THE DYNAMICS AND THERMODYNAMICS OF COMPRESSIBLE FLUID FLOW

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By

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IN TWO VOLUMES

VOLUME II

THE RONALD PRESS COMPANY , NEW YORK

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Library of Congress Catalog Card Number: 53-8869

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

During the past two decades a rapid growth of interest in the motion of compressible fluids has accompanied developments in high-speed flight, jet engines, rockets, ballistics, combustion, gas turbines, ram jets and other novel propulsive mechanisms, heat transfer at high speeds, and blast-wave phenomena. My purpose in writing this book is to make available to students, engineers, and applied physicists a work on compressible fluid motion which would be suitable as an introductory text in the subject as well as a reference work for some of its more advanced phases. The choice of subject matter has not been dictated by any particular field of engineering, but rather includes topics of interest to aeronautical engineers, mechanical engineers, chemical engineers, applied mechanicians, and applied physicists.

In selecting material from the vast literature of the field the basic objective has been to make the book of practical value for engineering purposes. To achieve this aim, I have followed the philosophy that the most practical approach to the subject of compressible fluid mechanics is one which combines theoretical analysis, clear physical reasoning, and empirical results, each leaning on the other for mutual support and advancement, and the whole being greater than the sum of the parts.

The analytical developments of this book comprise two types of treatments: those leading to design methods and those leading to exemplary methods. The design methods are direct and rapid, and easily applied to a variety of problems. Therefore, they are suited for use in the engineering office. The discussions of these design methods are detailed and illustrative examples are often given. The exemplary methods, on the other hand, comprise those theoretical analyses which are time consuming, which generally require mathematical invention, and which are not easily applied to a variety of problems. Such methods are primarily of value for yielding detailed answers to a small number of typical problems. Although they are not in themselves suitable for the engineering office, the examples which they permit to be worked out often provide important information about the behavior of fluids in typical situations. Thus they serve as guides to the designer in solving the many complex problems where even the so-called design methods are not sufficient. The treatment of exemplary methods in this book usually consists of a brief outline of the method, together with a presentation of those results obtained by the method which illuminate significant questions concerning fluid motion and which help to form the vital "feel" so desired by designers.

In keeping with the spirit of the several foregoing remarks, all the important results of the book have been reduced to the form of convenient charts and tables. Unless otherwise specified, the charts and tables are for a perfect gas with a ratio of specific heats (k) of 1.4.

In those parts of the book dealing with fundamentals, emphasis is placed on the introduction of new concepts in an unambiguous manner, on securing a clear physical understanding before the undertaking of an analysis, on the rigorous application of physical laws, and on showing fruitful avenues of approach in analytical thinking. The remaining part of the work proceeds at a more rapid pace befitting the technical maturity of advanced students and professionals.

The work is organized in eight parts. Part I sets forth the basic concepts and principles of fluid dynamics and thermodynamics from which the remainder of the book proceeds and also introduces some fundamental concepts peculiar to compressible flows. In Part II is a discussion of problems accessible by the most simple picture of fluid motion —the one-dimensional analysis. Part III constitutes a summary of the basic ideas and concepts necessary for the succeeding chapters on two- and three-dimensional flow. Parts IV, V, and VI then present in order comprehensive surveys of subsonic flows, of supersonic flows (including hypersonic flow), and of mixed subsonic-supersonic flows. In Part VII is an exposition of unsteady one-dimensional flows. Part VIII is an examination of the viscous and heat conduction effects in laminar and turbulent boundary layers, and of the interaction between shock waves and boundary layers. For those readers not already familiar with it, the mathematical theory of characteristic curves is briefly developed in Appendix A. Appendix B is a collection of tables which facilitate the numerical solution of problems.

The "References and Selected Bibliography" at the end of each chapter will, it is hoped, be a helpful guide for further study of the voluminous subject. Apart from specific references cited in each chapter, the lists include general references appropriate to the subject matter of each chapter. The choice of references has been based primarily on clarity, on completeness, and on the desirability of an English text, rather than on historical priority.

My first acknowledgment is to Professor Joseph H. Keenan, to whom I owe my first interest in the subject, and who, as teacher, friend, and colleague, has been a source of inspiration and encouragement.

In an intangible yet real way I am indebted to my students, who have made teaching a satisfying experience, and to my friends and colleagues

at the Massachusetts Institute of Technology who contributed the climate of constructive criticism so conducive to creative effort.

Many individuals and organizations have been cooperative in supplying me with helpful material and I hope that I have not failed to acknowledge any of these at the appropriate place in the text. The National Advisory Committee for Aeronautics and the M.I.T. Gas Turbine Laboratory have been especially helpful along these lines.

I was fortunate in being able to place responsibility for the important work of the drawings in the competent hands of Mr. Percy H. Lund, who, with Miss Prudence Santoro, has been most cooperative in this regard.

For help with the final revision and checking of the manuscript I wish to give thanks to Dr. Bruce D. Gavril and Dr. Ralph A. Burton.

Finally, but by no means least, I must express a word of appreciation to Sylvia, and to young Peter, Mardi, and Bunny, who, one and all, made it possible for me to escape from the office into the somewhat less trying atmosphere of the home, and there to carry this work forward to its completion.

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PART V

SUPERSONIC FLOW (Continued)

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Chapter 17

AXIALLY SYMMETRIC SUPERSONIC FLOW

17.1. Introductory Remarks

Important practical examples of axially symmetric supersonic flow are (i) the flow past the fuselages of supersonic aircraft, rockets, and ram jets, (ii) the flow past projectiles, and (iii) internal flow in ducts, nozzles, and diffusers of round cross section.

Even though there are only two space coordinates in an axially symmetric flow, the mathematical problems prove to be more difficult than for two-dimensional flow because an axi-symmetric flow is essentially a space flow, whereas a two-dimensional flow is essentially a plane flow. Likewise, the physical natures of the two types of flows are quite different.

The important analytical methods which have been developed and which are outlined in this chapter are (i) the classical Taylor-Maccoll exact solution for the flow past a cone, (ii) the approximate linearized theory, first proposed by von Kármán and Moore, and based on the elementary solution for an infinitesimal "source" in a uniform, parallel supersonic flow, and (iii) the method of characteristics, a procedure for stepwise construction applicable to any flow pattern, time permitting its use.

These analyses are based on the assumption of a frictionless, steady flow, isentropic along each streamline. The flow is taken to be axi-symmetric. In some cases, the flow is assumed irrotational, but in others it is necessary to take account of the vorticity in the fluid.

A final section in the chapter summarizes some typical experimental results.

Hypersonic Similarity Law. In Chapter 19 there is derived a *hypersonic similarity law* for flow with small perturbations at high supersonic speeds. The law is applicable when the *hypersonic similarity parameter* $\mathsf{K} \equiv \mathsf{M}_\infty \tau$ (τ = thickness ratio) is of the order of magnitude of unity or larger. It states that for a given class of affinely related axi-symmetric bodies the distribution of $C_p \mathsf{M}_\infty^2$ on the surface depends only on the parameter K.

This result is relevant to the present chapter because various investigations (see Chapter 19) have shown that the similarity law often prevails at Mach Numbers which are usually considered to be supersonic rather than hypersonic. In the present chapter several methods are presented for determining the pressure distribution on bodies of revolution. For the range in which the similarity law is valid, it is necessary to carry out these calculations (which are often tedious) only for a few values of $M_\infty \tau$. Then the similarity law gives, with little effort, the pressure distributions on members of the particular family of affinely related shapes investigated for all combinations of M_∞ and τ falling within the range of validity of the law.

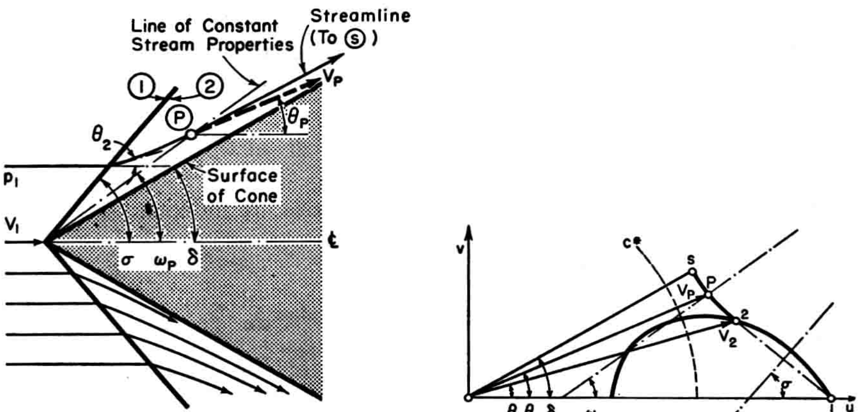
NOMENCLATURE

c	speed of sound	$\mathbf{V}$	vector velocity
C_p	pressure coefficient	x, y, z	Cartesian coordinates
F	see Eq. 17.34; also fineness ratio	α	Mach angle
G	see Eq. 17.34	β	$\sqrt{M_\infty^2 - 1}$
i	$\sqrt{-1}$	δ	semi-angle of cone; thickness ratio of body of revolution
K	similarity parameter, $M_1 \tau = M_1/F$	θ	flow direction
k	ratio of specific heats	ξ	x-coordinate of source
M	Mach Number	ρ	mass density
n	direction normal to streamline	σ	shock angle
p	pressure	τ	thickness ratio; $\tau = 1/F$
Q	source strength; also see Eq. 17.34a	φ	perturbation velocity potential
r	radius in spherical coordinates; also radius in cylindrical coordinates	Φ	velocity potential
		ω	angle in spherical coordinates
R	see Eq. 17.11; also radius in spherical coordinates; also radius to surface of axisymmetric body	$(\ )_1$	signifies conditions upstream of conical shock
		$(\ )_2$	signifies conditions downstream of conical shock
s	entropy per unit mass	$(\ )_s$	signifies conditions at surface of cone
S	cross-sectional area of axisymmetric body in plane normal to axis of symmetry	$(\ )_r$	signifies component in r-direction
u	x-component of perturbation velocity	$(\ )_\omega$	signifies component in ω-direction
U_∞	free-stream velocity	$(\ )_0$	signifies stagnation state
v	r-component of perturbation velocity	$(\ )_\infty$	signifies free-stream conditions
V	velocity	$(\ )_{I,II}$	signifies conditions on characteristic curves
V_{max}	maximum velocity for adiabatic flow		

17.2. Exact Solution for Flow Past a Cone

In plane supersonic flow a class of *simple-wave* solutions (Prandtl-Meyer flow, Chapter 15) was found, defined by the property that the two velocity components should be functions of each other. In the corresponding flow pattern all stream properties are uniform on straight lines in the physical plane, and these straight lines are identical with the Mach lines. If a simple-wave flow pattern defined by the same property is sought for an axi-symmetric flow, it is found again that all stream properties are uniform on straight lines in the physical plane. These lines are no longer the Mach lines, however, and the solution requires that these straight lines pass through a common point. Thus, because of the axial symmetry, all stream properties are constant on cones having a common vertex. The resulting flow pattern is in fact a special variety of the general class of *conical flows* discussed in Chapter 18.

General Nature of Flow Pattern. The type of simple-wave flow outlined above could be bounded only by a cone. By assuming that fluid properties are constant on cones having a common vertex, therefore, we obtain the flow pattern past a cone (Fig. 17.1a). The practical



(a) (b)

FIG. 17.1. Flow past cone.

(a) Shock cone and typical streamline.
(b) Hodograph image of streamline.

importance of this flow pattern is not limited to cones, however, since the solutions for a cone will be applicable to the region near the tip of any sharp-nosed body of revolution.

A continuous variation of fluid properties from free-stream conditions,

p_1 and V_1, to the surface of the cone, conditions p_s and V_s, proves to be impossible.[3] A shock is therefore necessary. But, since the flow downstream of the shock is by assumption conical, it follows that the shock itself must be conical and of uniform strength, and must be attached to the tip of the cone.

Consider a typical streamline (Fig. 17.1a) and its image in the hodograph plane (Fig. 17.1b). Across the shock cone there is a discontinuous change in direction and velocity from 1 to 2; points 1 and 2 therefore lie on a common hodograph shock polar originating at point 1. Between 2 and s is a region of conical flow in which all stream properties vary continuously, the streamline reaching point s on the surface of the cone only in infinite distance. The velocity vector to point s in the hodograph plane makes the cone semi-angle δ with the axis. A typical point P, with its corresponding angle ω_P is shown, together with its image point in the hodograph plane. All stream properties are constant on the cone ω_P, and hodograph point P is the image of the entire cone ω_P. Likewise, the line 1–2–P–s is the hodograph image of *all* streamlines. Since all streamlines experience the same entropy jump across the shock, the flow between the shock and the cone is isentropic and irrotational.

Governing Physical Equations. Let us use spherical coordinates r and ω, with corresponding velocity components V_r and V_ω (see Fig. 17.2a). Then, considering the toroidal-shaped control volume of Fig. 17.2b, for which the *in-going* mass flows are indicated, the equation of continuity states that the *net* outflux of mass is zero:

$$\frac{\partial}{\partial r}(2\pi\rho V_r r^2 \cdot d\omega \cdot \sin\omega)\, dr + \frac{\partial}{\partial \omega}(2\pi\rho V_\omega r \cdot dr \cdot \sin\omega)\, d\omega = 0$$

Simplifying, and noting that all stream properties are independent of r, i.e., $\partial/\partial r = 0$ and $\partial/\partial\omega = d/d\omega$, we have

$$2\rho V_r + \rho V_\omega \cot\omega + \rho\frac{dV_\omega}{d\omega} + V_\omega\frac{d\rho}{d\omega} = 0 \qquad (17.1)$$

Now, considering the velocity components of Fig. 17.2c, the condition of irrotationality is introduced by setting the circulation around the boundary of the control volume equal to zero:

$$V_r\, dr + \left(V_\omega + \frac{\partial V_\omega}{\partial r}\, dr\right)(r + dr)\, d\omega - \left(V_r + \frac{\partial V_r}{\partial\omega}\, d\omega\right) dr - V_\omega r\, d\omega = 0$$

which reduces to

$$V_\omega = \frac{dV_r}{d\omega} \qquad (17.2)$$

Euler's equation, the velocity of sound, and the energy equation are, respectively,

$$dp = -\rho V\, dV = -\rho(V_r\, dV_r + V_\omega\, dV_\omega) \tag{17.3}$$

$$dp/d\rho = c^2 \tag{17.4}$$

$$c^2 = \frac{k-1}{2}(V_{max}^2 - V^2) \tag{17.5}$$

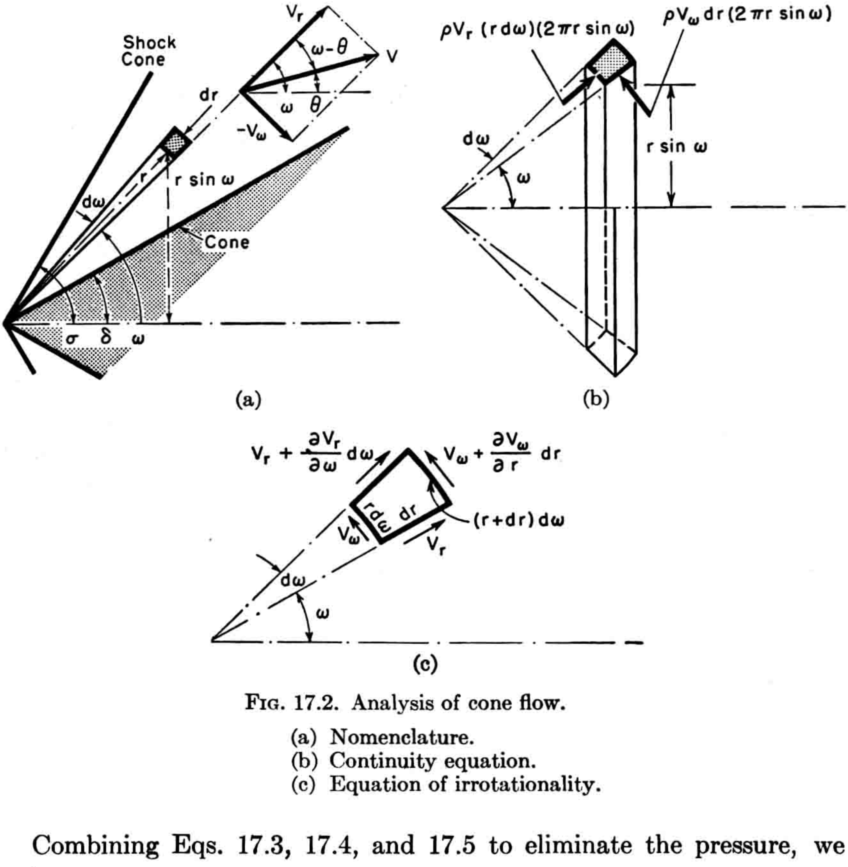


FIG. 17.2. Analysis of cone flow.

(a) Nomenclature.
(b) Continuity equation.
(c) Equation of irrotationality.

Combining Eqs. 17.3, 17.4, and 17.5 to eliminate the pressure, we get

$$V_r\, dV_r + V_\omega\, dV_\omega = -\frac{dp}{\rho} = -\frac{d\rho}{\rho}\frac{dp}{d\rho} = -c^2\frac{d\rho}{\rho}$$

$$= -\frac{k-1}{2}(V_{max}^2 - V^2)\frac{d\rho}{\rho}$$

whence

$$\frac{d\rho}{\rho} = -\frac{2}{k-1}\frac{V_r\, dV_r + V_\omega\, dV_\omega}{V_{max}^2 - V^2}$$

Inserting this expression into Eq. 17.1, and rearranging, we obtain

$$\frac{k-1}{2}\left(2V_r + V_\omega \cot\omega + \frac{dV_\omega}{d\omega}\right)(V_{max}^2 - V_\omega^2 - V_r^2)$$
$$= \left(V_r \frac{dV_r}{d\omega} + V_\omega \frac{dV_\omega}{d\omega}\right)V_\omega \qquad (17.6)$$

But, from Eq. 17.2,

$$V_\omega = \frac{dV_r}{d\omega}, \qquad \text{whence} \qquad \frac{dV_\omega}{d\omega} = \frac{d^2V_r}{d\omega^2}$$

With these it is now possible to eliminate terms in V_ω from Eq. 17.6. Thus we obtain, after rearrangement, an ordinary, nonlinear differential equation of second order for V_r in terms of ω:

$$-\frac{d^2V_r}{d\omega^2}\left[\frac{k+1}{2}\left(\frac{dV_r}{d\omega}\right)^2 - \frac{k-1}{2}(V_{max}^2 - V_r^2)\right] - \frac{k-1}{2}\left(\frac{dV_r}{d\omega}\right)^3 \cot\omega$$
$$- kV_r\left(\frac{dV_r}{d\omega}\right)^2 + \frac{k-1}{2}(V_{max}^2 - V_r^2)\frac{dV_r}{d\omega}\cot\omega$$
$$+ (k-1)V_r(V_{max}^2 - V_r^2) = 0 \qquad (17.7)$$

The integration of this equation was first done by Busemann [3] with a clever graphical construction in the hodograph plane, and subsequently by Taylor and Maccoll [4] by straightforward numerical integration. Let us consider the latter method first.

Numerical Integration of Taylor and Maccoll. The general procedure for arriving at solutions to Eq. 17.7 is indicated by the following schedule of operations:

(i) Select a value of δ (cone angle) and of $(V_r/V_{max})_s$, corresponding to the Mach Number at the cone surface.

(ii) Begin with $\omega = \delta$, at which $V_r = (V_r)_s$ and $V_\omega = 0$.

(iii) Integrate Eq. (17.7) stepwise, for small steps in ω, by replacing the differential equation by a finite-difference equation.

(iv) Having found the value of V_r/V_{max} corresponding to each value of ω, V_ω/V_{max} may be found by differentiation, using Eq. 17.2.

(v) The final step is to determine the appropriate shock angle σ and free-stream velocity V_1/V_{max}. This is done by cut-and-try. For each value of ω during the integration there is a corresponding flow angle θ and Mach Number M. Corresponding to each point of the integration the downstream Mach Number of a shock having a shock angle $\sigma = \omega$ and turning angle θ is compared with the Mach Number M of the integration. When these two Mach Numbers are found to be alike, the limit of integration has been reached and the correct shock strength has been found.

(vi) From the shock tables the approach Mach Number M_1 may then be found. Flow properties in the conical-flow region are finally computed by using the shock relations for the shock, the isentropic relations for the conical-flow region, and the values of V_r/V_{max} and V_ω/V_{max} as functions of ω found by the previous integration.

Graphical Construction of Busemann. A geometrical solution to the cone equations, leading to *apple curves* analogous to the hodograph shock polars for plane shocks, is due to Busemann.[3] The governing equations must first be put into a form involving only the hodograph variables V and θ.

HODOGRAPH EQUATIONS. From the geometry of Fig. 17.2a,

$$V_r = V\cos(\omega - \theta); \quad V_\omega = -V\sin(\omega - \theta) \tag{17.8}$$

or, differentiating with respect to ω,

$$\frac{dV_r}{d\omega} = -V\left(1 - \frac{d\theta}{d\omega}\right)\sin(\omega - \theta) + \frac{dV}{d\omega}\cos(\omega - \theta) \tag{17.9a}$$

$$\frac{dV_\omega}{d\omega} = -V\left(1 - \frac{d\theta}{d\omega}\right)\cos(\omega - \theta) - \frac{dV}{d\omega}\sin(\omega - \theta) \tag{17.9b}$$

From Eqs. 17.2 and 17.8, however,

$$\frac{dV_r}{d\omega} = V_\omega = -V\sin(\omega - \theta) \tag{17.9c}$$

When this is substituted into Eq. 17.9a, we obtain

$$\frac{d\theta}{d\omega} = -\frac{dV/d\omega}{V\tan(\omega - \theta)} \tag{17.9d}$$

Eliminating $d\theta/d\omega$ from Eq. 17.9b, and substituting the expressions for V_r, V_ω, $dV_r/d\omega$, and $dV_\omega/d\omega$ given by Eqs. 17.8, 17.9b, and 17.9c into Eq. 17.6, we get, after rearrangement,

$$\frac{dV}{d\omega} = \frac{V\sin(\omega - \theta)\dfrac{\sin\theta}{\sin\omega}}{1 - \dfrac{2V^2\sin^2(\omega - \theta)}{(k-1)(V_{max}^2 - V^2)}} \tag{17.10}$$

The geometrical relations for two neighboring points, P and P', on the same streamline in the hodograph plane are shown in Fig. 17.3. From Eq. 17.9d, it is evident that the tangent to the hodograph streamline makes the angle $(\omega - \theta)$ with the normal to the velocity vector. From the geometry of the figure, this may be interpreted as meaning that the vector change in velocity, $d\mathbf{V}$, must be normal to the line of constant ω. This result might have been reached on physical grounds since (i) the cones of constant ω are surfaces of constant pressure, (ii) the velocity gradient is normal to these cones, and therefore (iii) the vector change in $\mathbf{V}$ must lie normal to the line of constant ω.