Mathematics and Its Applications

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and P. M. Vasić (Eds.)

Means and Their Inequalities

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by

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SERIES EDITOR'S PREFACE

Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question.

'The Hermit Clad in Crane Feathers' in R. van Gulik's *The Chinese Maze Murders*.

It isn't that they can't see the solution. It is that they can't see the problem.

G.K. Chesterton. *The Scandal of Father Brown* 'The point of a Pin'.

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as "experimental mathematics", "CFD", "completely integrable systems", "chaos, synergetics and large-scale order", which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics. This programme, Mathematics and Its Applications, is devoted to new emerging (sub)disciplines and to such (new) interrelations as exempla gratia:

- a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

There are all kinds of small parts of mathematics which can be automatized, for which an expert system can be built. And certainly finding good algorithms for solving all kinds of problems and verifiable criteria for solvability is a main task of mathematicians. Thus working themselves out of work, were it not for the new tasks and problems that inevitably and unrelentingly come up.

One of the largely unheralded and largely unteachable abilities of the accomplished mathematician is that of finding good estimates for all kinds of things. Indeed sometimes I get the impression that this is the main art involved in certain parts of mathematics, especially in analysis and probability. In fact it is certainly a main aspect in all of mathematics; it also would appear to be one of the least automatizable parts of the trade.

A most important aspect of the art of obtaining estimates is certainly the eclectic use of all kinds of well known - even famous -, and lesser known, inequalities, itself an area of mathematics which tends to resist classification and systematization. It is also a very large field.

An absolutely astonishing - to me - number of inequalities are based on, or involve, means of various kinds. This is a part of the field of inequalities in general admitting a substantial amount of systematics and this is what this book is about. Systematic knowledge tends to be of much greater usefulness than an unsystematic jumble of facts. Thus I expect this book to enrich greatly the toolbox of all those whose use inequalities and estimates, i.e. practically every scientist and mathematician.

The unreasonable effectiveness of mathematics in science ...

Eugene Wigner

Well, if you know of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange.

Bussum, September 1987

Michiel Hazewinkel

PREFACE

There seem to be two types of books on inequalities. On the one hand there are the treatises that attempt to cover all or most aspects of the subject, and where an attempt is made to give all results in their best possible form, together with either a full proof or a sketch of the proof together with references to where a full proof or proofs can be found. Such books, aimed at the professional pure and applied mathematicians, are rare. The first such, that brought some order to this untidy field, is the classical "Inequalities" of Hardy-Littlewood and Pólya published in 1934. Important as this outstanding work was and still is it made no attempt at completeness; rather it consisted of the total knowledge of three front rank mathematicians in a field in which each had made fundamental contributions. Extensive as this combined knowledge was there were inevitably certain lacunae; some important results, such as the Steffensen Inequality were not mentioned at all; the works of certain schools of mathematicians were omitted and many important ideas were not developed, appearing as exercises at the ends of various chapters. The later "Inequalities" of Beckenbach and Bellman published in 1961 repairs many of these omissions. However this book is far from a complete coverage of the field either in depth or in scope. A much more definitive work is the recent "Analytic Inequalities" by D.S. Mitrinović, published in 1970, a work that is surprisingly complete considering the vast field to be covered.

On the other hand there are many works aimed at the student or non-mathematician. These introduce the reader to some particular section of the subject, giving him a feel for inequalities and enabling him to progress to the more advanced and detailed books mentioned above. Whereas the advanced books seem to exist only in English, excellent elementary books exist in several languages: "Analytic Inequalities" by Kazarinoff, "Geometric Inequalities" by Bottema, Djordjević, Janić, Mitrinović and Vasić in English, "Nejednakosti" by Mitrinović, "Sredine" by Mitrinović, Vasić in Serbo-Croat, to mention a few. Included in this group although slightly different are some books that list all the inequalities of a certain type—a sort of table of inequalities for reference; several of the books of D.S. Mitrinović are in part of this type.

Due to the wideness of the field and the variety of the applications none of the above mentioned books were complete on all of the topics they take up. Most inequalities depend on many parameters and what is the most natural domain for these parameters is not necessarily obvious and usually is not the widest possible range in which the inequality holds. Thus the author, even the most meticulous, is forced to choose; and what is omitted from the conditions of an inequality is just what is needed for some particular applications. What appears to be needed are works that pick some fairly restricted area from the vast subject of inequalities and treat it in depth. Such coherent parts of this discipline exists; as Hardy-Littlewood and Pólya showed, the subject of inequalities is not just a collection of results. However, no one seems to have written a treatise on some such limited but coherent area. The situation is different in the set of elementary books; several deal with certain fairly closely defined areas such as geometric inequalities, number theoretic inequalities, means, to mention a few.

It is the last mentioned area of means that is the topic of this book. Means are basic to the whole subject of inequalities and to many of the applications of inequalities to other fields. To take one example, the basic geometric-arithmetic mean inequality can be found lurking, often in an almost impenetrable disguise, behind inequalities in every area of the subject. The idea of mean is used extensively in probability and statistics, in the summations of series and integrals to mention but a few of the many applications of the subject. The object of this book is to provide as complete an account of the properties of means that occur in theory of inequalities as is within the authors' competence. The origin of this account is to be found in the much more modest "Sredine" mentioned above, which gives an elementary account of this topic.

A full discussion will be given of the various means that occur in the current literature of inequalities, together with a history of the origin of the various inequalities connecting these means, and a complete catalogue of all the important proofs of the basic results, as these indicate the many possible interpretations and applications that can be made. Also, all known inequalities involving means will be discussed. As it is in the nature of things that some omissions and errors will be made it is hoped that any reader who notices any will let the authors know, so that later editions can be more complete and more accurate.

An earlier version of this book was published in 1977 in Serbo-Croatian, "Sredine i sa Njima Povezane Nejednakosti". The present work is a complete revision and updating of that work.

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Some Basic References

There are some books on inequalities to which frequent reference will be made and which will be given short designations.

BB: Inequalities, E.F. Beckenbach and R. Bellman, Berlin-Heidelberg-New York, 1961.

HLP: Inequalities, G.H. Hardy, J.E. Littlewood and G. Pólya, Cambridge, 1934.

AI: Analytic Inequalities, D.S. Mitrinović (with the cooperation of P.M. Vasić), Berlin-Heidelberg-New York, 1970.

There are of course many other books on inequalities and many are listed in the Bibliography; Gini et al [1], Kazarinoff [1,2], Korovikin [1], Marshall and Olkin [2], Mitrinović [2,3,9], Mitrinović, Bullen and Vasić [1], Mitrinović and Vasić [11].

Many books contain important and useful sections on inequalities and a few of these are mentioned in the Bibliography; Aumann and Haupt [1], Borwein and Borwein [1], Bourbaki [1], Bromwich [1], Hardy [1], Littlewood [1], Melzak [1,3,4], Pólya [2], Whittaker and Watson [1].

From time to time conferences devoted to inequalities have published proceedings. In particular, there are the proceedings of three symposia held in the United States, and of four international conferences held at Oberwolfach. (Individual papers in these papers that are referred to in the text are listed under the various authors in the Bibliography.)

Inequalities, Proceedings of a Symposium held at Wright-Patterson Air Force Base, Ohio, August 19-27, 1965, ed. O Shisha, New York and London, 1967.

Inequalities II, Proceedings of the Second Symposium on Inequalities held at the United States Air Force Academy, Colorado, August 14-22, 1967, ed. O. Shisha, New York and London, 1970.

Inequalities III, Proceedings of the Third Symposium on Inequalities held at the University of California, Los Angeles, September 1-9, 1969, ed. O. Shisha, New York and London, 1972.

General Inequalities 1,2,3, Proceedings of First, Second and Third International Conferences on General Inequalities, Oberwolfach, 1976, 1978, 1981, ed. E.F. Beckenbach, Basel, 1978, 1980, 1983.

General Inequalities 4, Proceedings of Fourth International Conference on General Inequalities, in memoriam E.F. Beckenbach, Oberwolfach, 1983, ed. W. Walter, Basel, 1984.

NOTATIONS

1. Referencing. Theorems, definitions, lemmas, corollaries and formulae are listed consecutively in each section; remarks are numbered consecutively in each subsection. References to the same chapter list section, (subsection for remarks), followed by the detail; thus 3. Theorem 2, or 1.2 Remark (6). References to other chapters add the chapter number; thus I.3 Lemma 6(a), V 1.2. Remark (7).

2. For the notation of standard references see p. (vi).

3. Certain inequalities are given short names: B I.3(1); J I.5(4); GA II 2(1); H III.2(2); C III.2.1 Remark (10); M III.2(18); (r;s)III 3(1); S(r;s) IV 2(8). Most of these inequalities have various forms, the short name refers to that form appropriate to the context.

4. The notation $\mathbf{R}$, $\mathbf{Z}$, $\bar{\mathbf{R}}$ are standard; note that $\mathbf{N} = \{n;\ n \quad \mathbf{Z} \text{ and } n \geq 0\}$, $\mathbf{N}^* = \{n;\ n \in \mathbf{Z} \text{ and } n > 0\}$; $\mathbf{R}_+^* = \{x;\ x \in \mathbf{R} \text{ and } x > 0\}$, $\mathbf{R}_+ = \{x;\ x \in \mathbf{R} \text{ and } x \geq 0\}$.

5. Intervals are written $[a,b]$ (closed), $]a,b[$ (open), etc.

6. n-tples (= n-tuples) are written $\underline{a}$ and the usual vector notation is followed. In addition the following conventions are used: (i) $f(\underline{a}) = \big(f(a_1), \dots, f(a_n)\big)$ $f(\underline{a},\underline{b}) = \big(f(a_1,b_1), \dots, f(a_n,b_n)\big)$; in particular $\underline{a}\,\underline{b} = (a_1b_1, \dots, a_nb_n)$, $\max \underline{a} = \max_{1\leq i\leq n} a_i$, $\min \underline{a} = \min_{1\leq i\leq 1} a_i$: (ii) $\underline{a} \leq \underline{b}$ means $a_i \leq b_i$, $1 \leq i \leq n$; $m < \underline{a} < M$ means $m < a_i < M$, $1 \leq i \leq n$: (iii) $\underline{e}$ is the n-tple with $a_i = 1$, $1 \leq i \leq n$, $\underline{0}$ (or just 0) has $a_i = 0$, $1 \leq i \leq n$: (iv) $\underline{a}_i'$ is the (n-1)-tple $(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$: (v) $\underline{a} \sim \underline{b}$ means that $\underline{a}$ and $\underline{b}$ are linearly dependent: (vi) If $\underline{a} > 0$ we say $\underline{a}$ is positive, if $a_i = c$, $1 \leq i \leq n$, we say $\underline{a}$ is constant, or is a constant: (vii) $\underline{a} * \underline{b}$, see I.4

Definition 4: (viii) $\underline{a} \prec \underline{b}$, see I.4 Definition 12: (ix) If $\underline{W} = (w_1,\dots,w_n)$, $W_n = w_1+\dots+w_n = |\underline{w}|$.

7. Sequences are given the same notation as n-tples, and the same conventions are used.

8. If f is a function of k-variables, $\underline{a}$ an n-tple, $n \geq k$ then $\sum_k ! f(a_{i_1}, \dots, a_{i_k})$ means that the sum is taken over all permutations of k elements from $a_1, \dots, a_n$. In the case k = n this will be written $\sum ! f(\underline{a})$. (In a similar way $\prod_k !$).

The remaining notations are listed in order of appearance in the text.

$\Delta^k a_n$	I.4.1.(1)
W_I	I.5.1
$[\underline{a}; f]$	I.5.3
$A_n(\underline{a})$	II.1(1).
$A_n(a_i;\ 1 \leq i \leq n)$	I.1.1 Remark (1)
$A_n(a_1,\dots,a_n)$	I.1.1 Remark (1)
$A_{n-1}(\underline{a})$	II.1.1 Remark(1)
$A_n(\underline{a};\underline{w})$	II 1.(3)
$G_n(\underline{a};\underline{w})$	II.1. (4)
$H_n(\underline{a};\underline{w})$	II.1(5)
$G_n(\underline{a})$	II 1.2 Remark (1)
$H_n(\underline{a})$	II 1.2 Remark (1)
$A_I(\underline{a};\underline{w})$	II 3.2.2
$G_I(\underline{a};\underline{w})$	II 3.2.2
$\bar{\underline{a}}$	II. 3.2.2
$\bar{A}_m(\underline{a};\underline{w})$	II.3.2.2
$\bar{G}_m(\underline{a};\underline{w})$	II.3.2.2.
$M_n^{[r]}(\underline{a};\underline{w})$	III 1(1)
$M_n^{[r]}(\underline{a})$	III 1.1
$M_I^{[r]}(\underline{a};\underline{w})$	III 1.1.1
$Q_n(\underline{a};\underline{w})$	III 1.1
$S(r,\underline{a})$	III. 2.1

CONTENTS