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Critical Point Theory and Submanifold Geometry

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To Shiing-shen Chern

Scholar, Teacher, Friend

Preface

This book is divided into two parts. Part I is a modern introduction to the very classical theory of submanifold geometry. We go beyond the classical theory in at least one important respect; we study submanifolds of Hilbert space as well as of Euclidean spaces. Part II is devoted to critical point theory, and here again the theory is developed in the setting of Hilbert manifolds. The two parts are inter-related through the Morse Index Theorem, that is, the fact that the structure of the set of critical points of the distance function from a point to a submanifold can be described completely in terms of the local geometric invariants of the submanifold.

Now it is perfectly standard and natural to study critical point theory in infinite dimensions; one of the major applications of critical point theory is to the Calculus of Variations, where an infinite dimensional setting is essential. But what is the rationale for extending the classical theory of submanifolds to Hilbert space? The elementary theory of Riemannian Hilbert manifolds was developed in the 1960's, including for example the existence of Levi-Civita connections, geodesic coordinates, and some local theory of submanifolds. But Kuiper's proof of the contractibility of the group of orthogonal transformations of an infinite dimensional Hilbert space was discouraging. It meant that one could not expect to obtain interesting geometry and topology from the study of Riemannian Hilbert manifolds with the seemingly natural choice of structure group, and it was soon realized that a natural Fredholm structure was probably necessary for an interesting theory of infinite dimensional Riemannian manifolds. However, for many years there were few interesting examples to inspire further work in this area. The recent development of Kac-Moody groups and their representation theory has changed this picture. The coadjoint orbits of these infinite dimensional groups are nice submanifolds of Hilbert space with natural Fredholm structures. Moreover they arise in the study of gauge group actions and have a rich and interesting geometry and topology. Best of all from our point of view, they are isoparametric (see below) and provide easily studied explicit models that suggest good assumptions to make in order to extend classical Euclidean submanifold theory to a theory of submanifolds of Hilbert space.

One of the main goals of part I is to help graduate students get started doing research in Riemannian geometry. As a result we have tried to make it a reasonably self-contained source for learning the techniques of the subject. We do assume that the reader is familiar with the elementary theory of differentiable manifolds, as presented for example in Lang's book [La], and the basic theory of Riemannian geometry as in Hicks' book [Hk], or selected parts of Spivak's [Sp]. But in Chapter 1 we give a review of finite dimensional Riemannian geometry, with emphasis on the techniques of computation. We use Cartan's moving frame method, always trying to emphasize the intrinsic meaning behind seemingly non-invariant computations. We also give many exercises that are meant as an introduction to a variety of interesting research topics. The local geometry of submanifolds of R^n is treated in Chapter 2. In Chapter 3 we apply the local theory to study Weingarten surfaces in R^3 and S^3. The focal structure of submanifolds and its relation to the critical point structure of distance and height functions are explained in Chapter 4. The remaining chapters in part I are devoted to two problems, the understanding of which is a natural step towards developing a more general theory of submanifolds:

(1) Classify the submanifolds of Hilbert space that have the "simplest local invariants", namely the so-called isoparametric submanifolds. (A submanifold is called isoparametric if its normal curvature is zero and the principal curvatures along any parallel normal field are constant).
(2) Develop the relationship between the geometry and the topology of isoparametric submanifolds.

Many of these "simple" submanifolds arise from representation theory. In particular the generalized flag manifolds (principal orbits of adjoint representations) are isoparametric and so are the principal orbits of other isotropy representations of symmetric spaces. In fact it is now known that all homogeneous isoparametric submanifolds arise in this way, so that they are effectively classified. But there are also many non-homogeneous examples. In fact, problem (1) is far from solved, and the ongoing effort to better understand and classify isoparametric manifolds has given rise to a beautiful interplay between Riemannian geometry, algebra, transformation group theory, differential equations, and Morse theory.

In Chapter 5 we develop the basic theory of proper Fredholm Riemannian group actions (for both finite and infinite dimensions). In Chapter 6 we study the geometry of finite dimensional isoparametric submanifolds. In Chapter 7 we develop the basic theory of proper Fredholm submanifolds of Hilbert space (the condition "proper Fredholm" is needed in order to use the techniques of differential topology and Morse theory on Hilbert manifolds). Finally, in chapter 8, we use the Morse theory developed in part II to study the homology of isoparametric submanifolds of Hilbert space.

Part II of the book is a self-contained account of critical point theory on Hilbert manifolds. In Chapters 9 we develop the standard critical point theory for non-degenerate functions that satisfy Condition C: the deformation theorems, minimax principal, and Morse inequalities. We then develop the theory of linking cycles in Chapters 10; this is used in Chapter 8 of Part I to compute the homology of isoparametric submanifolds of Hilbert space. In Chapter 11, we apply our abstract critical point theory to the Calculus of Variations. We treat first the easy case of geodesics, where the abstract theory fits like a glove. We then consider a model example of the more complex "multiple integral" problems in the Calculus of Variations; the so-called Yamabe Problem, that arises in the conformal deformation of a metric to constant scalar curvature. Here we illustrate some of the major techniques that are required to make the abstract theory work in higher dimensions.

This book grew out of lectures we gave in China in May of 1987. Over a year before, Professor S.S. Chern had invited the authors to visit the recently established Nankai Mathematics institute in Tianjin, China, and lecture for a month on a subject of our choice. Word had already spread that the new Institute was an exceptionally pleasant place in which to work, so we were happy to accept. And since we were just then working together on some problems concerning isoparametric submanifolds, we soon decided to give two inter-related series of lectures. One series would be on isoparametric submanifolds; the other would be on aspects of Morse Theory, with emphasis on our generalization to the isoparametric case of the Bott-Samelson technique for calculating the homology and cohomology of certain orbits of group actions. At Professor Chern's request we started to write up our lecture notes in advance, for eventual publication as a volume in a new Nankai Institute sub-series of the Springer Verlag Mathematical Lecture Notes. Despite all good intentions, when we arrived in Tianjin in May of 1987 we each had only about a week's worth of lectures written up, and just rough notes for the rest. Perhaps it was for the best! We were completely surprised by the nature of the audience that greeted us. Eighty graduate students and young faculty, interested in geometry, had come to Tianjin from all over China to participate in our mini courses. From the beginning this was as bright and enthusiastic a group of students as we have lectured to anywhere. Moreover, before we arrived, they had received considerable background preparation for our lectures and were soon clamoring for us to pick up the pace. Perhaps we did not see as much of the wonderful city of Tianjin as we had hoped, but nevertheless we spent a very happy month talking to these students and scrambling to prepare appropriate lectures. One result was that the scope of these notes has been considerably expanded from what was originally planned. For example, the Hilbert space setting for the part on Morse Theory reflects the students desire to hear about the infinite dimensional aspects of the theory. And the part on isoparametric submanifolds was expanded to a general exposition of the modern theory of submanifolds of space forms, with material on orbital geometry and tight and taut immersions. We would like to take this opportunity to thank those many students at Nankai for the stimulation they provided.

We will never forget our month at Nankai or the many good friends we made there. We would like to thank Professor and Mrs. Chern and all of the faculty and staff of the Mathematics Institute for the boundless effort they put into making our stay in Tianjin so memorable.

After the first draft of these notes was written, we used them in a differential geometry seminar at Brandeis University. We would like to thank the many students who lectured in this seminar for the errors they uncovered and the many improvements that they suggested.

Both authors would like to thank The National Science Foundation for its support during the period on which we wrote and did research on this book. We would also like to express our appreciation to our respective Universities, Brandeis and Northeastern, for providing us with an hospitable envoironment for the teaching and research that led up to its publication.

And finally we would both like to express to Professor Chern our gratitude for his having been our teacher and guide in differential geometry. Of course there is not a geometer alive who has not benefited directly or indirectly from Chern, but we feel particularly fortunate for our many personal contacts with him over the years.

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Part I. Submanifold Theory.

Chapter 1.

Preliminaries.

In this chapter we review some basic facts concerning connections and the existence theory for systems of first order partial differential equations. These are basic tools for the study of submanifold geometry. A connection is defined both globally as a differential operator (Koszul's definition) and locally as connection 1-forms (Cartan's formulation). While the global definition is better for interpreting the geometry, the local definition is easier to compute with. A first order system of partial differential equations can be viewed as a system of equations for differential 1-forms, and the associated existence theory is referred to as the Frobenius theorem.

1.1 Connections on a vector bundle.

Let M be a smooth manifold, ξ a smooth vector bundle of rank k on M, and $C^\infty(\xi)$ the space of smooth sections of ξ.

1.1.1. Definition. A *connection* for ξ is a linear operator

$$\nabla : C^\infty(\xi) \to C^\infty(T^*M \otimes \xi)$$

such that

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

for every $s \in C^\infty(\xi)$ and $f \in C^\infty(M)$. We call $\nabla(s)$ the *covariant derivative* of s.

If ξ is trivial, i.e., $\xi = M \times \boldsymbol{R}^k$, then $C^\infty(\xi)$ can be identified with $C^\infty(M, \boldsymbol{R}^k)$ by $s(x) = (x, f(x))$. The differential of maps gives a trivial connection on ξ, i.e., $\nabla s(x) = (x, df_x)$. The collection of all connections on ξ can be described as follows. We call k smooth sections $s_1, \ldots, s_k$ of ξ a *frame field* of ξ if $s_1(x), \ldots, s_k(x)$ is a basis for the fiber ξ_x at every $x \in M$. Then every section of ξ can be uniquely written as a sum $f_1 s_1 + \ldots + f_k s_k$, where f_i are uniquely determined smooth functions on M. A connection ∇ on ξ is uniquely determined by $\nabla(s_1), \ldots, \nabla(s_k)$, and these can be completely arbitrary smooth sections of the bundle $T^*M \otimes \xi$. Each of the sections $\nabla(s_i)$ can be written uniquely as a sum $\sum \omega_{ij} \otimes s_j$, where (ω_{ij}) is an arbitrary $n \times n$ matrix of smooth real-valued one forms on M. In fact, given $\nabla(s_1), \ldots, \nabla(s_k)$ we can define ∇ for an arbitrary section by the formula

$$\nabla(f_1 s_1 + \cdots + f_k s_k) = \sum (df_i \otimes s_i + f_i \nabla(s_i)).$$

(Here and in the sequel we use the convention that $\sum$ always stands for the summation over all indices that appear twice).

Suppose U is a small open subset of M such that $\xi|U$ is trivial. A frame field $s_1, \ldots, s_k$ of $\xi|U$ is called a *local frame field* of ξ on U.

It follows from the definition that a connection ∇ is a local operator, that is, if s vanishes on an open set U then ∇s also vanishes on U. In fact, since $s(p) = 0$ and $ds_p = 0$ imply $\nabla s(p) = 0$, ∇ is a first order differential operator ([Pa3]).

Since a connection is a local operator, it makes sense to talk about its restriction to an open subset of M. If a collection of open sets U_α covers M such that $\xi|U_\alpha$ is trivial, then a connection ∇ on ξ is uniquely determined by its restrictions to the various U_α. Let $s_1, \ldots, s_k$ be a local frame field on U_α, then there exists unique $n \times n$ matrix of smooth real-valued one forms (ω_{ij}) on U_α such that $\nabla(s_i) = \sum \omega_{ij} \otimes s_j$.

Let $\boldsymbol{GL}(k)$ denote the Lie group of the non-singular $k \times k$ real matrices, and $gl(k)$ its Lie algebra. If s_i and s_i^* are two local frame fields of ξ on U, then there is a uniquely determined smooth map $g = (g_{ij}) : U \to \boldsymbol{GL}(k)$ such that $s_i^* = \sum g_{ij} s_j$. Let $g^{-1} = (g^{ij})$ denote the inverse of g, so that $s_i = \sum g^{ij} s_j^*$. Suppose

$$\nabla s_i = \sum \omega_{ij} \otimes s_j, \qquad \nabla s_i^* = \sum \omega_{ij}^* \otimes s_j^*.$$

Let $\omega = (\omega_{ij})$ and $\omega^* = (\omega_{ij}^*)$. Since

$$\begin{aligned}
\nabla s_i^* &= \nabla(\sum g_{im} s_m) = \sum dg_{im} s_m + g_{im} \nabla s_m \\
&= \sum_m (dg_{im} + \sum_k g_{ik} \omega_{km}) s_m \\
&= \sum_j (\sum_m dg_{im} g^{mj} + \sum_{m,k} g_{ik} \omega_{km} g^{mj}) s_j^* \\
&= \sum_j \omega_{ij}^* s_j^*,
\end{aligned}$$

we have

$$\omega^* = (dg) g^{-1} + g \omega g^{-1}.$$

Given an open cover U_α of M and local frame fields $\{s_i^\alpha\}$ on U_α, suppose $s_i^\alpha = \sum (g_{ij}^{\alpha\beta}) s_j^\beta$ on $U_\alpha \cap U_\beta$. Let $g^{\alpha\beta} = (g_{ij}^{\alpha\beta})$. Then a connection on ξ is defined by a collection of $gl(k)$–valued 1-forms ω^α on U_α, such that on $U_\alpha \cap U_\beta$ we have $\omega^\beta = (dg^{\alpha\beta})(g^{\alpha\beta})^{-1} + g^{\alpha\beta} \omega^\alpha (g^{\alpha\beta})^{-1}$.

Identify $T^*M \otimes \xi$ with $L(TM, \xi)$, and let $\nabla_X s$ denote $(\nabla s)(X)$. For $X, Y \in C^\infty(TM)$ and $s \in C^\infty(\xi)$ we define

$$K(X,Y)(s) = -(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})(s). \tag{1.1.1}$$

It follows from a direct computation that

$$K(Y,X) = -K(X,Y),$$

$$K(fX,Y) = K(X,fY) = fK(X,Y),$$
$$K(X,Y)(fs) = fK(X,Y)(s).$$

Hence K is a smooth section of $L(\xi \otimes \bigwedge^2 TM, \xi) \simeq L(\xi, \bigwedge^2 T^*M \otimes \xi)$.

1.1.2. Definition. This section K of the vector bundle $L(\xi, \bigwedge^2 T^*M \otimes \xi)$ is called the *curvature* of the connection ∇.

Recall that the bracket operation on vector fields and the exterior differentiation on p forms are related by

$$\begin{aligned} d\omega(X_0, \ldots, X_p) &= \sum_i (-1)^i X_i \omega(X_0, \ldots, \hat{X}_i, \ldots, X_p) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p). \end{aligned} \tag{1.1.2}$$

Suppose $s_1, \ldots, s_k$ is a local frame field on U, and $\nabla s_i = \sum \omega_{ij} \otimes s_j$. Then there exist 2-forms Ω_{ij} such that

$$K(s_i) = \sum \Omega_{ij} \otimes s_j.$$

Since

$$\begin{aligned} -K(X,Y)(s_i) &= \nabla_X \nabla_Y s_i - \nabla_Y \nabla_X s_i - \nabla_{[X,Y]} s_i \\ &= \nabla_X (\sum \omega_{ij}(Y) s_j) - \nabla_Y (\sum \omega_{ij}(X) s_j) \\ &\quad - \sum \omega_{ij}([X,Y]) s_j \\ &= \sum (X(\omega_{ij}(Y)) - Y(\omega_{ij}(X)) - \omega_{ij}([X,Y])) s_j \\ &\quad + \sum (\omega_{ij}(Y)\omega_{jk}(X) - \omega_{ij}(X)\omega_{jk}(Y)) s_k \\ &= \sum (d\omega_{ij} - \sum \omega_{ik} \wedge \omega_{kj})(X,Y)\, s_j, \end{aligned}$$

we have

$$-\Omega_{ij} = d\omega_{ij} - \sum \omega_{ik} \wedge \omega_{kj}.$$

Thus K can is locally described by the $k \times k$ matrix $\Omega = (\Omega_{ij})$ of 2-forms just as ∇ is defined locally by the matrix $\omega = (\omega_{ij})$ of 1-forms. In matrix notation, we have

$$-\Omega = d\omega - \omega \wedge \omega. \tag{1.1.3}$$

Let $g = (g_{ij}) : U \to \boldsymbol{GL}(k)$ be a smooth map and let $\omega = (dg)g^{-1}$. Then ω is a $gl(k)-$ valued 1-form on U, satisfying the so-called *Maurer-Cartan equation*

$$d\omega = \omega \wedge \omega.$$

Conversely, given a $gl(k)-$ valued 1-form on U with $d\omega = \omega \wedge \omega$., it follows from Frobenius theorem (cf. 1.4) that given any $x_0 \in U$ and $g_0 \in \boldsymbol{GL}(k)$ there is a neighborhood U_0 of x_0 in U and a smooth map $g = (g_{ij}) : U_0 \to \boldsymbol{GL}(k)$ such that $g(x_0) = g_0$ and $(dg)g^{-1} = \omega$. Thus $d\omega = \omega \wedge \omega$ is a necessary and sufficient condition for being able to solve locally the system of first order partial differential equations:

$$dg = \omega g. \tag{1.1.4}$$

Let e_i denote the i^{th} row of the matrix g and $\omega = (\omega_{ij})$. Then (1.1.4) can be rewritten as

$$de_i = \sum_j \omega_{ij} \otimes e_j.$$

1.1.3. Definition. A smooth section s of $\xi|U$ is *parallel* with respect to ∇ if $\nabla s = 0$ on U.

1.1.4. Definition. A connection is *flat* if its curvature is zero.

1.1.5. Proposition. *The connection ∇ on ξ is flat if and only if there exist local parallel frame fields.*

PROOF. Let s_i and $\omega = (\omega_{ij})$ be as before. Suppose $\Omega = 0$, then ω satisfies the Maurer- Cartan equation $d\omega = \omega \wedge \omega$. So locally there exists a $\boldsymbol{GL}(k)-$ valued map $g = (g_{ij})$ such that $(dg)g^{-1} = \omega$. Let $g^{-1} = (g^{ij})$, and $s_i^* = \sum g^{ij} s_j$. Then $\nabla s_i^* = \sum \omega_{ij}^* \otimes s_j^*$, and

$$\begin{aligned} \omega^* &= d(g^{-1})g + g^{-1}\omega g \\ &= -g^{-1}(dg)g^{-1}g + g^{-1}(dg)g^{-1}g = 0^{\cdot} \end{aligned}$$

So s_i^* is a parallel frame. ∎

1.1.6. Definition. A connection ∇ on ξ is called globally flat if there exists a parallel frame field defined on the whole manifold M.

1.1.7. Example. Let ξ be the trivial vector bundle $M \times \boldsymbol{R}^k$, and ∇ the trivial connection on ξ given by the differential of maps. Then a section $s(x) = (x, f(x))$ is parallel if and only if f is a constant map, so ∇ is globally flat.

1.1.8. Remarks.

(1) If ξ is not a trivial bundle then no connection on ξ can be globally flat.

(ii) A flat connection need not be globally flat. For example, let M be the Möbius band $[0,1] \times \boldsymbol{R}/\sim$ (where $(0,t) \sim (1,-t)$)). Then the trivial connection on $[0,1] \times R$ induces a flat connection on TM. But since TM is not a product bundle this connection is not globally flat.

Given $x_0 \in M$, a smooth curve $\alpha : [0,1] \to M$ such that $\alpha(0) = x_0$ and $v_0 \in \xi_{x_0}$ (the fiber of ξ over x_0), then the following first order ODE

$$\nabla_{\alpha'(t)} v = 0, \quad v(0) = v_0, \tag{1.1.5}$$

has a unique solution. A solution of (1.1.5) is called a parallel field along α, and $v(1)$ is called the parallel translation of v_0 along α to $\alpha(1)$. Let $P(\alpha) : \xi_{x_0} \to \xi_{x_0}$ be the map defined by $P(\alpha)(v_0) = v(1)$ for closed curve α such that $\alpha(0) = \alpha(1) = x_0$. The set of all these $P(\alpha)$ is a subgroup of $\boldsymbol{GL}(\xi_{x_0})$, that is called the *holonomy group* of ∇ with respect to x_0. It is easily seen that ∇ is globally flat if and only if the holonomy group of ∇ is trivial.

1.1.9. Definition. A local frame s_i of vector bundle ξ is called parallel at a point x_0 with respect to the connection ∇, if $\nabla s_i(x_0) = 0$ for all i.

1.1.10. Proposition. *Let ∇ be a connection on the vector bundle ξ on M. Given $x_0 \in M$, then there exist an open neighborhood U of x_0 and a frame field defined on U, that is parallel at x_0.*

PROOF. Let s_i be a local frame field, $\nabla s_i = \sum_j \omega_{ij} \otimes s_j$, and $\omega = (\omega_{ij})$. Let $x_1, \ldots, x_n$ be a local coordinate system near x_0, and $\omega = \sum_i f_i(x)\, dx_i$, for some smooth $gl(k)$ valued maps f_i. Let $a_i = f_i(x_0)$. Then $a_i \in gl(k)$, and $g^{-1}dg + \omega = 0$ at x_0, where $g(x) = \exp(\sum_i x_i a_i)$. So we have $dg\, g^{-1} + g\omega g^{-1} = 0$ at x_0, i.e., $s_i^* = \sum g_{ij} s_j$ is parallel at x_0, where $g = (g_{ij})$. ∎

Let $\boldsymbol{O}(m,k)$ denote the Lie group of linear isomorphism that leave the following bilinear form on $\boldsymbol{R}^{m+k}$ invariant:

$$(x, y) = \sum_{i=1}^{m} x_i y_i - \sum_{j=1}^{k} x_{m+j} y_{m+j}.$$

So an $(m+k)\times(m+k)$ matrix A is in $\boldsymbol{O}(m,k)$ if and only if

$$A^tEA = E, \quad \text{where} \quad E = \quad \text{diag}(1,\ldots,1,-1,\ldots,-1),$$

and its Lie algebra is:

$$o(m,k) = \{A \in gl(m+k) \mid A^tE + EA = 0\}.$$

1.1.11. Definition. A rank $(m+k)$ vector bundle ξ is called an $\boldsymbol{O}(m,k)-$ bundle (an orthogonal bundle if $k = 0$) if there is a smooth section g of $S^2(\xi^*)$ such that $g(x)$ is a non-degenerate bilinear form on ξ_x of index k for all $x \in M$. A connection ∇ on ξ is said to be *compatible* with g if

$$X(g(s,t)) = g(\nabla_X s, t) + g(s, \nabla_X t),$$

for all $X \in C^\infty(TM), s,t \in C^\infty(\xi)$.

Suppose $s_1,\ldots,s_{m+k}$ is a local frame field, $g(s_i,s_j) = g_{ij}$, and

$$\nabla s_i = \sum_j \omega_{ij} \otimes s_j.$$

Then ∇ is compatible with g if and only if

$$\omega G + G\omega^t = dG,$$

where $\omega = (\omega_{ij})$ and $G = (g_{ij})$. In particular, if $G = E$ as above, then

$$\omega E + E\omega^t = 0, \tag{1.1.6}$$

i.e., ω is an $o(m,k)-$ valued 1-form on M.

The collection of all connections on ξ does not have natural vector space structure. However it *does* have a natural affine structure. In fact if ∇_1 and ∇_2 are two connections on ξ and f is a smooth function on M then the linear combination $f\nabla_1 + (1-f)\nabla_2$ is again a well-defined connection on ξ, and $\nabla_1 - \nabla_2$ is a smooth section of $L(\xi, T^*M \otimes \xi)$.

Next we consider connections on induced vector bundles. Given a smooth map $\varphi : N \to M$ we can form the induced vector bundle $\varphi^*\xi$. Note that there are canonical maps

$$\varphi^* : C^\infty(\xi) \to C^\infty(\varphi^*\xi),$$

$$\varphi^* : C^\infty(T^*M) \to C^\infty(T^*N).$$