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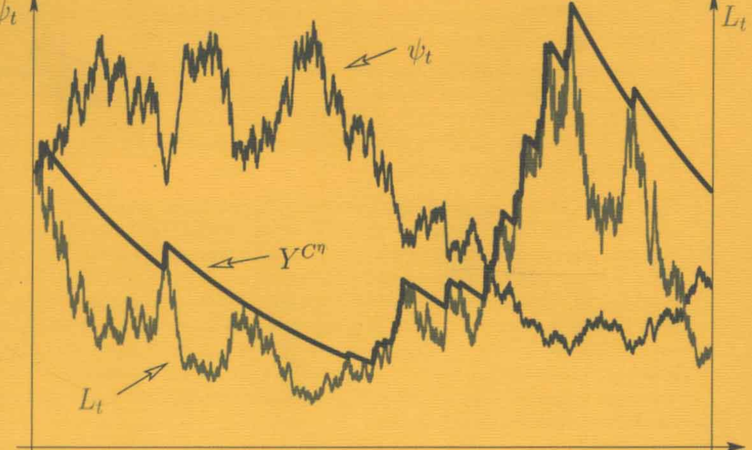

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on page VII]

Cover Figure: Typical paths for the deflator ψ, a universal consumption signal L, and the induced level of satisfaction Y^{C^η}, by courtesy of P. Bank and H. Föllmer

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Preface

This is the first volume of the Paris-Princeton Lectures in Financial Mathematics. The goal of this series is to publish cutting edge research in self-contained articles prepared by well known leaders in the field, or promising young researchers invited by the editors to contribute to a volume. Particular attention is paid to the quality of the exposition and we aim at articles that can serve as an introductory reference for research in the field.

The series is a result of frequent exchanges between researchers in finance and financial mathematics in Paris and Princeton. Many of us felt that the field would benefit from timely exposés of topics in which there is important progress. René Carmona, Erhan Cinlar, Ivar Ekeland, Elyes Jouini, José Scheinkman and Nizar Touzi will serve in the first editorial board of the Paris-Princeton Lectures in Financial Mathematics. Although many of the chapters in future volumes will involve lectures given in Paris or Princeton, we will also invite other contributions. Given the current nature of the collaboration between the two poles, we expect to produce a volume per year. Springer Verlag kindly offered to host this enterprise under the umbrella of the Lecture Notes in Mathematics series, and we are thankful to Catriona Byrne for her encouragement and her help in the initial stage of the initiative.

This first volume contains four chapters. The first one was written by Peter Bank and Hans Föllmer. It grew out of a seminar course at given at Princeton in 2002. It reviews a recent approach to optimal stopping theory which complements the traditional Snell envelop view. This approach is applied to utility maximization of a satisfaction index, American options, and multi-armed bandits.

The second chapter was written by Fabrice Baudoin. It grew out of a course given at CREST in November 2001. It contains an interesting, and very promising, extension of the theory of initial enlargement of filtration, which was the topic of his Ph.D. thesis. Initial enlargement of filtrations has been widely used in the treatment of asymetric information models in continuous-time finance. This classical view assumes the knowledge of some random variable in the almost sure sense, and it is well known that it leads to arbitrage at the final resolution time of uncertainty. Baudoin's chapter offers a self-contained review of the classical approach, and it gives a complete analysis of the case where the additional information is restricted to the distribution of a random variable.

The chapter contributed by Chris Rogers is based on a short course given during the *Montreal Financial Mathematics and Econometrics Conference* organized in June 2001 by CIRANO in Montreal. The aim of this event was to bring together leading experts and some of the most promising young researchers in both fields in order to enhance existing collaborations and set the stage for new ones. Roger's contribution gives an intuitive presentation of the duality approach to utility maximization problems in different contexts of market imperfections.

The last chapter is due to Mete Soner and Nizar Touzi. It also came out of seminar course taught at Princeton University in 2001. It provides an overview of the duality approach to the problem of super-replication of contingent claims under portfolio constraints. A particular emphasis is placed on the limitations of this approach, which in turn motivated the introduction of an original geometric dynamic programming principle on the initial formulation of the problem. This eventually allowed to avoid the passage from the dual formulation.

It is anticipated that the publication of this first volume will coincide with the *Blaise Pascal International Conference in Financial Modeling*, to be held in Paris (July 1-3, 2003). This is the closing event for the prestigious *Chaire Blaise Pascal* awarded to Jose Scheinkman for two years by the *Ecole Normale Supérieure de Paris*.

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Contents

American Options, Multi–armed Bandits, and Optimal Consumption Plans: A Unifying View

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Summary. In this survey, we show that various stochastic optimization problems arising in option theory, in dynamical allocation problems, and in the microeconomic theory of intertemporal consumption choice can all be reduced to the same problem of representing a given stochastic process in terms of running maxima of another process. We describe recent results of Bank and El Karoui (2002) on the general stochastic representation problem, derive results in closed form for Lévy processes and diffusions, present an algorithm for explicit computations, and discuss some applications.

Key words: American options, Gittins index, multi–armed bandits, optimal consumption plans, optimal stopping, representation theorem, universal exercise signal.
AMS 2000 subject classification. 60G07, 60G40, 60H25, 91B16, 91B28.

1 Introduction

At first sight, the optimization problems of exercising an American option, of allocating effort to several parallel projects, and of choosing an intertemporal consumption plan seem to be rather different in nature. It turns out, however, that they are all related to the same problem of representing a stochastic process in terms of running maxima of another process. This stochastic representation provides a new method for solving such problems, and it is also of intrinsic mathematical interest. In this survey, our purpose is to show how the representation problem appears in these different contexts, to explain and to illustrate its general solution, and to discuss some of its practical implications.

As a first case study, we consider the problem of choosing a consumption plan under a cost constraint which is specified in terms of a complete financial market

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model. Clearly, the solution depends on the agent's preferences on the space of consumption plans, described as optional random measures on the positive time axis. In the standard formulation of the corresponding optimization problem, one restricts attention to absolutely continuous measures admitting a rate of consumption, and the utility functional is a time–additive aggregate of utilities applied to consumption rates. However, as explained in [25], such time–additive utility functionals have serious conceptual deficiencies, both from an economic and from a mathematical point of view. As an alternative, Hindy, Huang and Kreps [25] propose a different class of utility functionals where utilities at different times depend on an index of satisfaction based on past consumption. The corresponding singular control problem raises new mathematical issues. Under Markovian assumptions, the problem can be analyzed using the Hamilton–Jacobi–Bellman approach; see [24] and [8]. In a general semimartingale setting, Bank and Riedel [6] develop a different approach. They reduce the optimization problem to the problem of representing a given process X in terms of running suprema of another process ξ:

$$X_t = \mathbb{E}\left[\int_{(t,+\infty]} f(s, \sup_{v\in[t,s)} \xi_v)\,\mu(ds)\,\middle|\,\mathcal{F}_t\right] \quad (t \in [0,+\infty))\,. \tag{1}$$

In the context of intertemporal consumption choice, the process X is specified in terms of the price deflator; the function f and the measure μ reflect the structure of the agent's preferences. The process ξ determines a minimal level of satisfaction, and the optimal consumption plan consists in consuming just enough to ensure that the induced index of satisfaction stays above this minimal level. In [6], the representation problem is solved explicitly under the assumption that randomness is modelled by a Lévy process.

In its general form, the stochastic representation problem (1) has a rich mathematical structure. It raises new questions even in the deterministic case, where it leads to a time–inhomogeneous notion of convex envelope as explained in [5]. In discrete time, existence and uniqueness of a solution easily follow by backwards induction. The stochastic representation problem in continuous time is more subtle. In a discussion of the first author with Nicole El Karoui at an Oberwolfach meeting, it became clear that it is closely related to the theory of Gittins indices in continuous time as developed by El Karoui and Karatzas in[17].

Gittins indices occur in the theory of multi–armed bandits. In such dynamic allocation problems, there is a a number of parallel projects, and each project generates a specific stochastic reward proportional to the effort spent on it. The aim is to allocate the available effort to the given projects so as to maximize the overall expected reward. The crucial idea of [23] consists in reducing this multi–dimensional optimization problem to a family of simpler benchmark problems. These problems yield a performance measure, now called the Gittins index, separately for each project, and an optimal allocation rule consists in allocating effort to those projects whose current Gittins index is maximal. [23] and [36] consider a discrete–time Markovian setting, [28] and [32] extend the analysis to diffusion models. El Karoui and Karatzas [17] develop a general martingale approach in continuous time. One of their results shows that Gittins indices can be viewed as solutions to a representation problem of the form (1). This connection turned out to be the key to the solution of the general representation problem in [5]. This representation result can be used as an alternative way to define Gittins indices, and it offers new methods for their computation.

As another case study, we consider American options. Recall that the holder of such an option has the right to exercise the option at any time up to a given deadline. Thus, the usual approach to option pricing and to the construction of replicating strategies has to be combined with an optimal stopping problem: Find a stopping time which maximizes the expected payoff. From the point of view of the buyer, the expectation is taken with respect to a given probabilistic model for the price fluctuation of the underlying. From the point of view of the seller and in the case of a complete financial market model, it involves the unique equivalent martingale measure. In both versions, the standard approach consists in identifying the optimal stopping times in terms of the Snell envelope of the given payoff process; see, e.g., [29]. Following [4], we are going to show that, alternatively, optimal stopping times can be obtained from a representation of the form (1) via a level crossing principle: A stopping time is optimal iff the solution ξ to the representation problem passes a certain threshold. As an application in Finance, we construct a universal exercise signal for American put options which yields optimal stopping rules simultaneously for all possible strikes. This part of the paper is inspired by a result in [18], as explained in Section 2.1.

The reduction of different stochastic optimization problems to the stochastic representation problem (1) is discussed in Section 2. The general solution is explained in Section 3, following [5]. In Section 4 we derive explicit solutions to the representation problem in homogeneous situations where randomness is generated by a Lévy process or by a one–dimensional diffusion. As a consequence, we obtain explicit solutions to the different optimization problems discussed before. For instance, this yields an alternative proof of a result by [33], [1], and [10] on optimal stopping rules for perpetual American puts in a Lévy model.

Closed–form solutions to stochastic optimization problems are typically available only under strong homogeneity assumptions. In practice, however, inhomogeneities are hard to avoid, as illustrated by an American put with finite deadline. In such cases, closed–form solutions cannot be expected. Instead, one has to take a more computational approach. In Section 5, we present an algorithm developed in [3] which explicitly solves the discrete–time version of the general representation problem (1). In the context of American options, for instance, this algorithm can be used to compute the universal exercise signal as illustrated in Figure 1.

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Notation. *Throughout this paper we fix a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a filtration* $(\mathcal{F}_t)_{t\in[0,+\infty]}$ *satisfying the usual conditions. By* $\mathcal{T}$ *we shall denote the set of all stopping times* $T \geq 0$. *Moreover, for a (possibly random) set* $A \subset [0, +\infty]$, $\mathcal{T}(A)$

will denote the class of all stopping times $T \in \mathcal{T}$ taking values in A almost surely. For instance, given a stopping time S, we shall make frequent use of $\mathcal{T}((S, +\infty])$ in order to denote the set of all stopping times $T \in \mathcal{T}$ such that $T(\omega) \in (S(\omega), +\infty]$ for almost every ω. For a given process $X = (X_t)$ we use the convention $X_{+\infty} = 0$ unless stated otherwise.

2 Reducing Optimization Problems to a Representation Problem

In this section we consider a variety of optimization problems in continuous time including optimal stopping problems arising in Mathematical Finance, a singular control problem from the microeconomic theory of intertemporal consumption choice, and the multi–armed bandit problem in Operations Research. We shall show how each of these different problems can be reduced to the same problem of representing a given stochastic process in terms of running suprema of another process.

2.1 American Options

An American option is a contingent claim which can be exercised by its holder at any time up to a given terminal time $\hat{T} \in (0, +\infty]$. It is described by a nonnegative, optional process $X = (X_t)_{t \in [0,\hat{T}]}$ which specifies the contingent payoff X_t if the option is exercised at time $t \in [0, \hat{T}]$.

A key example is the American put option on a stock which gives its holder the right to sell the stock at a price $k \geq 0$, the so–called strike price, which is specified in advance. The underlying financial market model is defined by a stock price process $P = (P_t)_{t \in [0,\hat{T}]}$ and an interest rate process $(r_t)_{t \in [0,\hat{T}]}$. For notational simplicity, we shall assume that interest rates are constant: $r_t \equiv r > 0$. The discounted payoff of the put option is then given by the process

$$X_t^k = e^{-rt}(k - P_t)^+ \quad (t \in [0, \hat{T}]) .$$

Optimal Stopping via Snell Envelopes

The holder of an American put–option will try to maximize the expected proceeds by choosing a suitable exercise time. For a general optional process X, this amounts to the following optimal stopping problem:

$$\text{Maximize } \mathbb{E} X_T \text{ over all stopping times } T \in \mathcal{T}([0, \hat{T}]) .$$

There is a huge literature on such optimal stopping problems, starting with [35]; see [16] for a thorough analysis in a general setting. The standard approach uses the theory of the *Snell envelope* defined as the unique supermartingale U such that

$$U_S = \operatorname*{ess\,sup}_{T \in \mathcal{T}([S,\hat{T}])} \mathbb{E}\,[\, X_T \,|\, \mathcal{F}_S]$$