Research Notes in Mathematics 63

J M Hill

Solution of differential equations by means of one-parameter groups

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Preface

The trouble with solving differential equations is that whenever we are successful we seldom stop to ask why. The concept of one-parameter transformation groups which leave the differential equation invariant provides the only unified understanding of all known special solution techniques. In these notes I have attempted to present a fairly concise and self-contained account of the use of one-parameter groups to solve differential equations. The presentation is formal and is intended to appeal to Applied Mathematicians and Engineers whose principal concern is obtaining solutions of differential equations. I have included only the essentials of the subject, sufficient to enable the reader to attempt the group approach when solving differential equations. I have purposely not included all known results since this would inevitably lead to unnecessarily reproducing large portions of existing accounts. For example, for ordinary differential equations, the account of the subject by L.E. Dickson, "Differential equations from the group standpoint", is still extremely readable and is recommended to the reader interested in pursuing the subject further. For partial differential equations the books by G.W. Bluman and J.D. Cole, "Similarity methods for differential equations", and by L.V. Ovsjannikov "Group properties of differential equations", contain several applications and examples which I have not reproduced here.

The first two chapters are introductory. Chapter 1 gives a general introduction with simple examples involving both ordinary and partial differential equations. In Chapter 2 the concepts of one-parameter groups and Lie series are introduced. Just as ordinary methods of solving differential equations often require a certain ingenuity so does the group approach. In order to establish some familiarity with the group method I have attempted to exploit our experience with linear equations. Most of us are aware that linear differential equations for $y(x)$ remain linear under the transformation $x_1 = f(x)$, $y_1 = g(x)y$ and Chapter 3 of these notes is devoted to implications of this result. In Chapters 4 and 5 I have tried to relate the usual theory for the group method with the results obtained in the third chapter. In this respect these notes differ from most accounts of the subject and I believe that a number of results given, especially in Chapter 3 are new.

The remaining two chapters are devoted to partial differential equations. For the most part the theory is illustrated with reference to diffusion related partial differential equations. The theory for linear partial differential equations is introduced in Chapter 6 for the classical diffusion or heat conduction equation and the Fokker-Planck equation. Non-linear equations are treated in Chapter 7. For partial differential equations the group approach is less satisfactory since for boundary value problems both the equation and boundary conditions must remain invariant. In these notes we principally consider only the invariance of the equation and view the group method as a means of systematically deducing solution types of a given partial differential equation.

Although these notes appear as a research monograph they actually represent advanced teaching material and in fact form the basis of a post-graduate course given at the University of Wollongong for the past six years. I have therefore included numerous examples and exercises. In addition to the exercises I have used the problems at the end of each chapter to conveniently locate standard results for differential equations. On occasions I have also used these problems to include summaries of theory which is already adequately described in the literature.

The existing theory of the solution of differential equations by means of one-parameter groups is by no means complete. Many of the inadequacies of the subject are highlighted in the text. When it does work it is very easy and it is therefore an area of knowledge which every Applied Mathematician ought to be aware of. Whatever the limitations of the group method may be, it will always represent a profoundly interesting idea towards solving differential equations. I hope these notes prove to be useful and complement the existing literature.

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Australia.

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Contents

1 Introduction

Although a good deal of research over the past two centuries has been devoted to differential equations our present understanding of them is far from complete. These notes are concerned with obtaining solutions of differential equations by means of one-parameter transformation groups which leave the equation invariant. This subject was initiated by Sophus Lie [1] over a hundred years ago. Such an approach is not always successful in deriving solutions. However it does provide a framework in which existing special methods of solution can be properly understood and also it is applicable to linear and non-linear equations alike. In formulating differential equations the Applied Mathematician inevitably makes certain assumptions. Using group theory these assumptions can be seen to hold the key to obtaining solutions of their equations.

The purpose of this chapter is to present a simple introduction to the subject for both ordinary and partial differential equations by means of simple familiar examples. For ordinary differential equations comprehensive accounts of the subject are given by Cohen [2], Dickson [3], Page [4] and more recently Bluman and Cole [5] and Chester [6]. For partial differential equations the reader may consult Bluman and Cole [5] and Ovsjannikov [7] where additional references may also be found.

1.1 ORDINARY DIFFERENTIAL EQUATIONS

In order to illustrate some of the ideas developed in these notes we consider a simple example. It is well known that the 'homogeneous' first order differential equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}, \tag{1.1}$$

can be made separable by the substitution $u(x,y) = y/x$ and the resulting solution is given by

$$\log x - \frac{1}{2}\left(\frac{y}{x}\right)^2 = C, \tag{1.2}$$

where C denotes an arbitrary constant. We might well ask the following questions:

Question 1 Why does the substitution $u(x,y) = y/x$ lead to a separable equation for u?

Question 2 How do we interpret the degree of freedom embodied in the arbitrary constant C in the solution?

Answers to these questions can be provided within the framework of transformations which leave the differential equation unaltered. Consider the following transformation,

$$x_1 = e^{\varepsilon}x, \qquad y_1 = e^{\varepsilon}y, \tag{1.3}$$

where ε is an arbitrary constant. We notice that (1.1) remains invariant under (1.3) in the sense that the differential equation in the new variables x_1 and y_1 is identical to the original equation, namely

$$\frac{dy_1}{dx_1} = \frac{x_1^2 + y_1^2}{x_1 y_1}. \tag{1.4}$$

Moreover we see that (1.3) satisfies the following:

(i) $\varepsilon = 0$ gives the identity transformation $x_1 = x$, $y_1 = y$,

(ii) $-\varepsilon$ characterizes the inverse transformation $x = e^{-\varepsilon}x_1$, $y = e^{-\varepsilon}y_1$,

(iii) if $x_2 = e^{\delta}x_1$, $y_2 = e^{\delta}y_1$ then the product transformation is also a member of the set of transformations (1.3) and moreover is characterized by the parameter $\varepsilon+\delta$, that is $x_2 = e^{\varepsilon+\delta}x$, $y_2 = e^{\varepsilon+\delta}y$.

A transformation satisfying these three properties is said to be a *one-parameter group of transformations*. We observe that the usual associativity law for groups follows from the property (iii). With this terminology established we might answer the above questions as follows:

Answer 1 The substitution $u(x,y) = y/x$ leads to a separable equation for u because $u(x,y)$ is an invariant of (1.3) in the sense that $u(x_1,y_1) = u(x,y)$ since,

$$u(x_1,y_1) = \frac{y_1}{x_1} = \frac{y}{x} = u(x,y) , \tag{1.5}$$

and it is this property which results in a simplification of (1.1). In general we shall see that if a differential equation is invariant under a one-parameter group of transformations then use of an invariant of the group results in a simplification of the differential equation. If the differential equation is of first order then it becomes separable while if the equation is of higher order then use of an invariant of the group permits a reduction in the order of the equation by one.

Answer 2 From (1.2) and (1.3) we see that we have

$$\log x_1 - \frac{1}{2}\left(\frac{y_1}{x_1}\right)^2 = C + \varepsilon , \tag{1.6}$$

so that the degree of freedom in the solution (1.2) resulting from the arbitrary constant C is related to the invariance of the differential equation (1.1) under the group of transformations (1.3) which is characterized by the arbitrary parameter ε. That is, the transformation (1.3) permutes the solution curves (1.2). In general we shall see that for every one-parameter group in two variables there are functions $u(x,y)$ and $v(x,y)$ such that the group becomes

$$u(x_1,y_1) = u(x,y) , \quad v(x_1,y_1) = v(x,y) + \varepsilon . \tag{1.7}$$

Moreover if a first order differential equation is invariant under this group then in terms of these new variables u and v it takes the form,

$$\frac{dv}{du} = \phi(u) \ , \tag{1.8}$$

and consequently has a solution of the form

$$v + \psi(u) = C \ , \tag{1.9}$$

for appropriate functions $\phi(u)$ and $\psi(u)$.

In order to give the reader some indication of the usefulness of the above we consider the following non-trivial equation,

$$y \frac{dy}{dx} = \frac{2}{x^3} - \frac{3}{x^2} y \ . \tag{1.10}$$

This is an Abel equation of the second kind (Murphy [8], page 25) which we see is not readily amenable to any of the standard devices. However the equation is clearly invariant under the group

$$x_1 = e^{\varepsilon} x \ , \quad y_1 = e^{-\varepsilon} y \ , \tag{1.11}$$

and therefore we choose $u(x,y) = xy$ as the new dependent variable and the differential equation (1.10) becomes,

$$xu \frac{du}{dx} = u^2 - 3u + 2 \ , \tag{1.12}$$

which can be readily integrated. It is worthwhile emphasizing that not all equations can be solved in such a simple manner. Consider for example,

$$y \frac{dy}{dx} = \left(\frac{2}{x^3} + 6\right) - \left(\frac{3}{x^2} + 6x\right) y \ , \tag{1.13}$$

which arises in finite elasticity (see Hill [9]). This equation is again an Abel equation of the second kind but in this case there is apparently no simple group such as (1.11) which leaves the equation invariant.

In this general introduction it may be appropriate to mention here possible research areas for which group theory has not yet been applied. The reader might well like to bear these problems in mind with a view to developing results in these areas.

Research area 1 Differential-difference equations.

It is well known that formal solutions of linear differential-difference equations, for example

$$\frac{dy(x)}{dx} = -y(x-x_0) , \tag{1.14}$$

where x_0 is a constant, can be expressed as

$$y(x) = \sum_j C_j e^{-\omega_j x} , \tag{1.15}$$

where C_j are arbitrary constants and ω_j denote the roots of $\omega = e^{\omega x_0}$.
If the equation is non-linear then there are no such general methods of solution. Consider for example Hutchinson's equation which can be written as

$$\frac{dy(x)}{dx} = y(x)[1 - y(x-x_0)] . \tag{1.16}$$

This equation arises in theory of populations (see Hutchinson [10]). What are the implications of group theory, if any, for equations of this type? (See problems 19 and 20 of Chapter 4).

Research area 2 Differential equations invariant under transformations which cannot be characterized as one-parameter groups.

A differential equation occurring in fluid dynamics is Tuck's equation (see Tuck [11]),

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} + \frac{(5+3x)}{4x(1+x)}\left(\frac{dx}{dt}\right)^2 + \frac{3x(1-x)}{(1+x)} . \tag{1.17}$$

It can be verified that if $x(t)$ is a solution then so is $x(t)^{-1}$. If in the usual way we let $y = dx/dt$ then (1.17) becomes

$$y\frac{dy}{dx} = \frac{3x(1-x)}{(1+x)} + 2y + \frac{(5+3x)}{4x(1+x)}y^2 , \tag{1.18}$$

which is again an Abel equation of the second kind. From the invariance property of (1.17) we can deduce that (1.18) remains invariant under the transformation

$$x_1 = \frac{1}{x}, \qquad y_1 = -\frac{y}{x^2}, \tag{1.19}$$

which clearly cannot be characterized as a one-parameter group. Can we use such invariance properties to determine solutions of differential equations?

Research area 3   Abel equation of the second kind.

As we have already indicated one of the most frequently occurring differential equations which is not always amenable to standard devices is the Abel equation of the second kind. The general equation can be expressed in the form (see Murphy [8], page 26)

$$y\frac{dy}{dx} = a(x) + b(x)y . \tag{1.20}$$

Equation (1.20) with arbitrary functions $a(x)$ and $b(x)$ would appear to be a problem worthwhile studying.

1.2 PARTIAL DIFFERENTIAL EQUATIONS

Unlike ordinary differential equations the success of the group approach for partial differential equations depends to a considerable extent on the accompanying boundary conditions. That is, the group approach is only effective in the solution of boundary value problems if both the equation and boundary conditions are left unchanged by the one-parameter group. For the most part we confine our attention to specific differential equations rather than boundary value problems. For any particular boundary value problem we should always first look for any simple invariance properties. These may be more apparent from the physical hypothesis of the problem rather than its mathematical formulation. If no such invariance can be found and if the problem merits a numerical solution then the group approach might still be relevant as a means of checking the numerical technique with artificially imposed boundary conditions which permit an exact analytic solution.

As an illustration we consider a boundary value problem for which both the partial differential equation and the boundary conditions are invariant under a simple one-parameter group. Consider the problem of determining the source solution for the one-dimensional diffusion or heat conduction equation for $c(x,t)$, namely

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} \qquad (t > 0,\ -\infty < x < \infty)\ . \tag{1.21}$$

The source solution of (1.21) is a solution which vanishes at infinity for all times and initially satisfies

$$c(x,0) = c_0\delta(x)\ , \tag{1.22}$$

where c_0 is a constant specifying the strength of the source and $\delta(x)$ is the usual Dirac delta function. We observe that both of (1.21) and (1.22) are left unchanged by the transformation

$$x_1 = e^{\varepsilon}x\ , \quad t_1 = e^{2\varepsilon}t\ , \quad c_1 = e^{-\varepsilon}c\ , \tag{1.23}$$

where ε denotes an arbitrary constant and we have made use of the elementary property of delta functions,

$$\delta(\lambda x) = \lambda^{-1}\delta(x)\ , \tag{1.24}$$

for any non-zero constant λ. Thus if $c = \phi(x,t)$ is the solution of (1.21) and (1.22) then we have also $c_1 = \phi(x_1,t_1)$. Clearly this is the case if $\phi(x,t)$ has the functional form

$$\phi(x,t) = t^{-\frac{1}{2}}\ \psi(xt^{-\frac{1}{2}})\ , \tag{1.25}$$

for some function ψ of the argument indicated. Upon substituting (1.25) into (1.21) we obtain the ordinary differential equation

$$2\psi''(\xi) + \xi\psi'(\xi) + \psi(\xi) = 0\ , \tag{1.26}$$

where ξ denotes $xt^{-\frac{1}{2}}$ and primes indicate differentiation with respect to ξ. Equation (1.26) can be reduced to the confluent hypergeometric equation (see Murphy [8], page 321). However the solution vanishing at infinity can be readily verified to be simply,

$$\psi(\xi) = Ae^{-\xi^2/4} , \tag{1.27}$$

where A denotes an arbitrary constant. This constant is determined from (1.22), namely

$$\int_{-\infty}^{\infty} c(x,t)dx = c_0 . \tag{1.28}$$

From this equation, (1.25) and (1.27) we find that the required solution of the boundary value problem (1.21) and (1.22) becomes

$$c(x,t) = c_0 \frac{e^{-x^2/4t}}{(4\pi t)^{\frac{1}{2}}} \qquad (t > 0, -\infty < x < \infty) . \tag{1.29}$$

This solution is of course well known. For our purposes it firstly serves as a specific non-trivial boundary value problem for which the differential equation and boundary conditions are both invariant under a one-parameter group. Secondly it serves to illustrate that knowledge of a one-parameter group leaving the equation invariant enables, at least in the case of two independent variables, the partial differential equation to be reduced to an ordinary differential equation. For more independent variables knowledge of a group leaving the equation unchanged reduces the number of independent variables by one.

In these notes we give the general procedure for determining the group such as (1.23) which leaves a specific equation invariant. We also give the general technique for establishing the functional form of the solution such as that given by (1.25).

PROBLEMS

1. Determine in each case the constants α and β such that the one-parameter group

$$x_1 = e^{\alpha\varepsilon}x , \qquad y_1 = e^{\beta\varepsilon}y ,$$

leaves the following differential equations invariant. Use an invariant of the group to integrate the equation.

(a) $\frac{dy}{dx} = \frac{A}{x^{3/2}} + By^3$ (A and B are constants) ,

(b) $x(x^4 - 2y^3)\frac{dy}{dx} + (2x^4 + y^3)y = 0$,

(c) $x(A + xy^n)\frac{dy}{dx} + By = 0$ (A, B and n are constants) .

2. Verify that,

$$x_1 = x + \varepsilon , \qquad y_1 = e^{-2\varepsilon}y ,$$

is a one-parameter group of transformations and hence integrate the differential equation

$$(1 - 2x - \log y)\frac{dy}{dx} + 2y = 0 .$$

3. Integrate the differential equation

$$(x-y)^2\frac{dy}{dx} = A^2 \quad \text{(A is a constant)} ,$$

by observing that the equation admits the group

$$x_1 = x + \varepsilon , \qquad y_1 = y + \varepsilon .$$

4. Given that $\rho(x)$ is a solution of the linear differential-difference equation (1.14) show that

$$y(x) = \frac{\rho(x-x_0)}{\rho(x)} ,$$

is a solution of the non-linear differential-difference equation

$$\frac{dy(x)}{dx} = y(x)[y(x) - y(x-x_0)] .$$