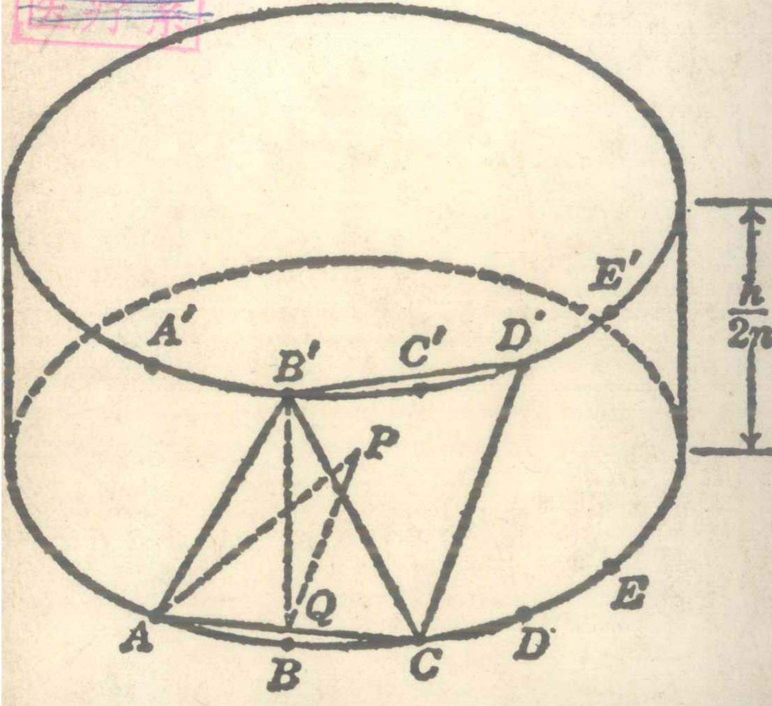
Riddles in Mathematics

Eugene P. Northrop



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PELICAN BOOKS

RIDDLES IN MATHEMATICS

Eugene P. Northrop was born in Danbury, Connecticut, in 1908. After graduating from the local high school he attended Robert College, Istanbul, where his uncle, Professor of Mathematics, converted him to a lifelong interest in the subject. Mr Northrop then took degrees of B.S., M.A., and Ph.D. at Yale, and also taught there.

From 1935 to 1943 he was mathematics master at Hotchkiss School, a private boarding school for boys. In 1943 he joined the College Faculty at the University of Chicago at the time when it was becoming nationally famous for its enlightened and adventurous approach to higher education. There, with a staff of young and energetic colleagues, he developed a radically different course in mathematics which has strongly influenced both college and secondary school teaching in the U.S.A. In 1953 he was named William Rainey Harper Professor.

In 1954–5 he was consultant in Washington to the National Science Foundation, and also to the Fund for the Advancement of Education. Since 1960 he has been the Ford Foundation's resident representative in Turkey, at the same time advising on developments in mathematics and science.

EUGENE P. NORTHROP

RIDDLES IN MATHEMATICS

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A BOOK OF PARADOXES

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Contents

Preface

POPULAR interest in mathematics is unquestionably increasing. Perhaps this is because of the fact that mathematics is a tool without which the applied sciences would cease to be sciences. On the other hand, the abstract aspect of mathematics is beginning to attract a large following of people who, weary of the complexities of the human equation in everyday activities, turn in their leisure to the simplicities of the mathematical equation. It is for these people that this book is written. Indeed, only two things are required of the prospective reader – an elementary training in mathematics, and an interest in matters mathematical. These two prerequisites are sufficient for an understanding of the first nine chapters of the book. The tenth – and last – chapter is specifically designed for the reader with more technical equipment.

Of all the problems dealt with in mathematics, paradoxes are among the most appealing and instructive. The appeal of a paradox is difficult to analyse in a word or two, but it probably arises from the fact that a contradiction comes as a complete surprise in what is generally thought of as the only 'exact' science. And a paradox is always instructive, for to unravel the troublesome line of reasoning requires a close scrutiny of the fundamental principles involved. In the light of these arguments it has seemed worth while to bring out a book devoted exclusively to some of the paradoxes which mathematicians, both amateur and professional, have found disconcerting.

The material for this book has been gathered from a wide variety of sources. Some of it has naturally appeared in other popular expositions of mathematics – such works as Ball's *Mathematical Recreations and Essays*, Steinhaus's *Mathematical Snapshots*, and Kasner and Newman's *Mathematics and the Imagination*, to mention only three. If this is a fault, it is not the fault of the author, but of the material he is trying to present. An attempt is made, in the majority of instances, to give references to original sources. This is not always possible, however, particularly when the same problem, in different forms, is to be found in a number of different places.

The author wishes to express his thanks to all who have

contributed to the development of this book. He is particularly indebted to Mr Henry C. Edgar, of the Hotchkiss School, for his painstaking study and criticism of the entire manuscript. Without his help many points, clear enough to the mathematician, would have remained obscure to the general reader. Special thanks are also due to the author's former teacher and colleague, Professor Einar Hille, of Yale University, who read and criticized the manuscript from the point of view of the mathematician.

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CHAPTER ONE

What is a Paradox?

TWO fathers and two sons leave town. This reduces the population of the town by three. False? No, true – provided the trio consists of father, son, and grandson.

A bookworm starts at the outside of the front cover of volume I of a certain set of books and eats his way to the outside of the back cover of volume III. If each volume is one inch thick, he must travel three inches in all. True? No, false. A moment's study of the accompanying figure shows that he has only to make his way through volume II – a distance of one inch.

FIG. 1

A man says, 'I am lying.' Is his statement true? If so, then he is lying, and his statement is false. Is his statement false? If so, then he is lying, and his statement is true.

The dictionaries define an island as 'a body of land completely surrounded by water' and a lake as 'a body of water completely surrounded by land'. But suppose the northern hemisphere were all land, and the southern hemisphere all water. Would you call the northern hemisphere an island, or would you call the southern hemisphere a lake?

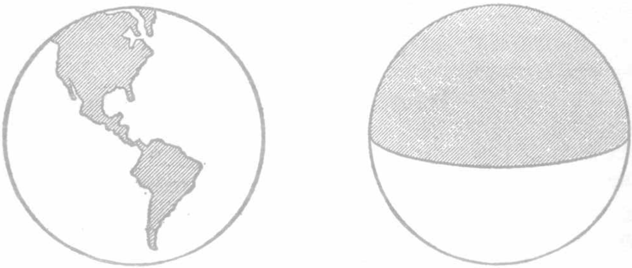

FIG. 2. If the northern hemisphere were all land and the southern hemisphere all water

It is of such brain-twisters as these that this book is composed. There are paradoxes for everyone – from the person who left mathematics behind in school (or who was left behind in school by mathematics) to the professional mathematician, who is still bothered by such a problem as that of the liar.

We shall use the word 'paradox', by the way, in the sense in which it is used in these examples. That is to say, a paradox is anything which offhand appears to be false, but is actually true; or which appears to be true, but is actually false; or which is simply self-contradictory. From time to time it may appear that we are straying from this meaning. But be patient – what seems crystal-clear to you may leave the next person completely confused.

*

If you are among those who at this point are saying, 'But we thought this book had to do with *mathematical* paradoxes – how about it?' then stay with us for a moment. If you are not interested in the answer to this question, you may as well skip to the next chapter.

A closer look at the difficulties encountered in our first examples will show that they are simply cases of very real difficulties encountered not only by the student of mathematics, but by the mature mathematician as well.

In the problem concerning fathers and sons, we find ourselves searching here and there for some instance in which the conditions of the problem will be fulfilled. It seems at first as though such an instance cannot possibly exist – common sense and intuition are all against it. But suddenly, there it is – as simple a solution as can be. This sort of thing happens time and again in mathematical research. The mathematician, working on the development of some theory or other, is suddenly confronted with a set of conditions which appear to be highly improbable. He begins looking for an example to fit the conditions, and it may be days, or weeks, or even longer, before he finds one. Frequently the solution of his difficulty is as simple as was ours – the kind of thing that makes him wonder why he hadn't thought of it before.

The problem of the bookworm's journey is a nice example of the way in which reason can be led astray by hasty judgement. The false conclusion is reached through failure to investigate carefully all aspects of the problem. There are many specimens of this sort – much more subtle ones, to be sure – in the literature of mathematics. A number of them enjoyed careers lasting many years before some doubting mathematician finally succeeded in discovering the trouble.

The case of the self-contradicting liar is but one of a whole string of logical paradoxes of considerable importance. Invented by the early Greek philosophers, who used them chiefly to confuse their opponents in debate, they have in more recent times served to bring about revolutionary changes in ideas concerning the nature and foundations of mathematics. In a later chapter we shall have more to say about problems of this kind.

The island-and-lake problem, which had to do with definitions and reasoning from definitions, is really typical of the development of any mathematical theory. The mathematician first defines the objects with which he is going to work – numbers, or points, or lines, or even just 'elements' of an unspecified nature. He then lays down certain laws – 'axioms', he calls them, or 'postulates' – which are to govern the behaviour of the objects he has defined. On this foundation he builds, through a series of logical arguments, a whole structure of mathematical propositions, each one resting on the conclusions established before it. He is not interested, by the way, in the *truth* of his definitions or axioms, but asks only that they be *consistent*, that is, that they lead to no real contradiction in the propositions (such, for example, as the contradiction in the problem of the liar). Bertrand Russell, in his *Mysticism and Logic*, has put what we are trying to say in the following words:

Pure mathematics consists entirely of assertions to the effect that if such and such a proposition is true of anything, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is of which it is supposed to be true. . . . Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

How is that, by the way, for a paradox?

CHAPTER TWO

Paradoxes for Everyone

MANY of the anecdotes and problems of this chapter are fairly well known. All of them have probably appeared in print in some form or other and at some time or other, and a few are so common that they can be found in almost any book on mathematical puzzles and games. It is next to useless to try to trace them to their original sources – most of them, like Topsy, 'just growed'.[1*]

*

We shall begin with a couple of lessons in geography. The first concerns a man who, you will say, must have been a crank. He designed a square house with windows on all four sides, each window having a view to the south. No bay windows (which would take care of three sides) or anything of that sort. Now how on earth can this be done? *Where* on earth would be more to the point, for there is indeed only one place where such a house can be built. Does that give it away? You've got it – it's the North Pole, of course, from which any direction is south.

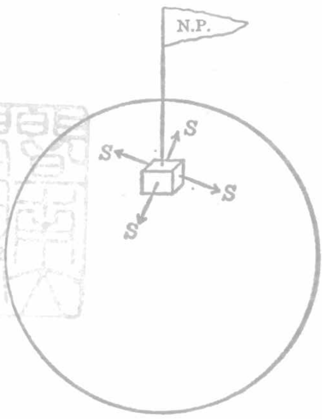

FIG. 3. Any direction from the North Pole is south

Without the foregoing discussion, the following problem strikes most people as quite paradoxical. A certain sportsman, experienced in shooting small game, was out on his first bear hunt. Suddenly he spotted a huge bear about a hundred yards due *east* of him. Seized with panic, the hunter ran – not directly away

* See page 227. Notes and references for all chapters will be found near the end of the book. They are inserted for the convenience of all who are interested in them and can be ignored safely by all who are not.

from the bear, but, in his confusion, due *north*. Having covered about a hundred yards, he regained his presence of mind, stopped, turned, and killed the bear – who had not moved from his original position – by shooting due *south*. Have you all the data clearly in mind? Very well, then; what colour was the bear?

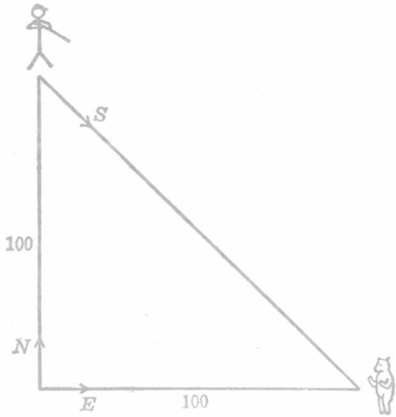

FIG. 4. Details of the bear hunt

The same problem can be put in another, although perhaps less startling form. Where can a man set out from his house, walk five miles due south, five miles due west, and five miles due north and find himself back home?

*

Charles L. Dodgson, better known to the general public as Lewis Carroll, the author of *Alice in Wonderland*, is recognized by mathematicians and logicians as one of their own number. We are indebted to him for the following paradox,[2] as well as for several others which appear in later parts of the book.

We can agree, can we not, that the better of two clocks is the one that more often shows the correct time? Now suppose we are offered our choice of two clocks, one of which loses a minute a day, while the other does not run at all. Which one shall we accept? Common sense tells us to take the one that loses a minute a day, but if we are to stick to our agreement, we shall have to take the one that doesn't run at all. Why? Well, the clock that loses a minute