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Edited by A. Dold and B. Eckmann

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Complete Intersections

Acireale 1983

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Lectures given at the 1st 1983 Session of the Centro Internationale Matematico Estivo (C.I.M.E.) held at Acireale (Catania), Italy, June 13–21, 1983

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cont

INTRODUCTION

This volume contains the proceedings of the CIME session on Complete Intersections held in Acireale (Catania, Italy) during the period 13-21 June, 1983.

The aim of the session was to present some ideas and techniques from Commutative Algebra, Algebraic Geometry and Analytic Geometry in connection with some problems on Complete Intersections.

The main courses were delivered by O. Forster, R. Lazarsfeld, L. Robbiano and G. Valla. The material developed in the lectures by Forster, Robbiano and Valla has been reshaped by the lecturers for these proceedings; the subject of Lazarsfeld's course is available elsewhere, hence the paper by Lazarsfeld included here contains some further developments and related topics, along with references for the lectures.

The volume contains also a number of original papers, chosen among the ones submitted for the proceedings.

Some of the results were announced during the meeting in special lectures delivered by C. Ciliberto, R. Froberg, S.Kleiman, D. Laksov, P. Valabrega, K. Watanabe.

We wish to thank all the contributors and participants, and the many referees for their collaboration. Our thanks must go also to the CIME for giving us the opportunity to have a meeting on this topic.

Silvio Greco
Rosario Strano

C.I.M.E. Session on "Complete Intersections"

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Complete intersections in affine algebraic varieties and Stein spaces

by

Otto Forster

Introduction. Let $(X,\mathcal{O}_X)$ be an affine algebraic variety (or an affine scheme, or a Stein space) and $Y \subset X$ a Zariski-closed (resp. analytic) subspace. We want to describe Y set-theoretically (or ideal-theoretically) by global functions, i.e. find elements $f_1,\ldots,f_N \in \Gamma(X,\mathcal{O}_X)$ such that

$$Y = \{x \in X: f_1(x) = \ldots = f_N(x) = 0\} ,$$

resp. such that $f_1,\ldots,f_N$ generate the ideal of Y (which is a stronger condition). The problem we consider here is how small the number N can be chosen. If in particular N can be chosen equal to the codimension of Y, then Y is called a set-theoretic (resp. ideal-theoretic) complete intersection.

In these lectures we discuss some results with respect to this problem in the algebraic and analytic case. In considering these cases simultaneously, it is interesting to note the analogies and differences of the methods and results. For this purpose we adopt also a more geometric point of view for the algebraic case. We hope that some proofs become more intuitive in this way.

1. Estimation of the number of equations necessary to describe an algebraic (resp. analytic) set

We begin with the following classical result on the set-theoretic description of an algebraic set.

1.1. Theorem (Kronecker 1882). Let $(X,\mathcal{O}_X)$ be an n-dimensional affine algebraic space (or the affine scheme of an n-dimensional noetherian ring) and $Y \subset X$ an algebraic subset. Then there exist functions $f_1,\ldots,f_{n+1} \in \Gamma(X,\mathcal{O}_X)$ such that

$$Y = V(f_1,\ldots,f_{n+1}) := \{x \in X: f_1(x) = \ldots = f_{n+1}(x) = 0\} .$$

Proof (due to Van der Waerden 1941). We prove by induction the following statement

(A.k) There exist $f_1,\dots,f_k \in \Gamma(X,\mathcal{O}_X)$ such that

$$V(f_1,\dots,f_k) = Y \cup Z_k ,$$

where Z_k is an algebraic subset of X with

$$\operatorname{codim} Z_k \geqq k.$$

The statement (A.0) is trivial, whereas (A.n+1) gives the theorem. So it remains to prove the induction step

(A.k) → (A.k+1). Let

$$Z_k = Z_k^1 \cup \dots \cup Z_k^s$$

be the decomposition of Z_k into its irreducible components. We may suppose that none of the Z_k^i is contained in Y. Choose a point $p_i \in Z_k^i \setminus Y$ for $i = 1,\dots,s$. Now it is easy to construct a function $f_{k+1} \in \Gamma(X,\mathcal{O}_X)$ with

$$f_{k+1}|Y = 0 \quad \text{and } f_{k+1}(p_i) \neq 0 \text{ for } i = 1,\dots,s.$$

Then $V(f_1,\dots,f_{k+1}) = Y \cup Z_{k+1}$ with

$$\operatorname{codim} Z_{k+1} > \operatorname{codim} Z_k \geqq k, \qquad \text{q.e.d.}$$

We want to give an example which shows that in general n equations do not suffice.

1.2. Example. Let $\overline{X}$ be an elliptic curve over $\mathbb{C}$, considered as a torus

$$\overline{X} = \mathbb{C}/\Gamma \ , \quad \Gamma \subset \mathbb{C} \text{ lattice.}$$

Let $p \in \overline{X}$ be an arbitrary point. Then

$$X := \overline{X} \setminus \{p\}$$

is a 1-dimensional affine algebraic variety. Let $Y := \{q\}$ with some $q \in X$. Let $P,Q \in \mathbb{C}$ be representatives of p and q respectively.

Claim. If $P-Q \notin \mathbb{Q}\cdot\Gamma$, then there exists no function $f \in \Gamma(X,\mathcal{O}_{X_{alg}})$ such that

$$Y = \{q\} = V(f).$$

Proof. Such a function f can be considered as a meromorphic function on $\overline{X}$, with poles only in p and zeros only in q. Let $k > 0$ be the vanishing order of f at q. Then k is also the order of the pole of f in p. Thus $k\cdot q - k\cdot p$ would be a principal divisor on $\overline{X}$. By the theorem of Abel, this implies

$$kQ - kP \in \Gamma .$$

But this contradicts our assumption $P - Q \notin \mathbb{Q}\cdot\Gamma$. Hence f cannot exist.

Remark. If we work in the analytic category, i.e. consider X as an open Riemann surface, then there exists a holomorphic function $f \in \Gamma(X,\mathcal{O}_{X_{an}})$ which vanishes precisely in q of order one. This is a special case of the theorem of Weierstraß for open Riemann surfaces, proved by Behnke/Stein 1948, that every divisor on an open Riemann surface is the divisor of a meromorphic function (see e.g.[ ]).

Open Riemann surfaces are special cases of Stein spaces, which are the analogue of affine algebraic varieties in complex analysis. A complex space $(X,\mathcal{O}_X)$ is called a Stein space, if the following conditions are satisfied:

i) X is holomorphically separable, i.e. given two points $x \neq y$ on X, there exists a holomorphic function $f \in \Gamma(X,\mathcal{O}_X)$ such that $f(x) \neq f(y)$.

ii) X is holomorphically convex, i.e. given a sequence $x_1, x_2, \ldots$ of points on X without point of accumulation, there exists $f \in \Gamma(X,\mathcal{O}_X)$ with $\limsup_{k\to\infty} |f(x_k)| = \infty$.

For the general theory of Stein spaces we refer to[ ].

In an n-dimensional Stein space, n equations always suffice to describe an analytic subset:

1.3. Theorem (Forster/Ramspott [12]). Let X be an n-dimensional Stein space and $Y \subset X$ a (closed) analytic subset. Then there exist n holomorphic functions $f_1,\ldots,f_n \in \Gamma(X,\mathcal{O}_X)$ such that

$$Y = V(f_1,\ldots,f_n).$$

Proof. We prove the theorem by induction on n. In order to do so, we have to prove a more precise version, namely, given a cherent ideal sheaf $\mathfrak{J} \subset \mathcal{O}_X$ with $V(\mathfrak{J}) = Y$, we can find functions $f_1,\ldots,f_n \in \Gamma(X,\mathfrak{J})$ such that $Y = V(f_1,\ldots,f_n)$.

n = 1. This is a little generalization of the Weierstraß theorem for open Riemann surfaces. It follows from the fact that for 1-dimensional

Stein spaces (which may have singularities) one has $H^1(X,\mathcal{O}_X^*) = H^1(X,\mathbb{Z}) = 0$.

<u>n-1 → n.</u> First one can find a function $f \in \Gamma(X,\mathfrak{I})$ such that

$$V(f) = Y \cup Z, \quad \text{where} \quad \dim Z \leq n-1.$$

Let $\mathfrak{J} \subset \mathcal{O}_Z$ be the image of $\mathfrak{I}$ under the restriction morphism $\mathcal{O}_X \to \mathcal{O}_Z$. Then $\mathfrak{J}$ is a coherent ideal sheaf with $V_Z(\mathfrak{J}) = Z \cap Y$, and we can apply the induction hypothesis to find $g_1,\ldots,g_{n-1} \in \Gamma(Z,\mathfrak{J})$ such that

$$Z \cap Y = V_Z(g_1,\ldots,g_{n-1}).$$

Since X is Stein, the morphism $\Gamma(X,\mathfrak{I}) \to \Gamma(Z,\mathfrak{J})$ is surjective. Let $f_1,\ldots,f_{n-1} \in \Gamma(X,\mathfrak{I})$ be functions that are mapped onto $g_1,\ldots,g_{n-1}$, then

$$Y = V_X(f_1,\ldots,f_{n-1},f), \qquad \text{q.e.d.}$$

As we have seen, in the algebraic case n equations do not suffice in general. However, if one can factor out an affine line from the affine algebraic variety, n equations will suffice.

<u>1.4. Theorem</u> (Storch [33], Eisenbud/Evans[7]). Let X be an affine algebraic space of the form $X = X_1 \times \mathbb{A}^1$, where X_1 is an affine algebraic space of dimension n-1 (or more generally X = Spec R[T] , where R is an (n-1)-dimensional noetherian ring). Then for every algebraic subset $Y \subset X$ there exist n functions $f_1,\ldots,f_n \in \Gamma(X,\mathcal{O}_X)$ such that

$$Y = V(f_1,\ldots,f_n).$$

In order to carry out the proof, we need a sharper version:
Let there be given an ideal $\mathfrak{a} \subset \Gamma(X,\mathcal{O}_X)$ such that $V(\mathfrak{a}) = Y$. Then the functions $f_1,\ldots,f_n$ can be chosen in $\mathfrak{a}$.
However, by the Hilbert Nullstellensatz the rough version of the theorem implies the sharper version.

<u>Proof</u> by induction on n. We may suppose X to be reduced.

<u>n = 1.</u> Then X_1 is a finite set of points, so X is a finite union of affine lines and the assertion is trivial.

<u>Induction step n-1 → n.</u> We have

$$(X,\mathcal{O}_X) = R[T] \ , \quad \text{where} \quad R = \Gamma(X_1,\mathcal{O}_{X_1}).$$

Let S be the set of non-zero divisors of R and

$K = Q(R) = S^{-1}$

the total quotient ring of R. We have

$K = K_1 \times \ldots \times K_r$

Where every K_j is a field. Let $\tilde{\mathfrak{a}} = \mathfrak{a}K[T]$. Since K[T] is a principal ideal ring, there is an $f \in R[T]$ such that $\tilde{\mathfrak{a}} = K[T]f$. Let $\mathfrak{b} = R[T]f$. Then there exists a certain $s \in S$ such that

(*) $\mathfrak{a} \supset \mathfrak{b} \supset s\mathfrak{a}$.

Let $X_2 := V_{X_1}(s)$. Then (*) implies

$Y \subset V(f) \subset Y \cup (X_2 \times \mathbb{A}^1)$.

We have $\dim X_2 \leqq n-2$. Applying the induction hypothesis to $X_2 \times \mathbb{A}^1$, the algebraic subset $(X_2 \times \mathbb{A}^1) \cap Y$ and the ideal $\mathfrak{a}_2 := \mathrm{Im}(\mathfrak{a} \to (R/s)[T])$, we get the theorem.

1.5. Corollary. In affine n-space $\mathbb{A}^n$, every algebraic subset is the set of zeros of n polynomials.

Remark. Also in projective n-space $\mathbb{P}^n$ every algebraic set can be described (set-theoretically) by n homogeneous polynomials. This can be proved by methods similar to the affine case, cf. Eisenbud/Evans [7]. For n = 3 this had been already proved by Kneser [21] .

To conclude this section, we formulate the following

Problem. Find a smooth n-dimensional affine algebraic variety X and a hypersurface $Y \subset X$ that cannot be described set-theoretically by less then n + 1 functions.

Example 1.2 is the case n = 1. In higher dimensions the problem appears to be much more difficult.

2. Estimation of the number of elements necessary to generate a module over a noetherian ring

Let R be a noetherian ring and M a finitely generated R-module. We want to estimate the minimal number of generators of M over R by a local-global principle. For this purpose, we associate to R the maximal ideal space

$X = \mathrm{Specm}(R)$,

indowed with the Zariski topology. Localization of R gives us a sheaf of rings $\mathcal{O}_X$ on X such that

$$R = \Gamma(X,\mathcal{O}_X).$$

To the R-module M there is associated a coherent $\mathcal{O}_X$-module sheaf $\mathcal{M}$ such that

$$M = \Gamma(X,\mathcal{M}).$$

We use the well-known fact: A system of elements $f_1,\dots,f_m \in M$ generates M over R iff the germs $f_{1x},\dots,f_{mx} \in \mathcal{M}_x$ generate $\mathcal{M}_x$ over $\mathcal{O}_{X,x} = R_x$ for every $x \in X = \mathrm{Specm}(R)$.

Let us introduce some further notations:
For $x \in X$ we denote by $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ the maximal ideal of the local ring $\mathcal{O}_{X,x}$ and by $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ its residue field. Further let

$$L_x(M) := \mathcal{M}_x/\mathfrak{m}_x\mathcal{M}_x .$$

This is a vector space over k(x). By the Lemma of Nakayama

$$d_x(M) := \dim_{k(x)} L_x(M)$$

is equal to the minimal number of generators of $\mathcal{M}_x$ over $\mathcal{O}_{X,x}$. More precisely:

$\phi_1,\dots,\phi_m \in \mathcal{M}_x$ generate $\mathcal{M}_x$ over $\mathcal{O}_{X,x}$ iff $\phi_1(x),\dots,\phi_m(x) \in L_x(M)$ generate $L_x(M)$ over $k(x)$.

Here we denote by $\phi_j(x)$ the image of ϕ_j under the morphism $\mathcal{M}_x \to L_x(M)$. For $f \in M$ we will denote by $f(x) \in L_x(M)$ the image of f under $M \to \mathcal{M}_x \to L_x(M)$.

The module M over R induces a certain stratification of X = Specm(R), which will be essential for us.

Definition. For $k \in \mathbb{N}$ let

$$X_k(M) := \{x \in X: d_x(M) \geq k\} .$$

It is easy to prove that $X_k(M)$ is a Zariski-closed subset of X. We have

$$X = X_0(M) \supset X_1(M) \supset \dots \supset X_r(M) \supset X_{r+1}(M) = \emptyset,$$

where $r := \sup\{\dim_{k(x)} L_x(M): x \in X\}$. (Since M is finitely generated, $r < \infty$.)

Let us consider some examples:

a) Suppose M is a projective module of rank r over R. Then the associated sheaf $\mathcal{M}$ is locally free of rank r (and defines by definition a vector bundle of rank r over X). We have

$$X = X_o(M) = \ldots = X_r(M) \supset X_{r+1}(M) = \emptyset.$$

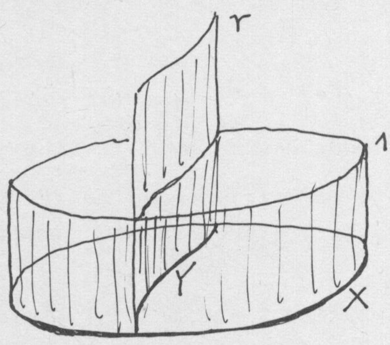
b) Let R be a regular noetherian ring and $I \subset R$ a locally complete intersection ideal of height r. If r=1, the ideal I is a projective R-module (example a), so suppose $r \geq 2$. Let

$$X = \text{Specm}(R), \quad Y = V_X(I) = \text{Specm}(R/I)$$

and let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf associated to I. For $x \in X \setminus Y$, we have $\mathcal{J}_x = \mathcal{O}_{X,x}$, hence $d_x(I) = 1$. For $y \in Y$, the minimal number of generators of $\mathcal{J}_y$ equals r, hence $d_y(I) = r$. This implies

$$X = X_o(I) = X_1(I) \supset X_2(I) = \ldots = X_r(I) \supset X_{r+1}(I) = \emptyset.$$

We visualize the situation by the following picture.



We remark that the topological space X = Specm(R) has a certain combinatorical dimension (finite or infinite). This dimension is less or equal to the dimension of Spec(R), which is the Krull dimension of R. In particular, if $R = k[T_1,\ldots,T_n]$ is a polynomial ring in n indeterminates over a field, dim Specm(R) = dim Spec(R) = n. For a local ring R we have always dim Specm(R)=0.

<u>2.1. Theorem</u> (Forster [9] , Swan [36]). Let R be a noetherian ring and M a finitely generated R-module. Set

$$b(M) := \sup\{k + \dim X_k(M) : k \geq 1, X_k(M) \neq \emptyset\}.$$

Then M can be generated by b(M) elements.

(We set b(M) = 0, if $X_1(M) = \emptyset$.)

Proof by induction on b(M). We may suppose $b(M) < \infty$, since otherwise there is nothing to prove.

If b(M) = 0, we have $\mathcal{M}_x = 0$ for all $x \in X = \mathrm{Specm}(R)$. This implies M = O, hence M is generated by 0 elements.

Induction step. Let us abbreviate $X_k(M)$ by X_k. We denote by X_k^j the (finitely many) irreducible components of X_k. Let J be the set of all pairs (k,j) such that

$$k \geq 1 \quad \text{and} \quad k + \dim X_k^j = b(M).$$

Then $X_k^j \not\subset X_{k+1}$, since otherwise we would have $(k+1) + \dim X_{k+1} > b(M)$, contradicting the definition of b(M). Choose a point

$$x_{kj} \in X_k^j \setminus X_{k+1}.$$

We have $\dim L_{x_{kj}}(M) = k > 0$ and it is easy to construct an element $f \in M$ such that

$$f(x_{kj}) \neq 0 \quad \text{for all } (k,j) \in J.$$

We consider the quotient module N := M/Rf. By the choice of f it follows that

$$\dim L_{x_{kj}}(N) = k - 1 \quad \text{for all } (k,j) \in J,$$

i.e. $x_{kj} \notin X_k(N)$. This implies $k + \dim X_k(N) < b(M)$. By induction hypothesis, N can be generared by b(M) - 1 elements, hence M can be generated by b(M) elements.

2.2. Corollary. Let M be a finitely generated projective module of rank r over a noetherian ring R and n := dim Specm(R). Then M can be generated by n + r elements.

2.3. Corollary. Let R be a regular noetherian ring and I be a locally complete intersection ideal of height r. Set

$$n := \dim \mathrm{Specm}(R), \quad k := \dim \mathrm{Specm}(R/I).$$

Then I can be generated by b(I) = max(n+1, k+r) elements