(r)-2$. Note that $X \in W(x(r+2),g(r+2))$ by sect.I and |2|, I.4. We want to show that $h^0(\mathcal{I}_{X \cup T \cup T'}(r+2))=0$ and that $X \cup T \cup T'$ can be deformed to a configuration satisfying $R(r+2)$. Set $K=[Y \cap (Q \setminus (D \cup D' \cup \bigcup_{i=1}^{z} D_i)] \cup T \cup T'$. We claim that $h^0(\mathcal{I}_{K,Q}(r,r+2-z))=0$. This follows from |2| §8. Indeed note that $r+2-z \geq 2$ (VII) and that we never are in an exceptional case of |2| §8 because $\deg(Z)$ is large. It follows that if $f \in H^0(\mathcal{I}_{X \cup T \cup T'}(r+2))$ then $f|Q=0$. Thus $f=qf'$ with $f' \in H^0(\mathcal{I}_{\mathrm{Res}_Q(X)}(r))$. Since $\mathrm{Res}_Q(X)=Y \cup p(F')$ we have $f'=0$ hence $f=0$ as wanted. Now let L_{-3} , L_{-2} be lines in $Y \setminus Z$ such that $L_i \neq L_j$ for $i \neq j$, $j \in \{-3,-2,-1,0\}$ (it is possible to find such lines because