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Functional methods in differential equations

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Dedicated to Professor Wolfgang L. Wendland on the occasion of his 65th birthday

Introduction

In recent years, functional methods have become central to the study of theoretical and applied mathematical problems. An advantage of such an approach is its generality and its potential unifying effect of particular results and techniques.

Functional analysis emerged as an independent discipline in the first half of the 20th century, primarily as a result of contributions of S. Banach, D. Hilbert, and F. Riesz. Significant advances have been made in different fields, such as spectral theory, linear semigroup theory (developed by E. Hille, R.S. Phillips, and K. Yosida), the variational theory of linear boundary value problems, etc. At the same time, the study of nonlinear physical models led to the development of nonlinear functional analysis. Today, this includes various independent subfields, such as convex analysis (where H. Brézis, J.J. Moreau, and R.T. Rockafellar have been major contributors), the Leray-Schauder topological degree theory, the theory of accretive and monotone operators (founded by G. Minty, F. Browder, and H. Brézis), and the nonlinear semigroup theory (developed by Y. Komura, T. Kato, H. Brézis, M.G. Grandall, A. Pazy, etc.).

As a consequence, there has been significant progress in the study of nonlinear evolution equations associated with monotone or accretive operators (see, e.g., the monographs by H. Brézis [Brézis1], and V. Barbu [Barbu1]). The most important applications of this theory are concerned with boundary value problems for partial differential systems and functional differential equations, including Volterra integral equations. The use of functional methods leads, in some concrete cases, to better results as compared to the ones obtained by classical techniques. In this context, it is essential to choose an appropriate functional framework. As a byproduct of this approach, we will sometimes arrive at mathematical models that are more general than the classical ones, and better describe concrete physical phenomena; in particular, we shall reach a concordance between the physical sense and the mathematical sense for the solution of a concrete problem.

The purpose of this monograph is to emphasize the importance of functional methods in the study of a broad range of boundary value problems, as well as that of various classes of abstract differential equations.

Chapter 1 is dedicated to a review of basic concepts and results that are used throughout the book. Most of the results are listed without proofs. In some instances, however, the proofs are included, particularly when we could not identify an appropriate reference in literature.

Chapters 2 through 6 are concerned with concrete elliptic, parabolic, or hyperbolic boundary value problems that can be treated by appropriate functional methods.

In Chapter 2, we investigate various classes of, mainly one-dimensional, elliptic boundary value problems. The first section deals with nonlinear nondegenerate boundary value problems, both in variational and nonvariational cases. The approach relies on convex analysis and the monotone operator theory. In the second section, we start with a two-dimensional capillarity problem. In the special case of a circular tube, we obtain a degenerate one-dimensional problem. A more general, doubly nonlinear multivalued variant of this problem is thoroughly analyzed under minimal restrictions on the data.

Chapter 3 is concerned with nonlinear parabolic problems. We consider a so-called algebraic boundary condition that includes, as special cases, conditions of Dirichlet, Neumann, and Robin-Steklov type, as well as space periodic boundary conditions. The term "algebraic" indicates that the boundary condition is an algebraic relation involving the values of the unknown and its space derivative on the boundary. The theory covers various physical models, such as heat propagation in a linear conductor and diffusion phenomena. We treat the cases of homogeneous and nonhomogeneous boundary conditions separately, since in the second case we have a time-dependent problem. The basic idea of our approach is to represent our boundary value problem as a Cauchy problem for an ordinary differential equation in the L^2-space. As a special topic, we investigate in the last section of this chapter, the problem of the higher regularity of solutions.

In Chapter 4 we consider the same nonlinear parabolic equation as in Chapter 3, but with algebraic-differential boundary conditions. This means that we have an algebraic boundary condition as in the previous chapter, as well as a differential boundary condition that involves the time derivative of the unknown. This problem is essentially different from the one in Chapter 3, and a new framework is needed in order to solve it. Specifically, we arrive at a Cauchy problem in the space $L^2(0,1)\times\mathbb{R}$ (see (4.1.6)-(4.1.7)). Actually, this Cauchy problem is a more general model, since it describes physical situations that are not covered by the classical theory. More precisely, if the Cauchy problem has a strong solution (u,ξ), then necessarily $\xi(t) = u(1,t)$; in other words, the second component of the solution is the trace of the first one on the boundary. Otherwise, $\xi(t) \neq u(1,t)$, but it still describes an evolution on the boundary. This is important in concrete cases, such as dispersion or diffusion in chemical substances. As in the preceding chapter, we study the case of a homogeneous algebraic boundary condition separately from the nonhomogeneous one. The higher regularity of solutions is also discussed.

Chapter 5 is dedicated to a class of semilinear hyperbolic partial differential systems with a general nonlinear algebraic boundary condition. We first study the existence, uniqueness, and asymptotic behavior of solutions as $t \to \infty$, by using the product space $L^2(0,1)^2$ as a basic functional setup. The theory has applications in physics and engineering (e.g., unsteady fluid flow with nonlinear pipe friction, electrical transmission phenomena, etc.). Unlike the parabolic case, we do not separate the homogeneous and nonhomogeneous cases, since we can always homogenize the problem. Although this leads to a time-dependent system, we can easily handle it by appealing to classical results on nonlinear nonautonomous evolution equations. In the second section of this chapter, we discuss the higher regularity of solutions. This is important, for instance, for the singular perturbation analysis of such problems. The natural functional framework for this theory seems to be the C^k-space. It is also worth noting that the method we use to obtain regularity results is different from the one in Chapters 3 and 4, and involves some classical tools such as D'Alembert type formulae, and fixed point arguments.

In Chapter 6, we consider the same hyperbolic partial differential systems as in the preceding chapter, but with algebraic-differential boundary conditions. Such conditions are suggested by some applications arising in electrical engineering. As before, we restrict our attention to the homogeneous case only. This problem has distinct features, as compared to the one involving just algebraic boundary conditions. We now consider a Cauchy problem in the product space $L^2(0,1)^2 \times \mathbb{R}$. In the case of strong solutions, we recover the original problem, but in general, this incorporates a wider range of applications. Moreover, the weak solution of this Cauchy problem can be viewed as a generalized solution of the original model.

The remainder of the book is dedicated to abstract differential and integro-differential equations to which functional methods can be applied.

In Chapter 7, the classical Fourier method is used in the study of first and second order linear differential equations in a Hilbert space H. The operator appearing in the equations is assumed to be linear, symmetric, and coercive. In order to use a more general concept of solution, we replace the abstract operator in the equation by its "energetic" extension. A basic assumption is that the corresponding energetic space is compactly embedded into H. This guarantees the existence of orthonormal bases of eigenvectors, and enables us to employ Fourier type methods. Existence and regularity results for the solution are established. In the case of partial differential equations, our solutions reduce to generalized (Sobolev) solutions. Finally, nonlinear functional perturbations are handled by a fixed-point approach. As applications various parabolic and hyperbolic partial differential equations are considered. Since the perturbations are functional, a large class of integro-differential equations is also covered.

In Chapter 8, we discuss the existence and regularity of solutions for first order linear differential equations in Banach spaces with nonlinear functional perturbations. The main methods are the variation of constants formula for linear semigroups and the Banach fixed-point theorem. The theory is applied to the study of a class of hyperbolic partial differential equations with nonlinear boundary conditions.

In Chapter 9, we consider first order nonlinear, nonautonomous differential equations in Hilbert spaces. The equations involve a time-dependent unbounded subdifferential with time-dependent domain, perturbed by time-dependent maximal monotone operators and functionals that can be typically integrals of the unknown function. The treatment of the problem without functional perturbation relies on the methods of H. Brézis [Brézis1]; the problem with functional perturbation is handled by a fixed-point reasoning. As an application, a nonlinear parabolic partial differential equation with nonlinear boundary conditions is studied.

Chapter 10 is concerned with implicit differential equations in Hilbert spaces. Results on the existence, uniqueness, and continuous dependence of solutions for related initial value problems are presented. The study of implicit differential equations is motivated by the two phase Stefan problem, which has recently attracted attention because of its importance for the optimal control of continuous casting of steel.

We continue with some general remarks regarding the structure of the book. The material is divided into chapters, which, in turn, are divided into sections. The main definitions, theorems, propositions, etc. are denoted by three digits: the first indicates the chapter, the second the corresponding section, and the third the position of the respective item in the section. For example, Proposition 1.2.3 denotes the third proposition of Section 2 in Chapter 1. Each chapter has its own bibliography but the labels are unique throughout the book.

We also note that many results are only sketched, in order to keep the book length within reasonable limits. On the other hand, this requires an active participation of the reader.

With the exception of Chapter 1, the book contains material mainly due to the authors, as considerably revised or expanded versions of earlier works. An earlier book by one of the authors must be here quoted [Moro6].

We would like to mention that the contribution of the former author was partly accomplished at Ohio University in Athens, Ohio, USA, in the winter of 2001. The work of the latter author was completed during his visits at Ohio University in Athens, Ohio, USA (fall 2000) and the University of Stuttgart, Germany (2001).

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Contents

Chapter 1

Preliminaries

This chapter has an introductory character. Its aim is to remind the reader of some basic concepts, notations, and results that will be used in the next chapters. In general, we shall not insist very much on the notations and concepts because they are well known. Also, the proofs of most of the theorems will be omitted, the appropriate references being indicated. However, there are a few exceptions, namely Propositions 1.2.1, 1.2.2, and 1.2.3, which might be known, but we could not identify them in literature. The material of this chapter is divided into several sections and subsections.

1.1 Function and distribution spaces

The L^p-spaces

We denote $\mathbb{R} = (-\infty, \infty)$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\mathbb{N}^* = \{1, 2, \ldots\}$. Let X be a real Banach space with norm $\|\cdot\|_X$. If $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}^*$, is a Lebesgue measurable set, we denote, as usual, by $L^p(\Omega; X)$, $1 \le p < \infty$, the space of all equivalence classes (with respect to the equality a.e. in Ω) of (strongly) measurable functions $f\colon \Omega \to X$ such that $x \mapsto \|f(x)\|_X^p$ is Lebesgue integrable over Ω. In general, every class of $L^p(\Omega; X)$ is identified with one of its representatives. $L^p(\Omega; X)$ is a real Banach space with the norm

$$\|u\|_{L^p(\Omega;X)} = \left(\int_\Omega \|u(x)\|_X^p \, dx \right)^{\frac{1}{p}}.$$

We shall denote by $L^\infty(\Omega; X)$ the space of all equivalence classes of measurable functions $f\colon \Omega \to X$ such that $x \mapsto \|f(x)\|_X$ are essentially bounded in Ω. Again, every class of $L^\infty(\Omega; X)$ is identified with one of its representatives. $L^\infty(\Omega; X)$ is a real Banach space with the norm

$$\|u\|_{L^\infty(\Omega;X)} = \operatorname*{ess\,sup}_{x \in \Omega} \|u(x)\|_X.$$

In the case $X = \mathbb{R}$ we shall simply write $L^p(\Omega)$ instead of $L^p(\Omega; \mathbb{R})$, for every $1 \leq p \leq \infty$. On the other hand, if Ω is an interval of real numbers, say $\Omega = (a, b)$ where $-\infty \leq a < b \leq \infty$, then we shall write $L^p(a, b; X)$ instead of $L^p\big((a, b); X\big)$. We shall also denote by $L^p_{loc}(\mathbb{R}; X)$, $1 \leq p \leq \infty$, the space of all (equivalence classes of) measurable functions $u: \mathbb{R} \to X$ such that the restriction of u to every bounded interval $(a, b) \subset \mathbb{R}$ is in $L^p(a, b; X)$. If $X = \mathbb{R}$, then this space will be denoted by $L^p_{loc}(\mathbb{R})$.

The theory of L^p-spaces is well known. So, classical results, such as Fatou's lemma, the Lebesgue Dominated Convergence Theorem, etc., will be used in the text without recalling them here.

Scalar distributions. Sobolev spaces

In the following we assume that Ω is a nonempty open subset of $\mathbb{R}^N$. Denote, as usual, by $C^k(\Omega)$, $k \in \mathbb{N}$, the space of all functions $f: \Omega \to \mathbb{R}$ that are continuous on Ω, and their partial derivatives up to the order k exist and are all continuous on Ω. Of course, $C^0(\Omega)$ will simply be denoted by $C(\Omega)$. In addition, we shall need the following spaces

$$C^\infty(\Omega) = \{\phi \in C(\Omega) \mid \phi \text{ has continuous partial derivatives of any order}\},$$
$$C_0^\infty(\Omega) = \{\phi \in C^\infty(\Omega) \mid \text{supp}\,\phi \text{ is a compact set included in } \Omega\},$$

where supp ϕ is the support of ϕ, i.e., the closure of the set $\{x \in \Omega \mid \phi(x) \neq 0\}$. When $C_0^\infty(\Omega)$ is endowed with the usual inductive limit topology, then it is denoted by $\mathcal{D}(\Omega)$.

DEFINITION 1.1.1 *A linear continuous functional $u: \mathcal{D}(\Omega) \to \mathbb{R}$ is said to be a distribution on Ω. The linear space of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$.*

Actually, $\mathcal{D}'(\Omega)$ is nothing else but the dual of $\mathcal{D}(\Omega)$. Notice that if $u \in L^1_{loc}(\Omega)$ (i.e., u is Lebesgue integrable on every compact subset of Ω), then the functional defined by

$$\mathcal{D}(\Omega) \ni \phi \mapsto \int_\Omega u(x)\phi(x)\,dx$$

is a distribution on Ω. Such a distribution will always be identified with the corresponding function u and so it will be denoted by u.

Now, recall that the partial derivative of a distribution $u \in D'(\Omega)$ with respect to x_j is defined by

$$\frac{\partial u}{\partial x_j}(\phi) = -u\left(\frac{\partial \phi}{\partial x_j}\right) \text{ for all } \phi \in D(\Omega),$$

and the higher order partial derivatives of u are defined iteratively, i.e.,

$$D^\alpha u(\phi) = (-1)^{|\alpha|} u(D^\alpha \phi) \text{ for all } \phi \in \mathcal{D}(\Omega),$$

where $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{N}^N$ is a so-called multiindex and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_N$. If $\alpha = (0, 0, \ldots, 0)$, then, by convention, $D^\alpha u = u$.

DEFINITION 1.1.2 *Let $1 \leq p \leq \infty$ and let $k \in \mathbb{N}^*$ be fixed. Then, the set*

$$W^{k,p}(\Omega) = \{u \colon \Omega \to \mathbb{R} \mid D^\alpha u \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^N \text{ with } |\alpha| \leq k\}$$

(where $D^\alpha u$ are the derivatives of u in the sense of distributions) is said to be a Sobolev space of order k.

Recall that, for each $1 \leq p < \infty$ and $k \in \mathbb{N}^*$, $W^{k,p}(\Omega)$ is a real Banach space with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \Big(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}}.$$

$W^{k,\infty}(\Omega)$ is also a real Banach space with the norm

$$\|u\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

The completion of $\mathcal{D}(\Omega)$ with respect to the norm of $W^{k,p}(\Omega)$ is denoted by $W_0^{k,p}(\Omega)$. In general, $W_0^{k,p}(\Omega)$ is strictly included in $W^{k,p}(\Omega)$. In the case $p = 2$ we have the notation

$$H^k(\Omega) := W^{k,2}(\Omega), \; H_0^k(\Omega) := W_0^{k,2}(\Omega).$$

These are both Hilbert spaces with respect to the scalar product

$$(u, v)_k := \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u(x) D^\alpha v(x)\, dx.$$

The dual of $H_0^k(\Omega)$ is denoted by $H^{-k}(\Omega)$. If Ω is an open bounded subset of $\mathbb{R}^N$, with a sufficiently smooth boundary $\partial\Omega$, then

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \mid \text{ the trace of } u \text{ on } \partial\Omega \text{ vanishes } \}.$$

If, in particular, Ω is an interval of real numbers, say $\Omega = (a, b)$ with $a < b$, then we shall write $C_0^\infty(a, b)$, $W^{k,p}(a, b)$, $H^k(a, b)$, and $W_0^{k,p}(a, b)$ instead of $C_0^\infty\big((a, b)\big)$, $W^{k,p}\big((a, b)\big)$, $H^k\big((a, b)\big)$, and $W_0^{k,p}\big((a, b)\big)$, respectively. If a, b are finite numbers, then every element of $W^{k,p}(a, b)$, $k \in \mathbb{N}^*$, $1 \leq p \leq \infty$, can be identified with an absolutely continuous function $f: [a,b] \to \mathbb{R}$ such that $d^j f/dt^j$, $1 \le j \le k-1$, exist and are all absolutely continuous on $[a,b]$, and $d^k f/dt^k$ (that obviously exists a.e. in (a,b)) belongs to $L^p(a,b)$ (more precisely, the equivalence class of $d^k f/dt^k$, with respect to the equality a.e. on (a,b) belongs to $L^p(a,b)$). Moreover, every element of $W_0^{k,p}(a,b)$ can be identified with such a function f, which also satisfies the conditions

$$\frac{d^j f}{dt^j}(a) = \frac{d^j f}{dt^j}(b) = 0,\ 0 \le j \le k-1.$$

Recall that if $-\infty < a < b < \infty$ and $k \in \mathbb{N}^*$, then $W^{k,1}(a,b)$ is continuously embedded into $C^{k-1}[a,b]$ (in particular, $W^{1,1}(a,b)$ is continuously embedded in $C[a,b]$). Finally, we set for $k \in \mathbb{N}^*$ and $1 \le p \le \infty$,

$$W^{k,p}_{loc}(\mathbb{R}) = \{u: \mathbb{R} \to \mathbb{R} \mid D^\alpha u \in L^p_{loc}(\mathbb{R}) \text{ for all } \alpha \in \mathbb{N} \text{ with } \alpha \le k\}.$$

Vectorial distributions. The spaces $W^{k,p}(a,b;X)$

Let Ω be an open interval (a,b) with $-\infty \le a < b \le \infty$ and denote by $\mathcal{D}'(a,b;X)$ the space of all continuous linear operators from $\mathcal{D}(a,b) := \mathcal{D}\big((a,b)\big)$ to X. The elements of $\mathcal{D}'(a,b;X)$ are called *vectorial distributions* on (a,b) with values in X. If $u: (a,b) \to X$ is integrable (in the sense of Bochner) over every bounded interval $I \subset (a,b)$ (i.e., equivalently, $t \mapsto \|u(t)\|_X$ belongs to $L^1(I)$, for every bounded subinterval I), then u defines a vectorial distribution, again denoted by u, as follows,

$$u(\phi) := \int_a^b \phi(t)u(t)\,dt \text{ for all } \phi \in \mathcal{D}(a,b).$$

The distributional derivative of order $j \in \mathbb{N}$ of $u \in \mathcal{D}'(a,b;X)$ is the distribution defined by

$$u^{(j)}(\phi) := (-1)^j u\Big(\frac{d^j\phi}{dt^j}\Big), \text{ for all } \phi \in \mathcal{D}(a,b),$$

where $d^j\phi/dt^j$ is the j-th ordinary derivative of ϕ. By convention, $u^{(0)} = u$.

Now, for $k \in \mathbb{N}^*$ and $1 \le p \le \infty$, we set

$$W^{k,p}(a,b;X) = \{u \in L^p(a,b;X) \mid u^{(j)} \in L^p(a,b;X),\ j = 1,2,\dots,k\},$$

where $u^{(j)}$ is the j-th distributional derivative of u. For each $k \in \mathbb{N}^*$ and $1 \le p < \infty$, $W^{k,p}(a,b;X)$ is a Banach space with the norm

$$\|u\|_{W^{k,p}(a,b;X)} = \Big(\sum_{j=0}^{k} \|u^{(j)}\|^p_{L^p(a,b;X)}\Big)^{\frac{1}{p}}.$$

Also, for each $k \in \mathbb{N}^*$, $W^{k,\infty}(a,b;X)$ is a Banach space with the norm

$$\|u\|_{W^{k,\infty}(a,b;X)} = \max_{0\le j\le k} \|u^{(j)}\|_{L^\infty(a,b;X)}.$$

As in the scalar case, for $p=2$, we may use the notation $H^k(a,b;X)$ instead of $W^{k,2}(a,b;X)$. Recall that, if X is a real Hilbert space with its scalar product denoted by $(\cdot,\cdot)_X$, then for each $k \in \mathbb{N}^*$, $H^k(a,b;X)$ is also a Hilbert space with respect to the scalar product

$$(u,v)_{H^k(a,b;X)} = \sum_{j=0}^{k} \int_a^b \left(u^{(j)}(t), v^{(j)}(t)\right)_X dt.$$

As usual, for $k \in \mathbb{N}^*$ and $1 \le p \le \infty$, we set

$$W^{k,p}_{loc}(a,b;X) = \{u \in \mathcal{D}'(a,b;X) \mid u \in W^{k,p}(t_1,t_2;X), \text{ for every } t_1,t_2 \in (a,b) \text{ with } t_1 < t_2\}.$$

In what follows, we shall assume that $-\infty < a < b < \infty$. For $k \in \mathbb{N}^*$ and $1 \le p \le \infty$, denote by $A^{k,p}([a,b];X)$ the space of all absolutely continuous functions $f\colon [a,b] \to X$ for which $d^j f/dt^j$, $1 \le j \le k-1$, exist, are all absolutely continuous, and (the class of) $d^k f/dt^k \in L^p(a,b;X)$.

If X is a reflexive Banach space and $v\colon [a,b] \to X$ is absolutely continuous, then v is differentiable a.e. on (a,b), $dv/dt \in L^1(a,b;X)$, and

$$v(t) = v(a) + \int_a^t \frac{dv}{ds}(s)\, ds, \ a \le t \le b.$$

Therefore, if X is reflexive, then $A^{1,1}(a,b;X)$ coincides with the space of all absolutely continuous functions $v\colon [a,b] \to X$, i.e.,

$$A^{1,1}([a,b];X) = AC([a,b];X).$$

We also recall the following result.

THEOREM 1.1.1

Let $1 \le p \le \infty$ and $k \in \mathbb{N}^*$ be fixed and let $u \in L^p(a,b;X)$ with $-\infty < a < b < \infty$. Then $u \in W^{k,p}(a,b;X)$ if and only if u has a representative in $A^{k,p}([a,b];X)$.

So, $W^{k,p}(a,b;X)$ will be identified with $A^{k,p}([a,b];X)$. If X is reflexive, then $W^{1,1}(a,b;X)$ can be identified with $AC([a,b];X)$, while $W^{1,\infty}(a,b;X)$ can be identified with $Lip([a,b];X)$ (the space of all Lipschitz continuous functions $v\colon [a,b] \to X$).

THEOREM 1.1.2

Let X be a real reflexive Banach space and let $u \in L^p(a,b;X)$ with $-\infty < a < b < \infty$ and $1 < p < \infty$. Then, the following two conditions are equivalent:

(i) $u \in W^{1,p}(a,b;X)$;

(ii) There exists a constant $C > 0$ such that

$$\int_a^{b-\delta} \|u(t+\delta) - u(t)\|_X^p \, dt \le C\delta^p \text{ for all } \delta \in (0, b-a].$$

Moreover, if $p = 1$ then (i) implies (ii) (actually, (ii) is true for $p = 1$ if one representative of $u \in L^1(a,b,X)$ is of bounded variation on $[a,b]$, where X is a general Banach space, not necessarily reflexive).

Now, let V and H be two real Hilbert spaces such that V is densely and continuously embedded in H. If H is identified with its own dual, then we have $V \subset H \subset V^*$, algebraically and topologically, where V^* denotes the dual of V. Denote by $\langle \cdot, \cdot \rangle$ the dual pairing between V and V^*, i.e., $\langle v, v^* \rangle = v^*(v)$, $v \in V$, $v^* \in V^*$. For $v^* \in H^* = H$, $\langle v, v^* \rangle$ reduces to the scalar product in H of v and v^*.

Now, for some $-\infty < a < b < \infty$, we set

$$W(a,b) := \{ u \in L^2(a,b;V) \mid u' \in L^2(0,T;V^*) \},$$

where u' is the distributional derivative of u. Obviously, every $u \in W(a,b)$ has a representative $u_1 \in A^{1,2}([a,b];V^*)$ and so u is identified with u_1. Moreover, we have:

THEOREM 1.1.3

Every $u \in W(a,b)$ has a representative $u_1 \in C([a,b];H)$ and so u can be identified with such a function. Furthermore, if u, $\tilde{u} \in W(a,b)$, then the function $t \mapsto (u(t), \tilde{u}(t))_H$ is absolutely continuous on $[a,b]$ and

$$\frac{d}{dt}\big(u(t), \tilde{u}(t)\big)_H = \langle u(t), \tilde{u}'(t) \rangle + \langle \tilde{u}(t), u'(t) \rangle \text{ for a.a. } t \in (a,b).$$

Hence, in particular,

$$\frac{d}{dt}\|u(t)\|_H^2 = 2\langle u(t), u'(t) \rangle \text{ for a.a. } t \in (a,b).$$