Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Approximate Identities and Factorization in Banach Modules

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Authors
Robert S. Doran
Department of Mathematics
Texas Christian University
Fort Worth, Texas 76129
USA

Josef Wichmann
Department of Mathematical Sciences
University of Petroleum & Minerals
Dhahran, Saudi Arabia

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PREFACE

In recent years Banach algebras with an approximate identity have received an increasing amount of attention. Many results known for C^*-algebras or for algebras with an identity have been extended to algebras with an approximate identity. This is an important extension since it includes the group algebra $L^1(G)$ of a locally compact group G and many other naturally occuring algebras.

Surprisingly little is known about the approximate identities themselves. In this monograph we have tried to collect all basic results about them with the aim of stimulating further research in this direction. As the main tool in the study of Banach algebras with bounded approximate identity we present Cohen-Hewitt's factorization theorem for Banach modules and its many refinements. Recognizing that factorization theory is a subject of great importance in its own right, we have included the most recent and up-to-date results in this area that we had knowledge of.

The level of exposition should be appropriate for those who are familiar with basic real and complex analysis, the elementary theory of commutative Banach algebras, and first results concerning C^*-algebras (say, a nodding acquaintence with the first two chapters of Dixmier [70]). Granted these, the monograph contains complete proofs, although we should warn the reader that some of the arguments are a bit tedious and will require some diligence on his part. We have assumed throughout that all algebras are over the complex field. The interested reader can determine, by examining a given definition or theorem, if the complexes can be replaced by the reals.

Examples and counterexamples are discussed whenever we knew of them. There is much further research to be done, and we have indicated in the last chapter a number of unsolved problems. The reader will also find in this last chapter a description of many interesting results, with references, which could not be included in the text proper without greatly increasing the size of this volume. A comprehensive bibliography concerning approximate identities and factorization has been assembled with both the Mathematical Review number and Zentralblatt number attached to aid the reader in his study of the subject. Although a reasonable attempt has been made in the last chapter to cite appropriate sources, omissions have undoubtedly occured and we apologize in advance to those we may have overlooked.

The authors are deeply grateful to Robert B. Burckel and Barry E. Johnson for their contributions to the present volume. Dr. Burckel carefully read two earlier versions of the manuscript and suggested many corrections, additions, and improvements; to him we offer our deepest thanks. Dr. Johnson also read an earlier version of the manuscript and made many helpful suggestions which have been incorporated into this volume. In addition, we wish to thank Peter G. Dixon for valuable correspondence, preprints, and reprints. Much of what is in this volume concerning approximate identities is due to him. His willingness to communicate his work, even when very busy, has been a big help to us. We thank him too for pointing out the recent work of Allan M. Sinclair, and then we must thank Dr. Sinclair for graciously allowing us to include some of his beautiful results prior to their publication. Many other friends have sent us their work, and to them we are also grateful. We wish to thank Ronald L. Morgan, a student of the first author, for reading the final manuscript and pointing out a number of slips which had gone undetected. Of course, any remaining errors or inaccuracies are the sole responsibility of the authors.

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Robert S. Doran

Josef Wichmann

CONTENTS

CHAPTER III. MORE ABOUT APPROXIMATE IDENTITIES

CHAPTER I

APPROXIMATE IDENTITIES IN NORMED ALGEBRAS

Our purpose in this chapter is to set forth in a systematic way some of the main properties of approximate identities in a normed algebra. Section 1 contains definitions and a useful reformulation of the definition of approximate identity. In Section 2 the relationship between left, right, and two-sided approximate identities is examined. Sections 3, 4 and 5 center on the problem of finding an analogue for approximate identities of the classical result that every normed algebra with identity has an equivalent algebra norm in which the identity has norm one. Sections 7 and 8 give basic results on approximate identities in quotient algebras and the tensor product of two normed algebras.

The notion of bounded approximate unit is defined and studied in Section 9 and we show, among other things, that a normed algebra has a bounded approximate unit if and only if it has a bounded approximate identity. The relationships between topological divisors of zero, topologically nilpotent elements and approximate identities are studied in Sections 10 and 11. Approximate identities in C^*-algebras is the topic of Section 12. The main result is that every C^*-algebra contains an increasing approximate identity bounded by one. Finally we end the chapter with a brief look at approximate identities in the group algebra of a locally compact group.

I. APPROXIMATE IDENTITIES IN NORMED ALGEBRAS

1. Approximate identities.

(1.1) Definitions. Let A be a normed algebra. A net $\{e_\lambda\}_{\lambda\varepsilon\Lambda}$ in A is called a left (resp. right, two-sided) approximate identity, abbreviated l.a.i. (resp. r.a.i., t.a.i.), if for all $x \varepsilon A$,

$$\lim_{\lambda\varepsilon\Lambda} e_\lambda x = x,$$

(resp. $\lim_{\lambda\varepsilon\Lambda} xe_\lambda = x$, $\lim_{\lambda\varepsilon\Lambda} e_\lambda x = x = \lim xe_\lambda$). It is said to be bounded if there is a constant K such that $||e_\lambda|| \leq K$ for all $\lambda \varepsilon \Lambda$; in this case, we define the bound of $\{e_\lambda\}_{\lambda\varepsilon\Lambda}$ by $\sup \{||e_\lambda||: \lambda \varepsilon \Lambda\}$, and the norm by

$$||\{e_\lambda\}|| = \limsup_\lambda ||e_\lambda||.$$

An approximate identity $\{e_\lambda\}_{\lambda\varepsilon\Lambda}$ is sequential if Λ is the set of positive integers with the usual order, and is said to be countable if it has countable range. It is abelian (or commuting) if e_{λ_1} and e_{λ_2} commute for all $\lambda_1, \lambda_2 \varepsilon \Lambda$.

Remark. The notions of bound and norm of an approximate identity are closely related: if $\{e_\lambda\}$ is a l.a.i. of norm K, with $e_\lambda \neq 0$ for each λ, then $\{K||e_\lambda||^{-1}e_\lambda\}$ is a l.a.i. of bound K. The norm of an approximate identity is always ≥ 1. Indeed,

$$||x|| = \lim_\lambda ||e_\lambda x|| \leq \limsup_\lambda ||e_\lambda||\cdot||x|| \quad \text{for all } x \varepsilon A,$$

so $1 \leq \limsup_\lambda ||e_\lambda||$.

§1. APPROXIMATE IDENTITIES

(1.2) Proposition. A normed algebra A has a l.a.i. (bounded by K) if and only if for every finite set $\{x_1,\ldots,x_n\}$ of elements in A and every $\varepsilon > 0$ there exists an element $e \in A$ (with $||e|| \leq K$) such that $||x_i - ex_i|| < \varepsilon$ for $i = 1,\ldots,n$.

Proof. Let A be a normed algebra with a l.a.i. $\{e_\lambda\}_{\lambda\in\Lambda}$ (bounded by K). Consider a finite set $\{x_1,\ldots,x_n\}$ of elements in A and choose $\varepsilon > 0$. Since

$$\lim_{\lambda\in\Lambda} e_\lambda x_i = x_i \quad \text{for} \quad i = 1,\ldots,n$$

there exist $\lambda_i \in \Lambda$ $(i = 1,\ldots,n)$ such that for $i = 1,\ldots,n$

$$||x_i - e_\lambda x_i|| < \varepsilon \quad \text{for all} \quad \lambda \geq \lambda_i.$$

Since Λ is a directed set there exists a $\lambda_o \in \Lambda$ with

$$\lambda_o \geq \lambda_i \quad \text{for all} \quad i = 1,\ldots,n.$$

Then for $e_{\lambda_o} \in A$ (with $||e_{\lambda_o}|| \leq K$) we have

$$||x_i - e_{\lambda_o} x_i|| < \varepsilon \quad \text{for} \quad i = 1,\ldots,n.$$

Conversely, assume that A is a normed algebra with the property that (there exists a constant K such that) for every finite set $\{x_1,\ldots,x_n\}$ of elements in A and every $\varepsilon > 0$ there exists an element $e \in A$ (with $||e|| \leq K$) such that

$$||x_i - ex_i|| < \varepsilon \quad \text{for all} \quad i = 1,\ldots,n.$$

Denote by Λ the set of all pairs $\lambda = (F,n)$ with F a finite subset of A and $n = 1,2,\ldots$; Λ is a directed set with respect to the partial ordering $\leq$ defined by: $(F_1,n_1) \leq (F_2,n_2)$ iff $F_1 \subset F_2$ and $n_1 \leq n_2$. Then for each $\lambda = (F,n)$ in Λ there is an element $e_\lambda \in A$ (with $||e_\lambda|| \leq K$) such that

$$||x - e_\lambda x|| < \frac{1}{n} \quad \text{for all} \quad x \in F.$$

Thus for every $x \in A$ and $\varepsilon > 0$ there is a $\lambda_o \in \Lambda$ such that

$$||x - e_\lambda x|| < \varepsilon \quad \text{for all} \quad \lambda \geq \lambda_o;$$

i.e., the net $\{e_\lambda\}_{\lambda\in\Lambda}$ (bounded by K) has the property:

$$\lim_{\lambda\in\Lambda} e_\lambda x = x \quad \text{for all} \quad x \in A.$$

Hence $\{e_\lambda\}_{\lambda\in\Lambda}$ is a l.a.i. for A (bounded by K). □

(1.3) <u>Proposition</u>. A normed algebra A has a t.a.i. (bounded by K) if and only if for every finite set $\{x_1,\ldots,x_n\}$ of elements in A and every $\varepsilon > 0$ there exists an element $e \in A$ (with $||e|| \leq K$) such that $||x_i - ex_i|| < \varepsilon$ and $||x_i - x_ie|| < \varepsilon$ for all $i = 1,\ldots,n$.

<u>Proof</u>. Analogous to the proof of Proposition (1.2). □

The following elementary result will be very helpful.

(1.4) <u>Lemma</u>. Let D be a dense subset of a normed algebra A. If A has a l.a.i. (r.a.i., t.a.i.) $\{e_\lambda\}_{\lambda\in\Lambda}$ then A has a l.a.i. (r.a.i., t.a.i.) $\{f_\mu\}_{\mu\in M}$ such that $f_\mu \in D$ ($\mu \in M$). If, further, $\{e_\lambda\}_{\lambda\in\Lambda}$ is sequential or bounded by K, then $\{f_\mu\}_{\mu\varepsilon M}$ may be chosen to have the same property. Conversely, if there is in D a net $\{g_\nu\}$ which is a bounded l.a.i. (r.a.i., t.a.i.) for D, then $\{g_\nu\}$ is a bounded l.a.i. (r.a.i., t.a.i.) for A.

Proof. This is straightforward. □

(1.5) Proposition. Let A be a separable normed algebra. If A has a bounded l.a.i., then it has a sequential l.a.i. bounded by the same constant.

Proof. Let $\{x_n\}_{n=1}^\infty$ be a countable dense subset of A and let $\{e_\lambda\}_{\lambda\varepsilon\Lambda}$ be a bounded l.a.i. in A. For each n = 1,2,... choose $\lambda_n \varepsilon \Lambda$ such that $||x_i - e_{\lambda_n} x_i|| < \frac{1}{n}$ for all i = 1,...,n. Then $\{e_{\lambda_n}\}_{n=1}^\infty$ is a sequential l.a.i. bounded by the same constant. □

2. One-sided and two-sided.

We shall now study the relationship between left, right, and two-sided approximate identities.

In general the existence of a l.a.i. and a r.a.i. in an algebra does not imply the existence of a t.a.i. Our counterexample will be the semigroup algebra of a semigroup which is itself defined by specifying generators and relations. To give an explicit description of the elements of the semigroup we shall need the following lemma.

(2.1) Lemma. Let S be the semigroup on a set of generators T with relations

(1) $$t_1t_2 = \gamma(t_1,t_2) \in T \quad ((t_1,t_2) \in R \subset T \times T).$$

Then the following are equivalent:

i) whenever $(t_1,t_2) \in R$ and $(t_2,t_3) \in R$ we have

a) $(t_1,t_2t_3) \in R$ if and only if $(t_1t_2,t_3) \in R$; and

b) if $(t_1,t_2t_3) \in R$ and $(t_1t_2,t_3) \in R$, then $\gamma(t_1,t_2t_3) = \gamma(t_1t_2,t_3)$; and, if $(t_1,t_2t_3) \notin R$ and $(t_1t_2,t_3) \notin R$, then $\gamma(t_1,t_2) = t_1$ and $\gamma(t_2,t_3) = t_3$ (i.e., $t_1t_2 = t_1$ and $t_2t_3 = t_3$).

ii) the distinct elements of S are those words $t_1 \dots t_n$ $(t_1,\dots,t_n \in T)$ with $(t_j,t_{j+1}) \notin R$ $(1 \leq j \leq n-1)$.

Proof. Clearly ii) implies i).

Conversely, assume i). Suppose that a word

$$w = t_1 \dots t_n$$

reduces in two different ways by single applications of (1):

$$w_1 = t_1 \dots (t_it_{i+1}) \dots t_n,$$

$$w_2 = t_1 \dots (t_jt_{j+1}) \dots t_n.$$

Then each of the words w_1 and w_2 can be reduced, by single applications of (1), to a word w_3. This step uses i) if $i = j \pm 1$. An easy inductive argument shows that the full reduction of w to an irreducible word is unique. □

Condition ii) says that the elements of S are those words which cannot be reduced in length by applying (1). It is clear that every element of S can be written in this form, but the crucial point of ii) is that different irreducible words give different elements of S. The lemma asserts that to obtain this, it is sufficient to check that every word of length three reduces uniquely.

(2.2) Example. There is a Banach algebra with an abelian sequential unbounded l.a.i., an abelian sequential unbounded r.a.i., and no t.a.i.

Proof. Let A_o be the complex associative algebra generated by $\{t_i: i = 1,2,3,\ldots\}$ subject to the relations

$$t_it_j = t_{\min\{i,j\}},$$

unless i is odd and j is even.

Using Lemma (2.1), we find that every element of A_o is uniquely expressible in the form

$$x = \sum_{n=1}^{\infty} \alpha_n t_n + \sum_{i,j} \alpha_{ij} t_i t_j, \tag{1}$$

where almost all of the scalars α_n, α_{ij} are zero, and the second summation is over odd values of i and even values of j.

For the rest of this proof, h, i will always denote odd numbers and j, k even numbers.

We define a norm on A_o by

$$||x|| = \sum_{n=1}^{\infty} |\alpha_n| 2^n + \sum_{i,j} |\alpha_{ij}| 2^{i+j}, \tag{2}$$

and form the completion A. A typical element x in A is of the form (1), without the restriction that almost all the α_n, α_{ij} should vanish, but with $||x||$, as in (2), being finite (for a proof that the norm in (2) is submultiplicative, see Appendix.)

It is now easy to check that $\{t_i\}$ is an abelian sequential unbounded r.a.i. in A, and $\{t_j\}$ an abelian sequential unbounded l.a.i. in A.

Suppose A has a t.a.i. $\{e_\lambda\}_{\lambda\varepsilon\Lambda}$. Let

$$e_\lambda = \sum_{n=1}^{\infty} \alpha_n^{(\lambda)} t_n + \sum_{i,j} \alpha_{ij}^{(\lambda)} t_i t_j .$$

The coefficient of t_h in $t_h e_\lambda - t_h$ is $\sum_{i\geq h} \alpha_i^{(\lambda)} - 1$. This must tend to zero as λ runs through Λ. So

$$\lim_{\lambda\varepsilon\Lambda} \sum_{i\geq h} \alpha_i^{(\lambda)} = 1. \tag{3}$$

Similarly, considering the sum of the $t_i t_j$ terms in $xe_\lambda - x$, where $x = \sum_k 2^{-2k} t_k$, we obtain

$$\lim_{\lambda\varepsilon\Lambda} \sum_{i,j} |\alpha_{ij}^{(\lambda)}| 2^{j-i} = 0. \tag{4}$$

Finally, consider

$$e_\lambda x - x = \left(\sum_{n=1}^{\infty} \alpha_n^{(\lambda)} t_n + \sum_{i,j} \alpha_{ij}^{(\lambda)} t_i t_j \right)\left(\sum_k 2^{-2k} t_k\right) - \sum_k 2^{-2k} t_k .$$

The sum of the $t_i t_{i+1}$ terms in $e_\lambda x - x$ is

$$\alpha_i^{(\lambda)} t_i 2^{-2(i+1)} t_{i+1} + \alpha_{ii+1}^{(\lambda)} t_i t_{i+1} \left(\sum_{k\geq i+1} 2^{-2k} t_k \right) + \sum_{j>i+1} \alpha_{ij}^{(\lambda)} t_i t_j 2^{-2(i+1)} t_{i+1}$$

$$= \alpha_i^{(\lambda)} 2^{-2(i+1)} t_i t_{i+1} + \alpha_{ii+1}^{(\lambda)} \Big(\sum_{k \geq i+1} 2^{-2k} \Big) t_i t_{i+1} + \sum_{j>i+1} \alpha_{ij}^{(\lambda)} 2^{-2(i+1)} t_i t_{i+1}$$

$$= 2^{-2(i+1)} \Big[\alpha_i^{(\lambda)} + \alpha_{ii+1}^{(\lambda)} \Big(\sum_{k \geq 0} 2^{-2k} \Big) + \sum_{j>i+1} \alpha_{ij}^{(\lambda)} \Big] t_i t_{i+1}$$

$$= 2^{-2(i+1)} \Big[\alpha_i^{(\lambda)} + \frac{16}{15} \alpha_{ii+1}^{(\lambda)} + \sum_{j>i+1} \alpha_{ij}^{(\lambda)} \Big] t_i t_{i+1}.$$

Hence

$$\lim_{\lambda \in \Lambda} \sum_i \Big| \alpha_i^{(\lambda)} + \frac{1}{15} \alpha_{ii+1}^{(\lambda)} + \sum_{j>i} \alpha_{ij}^{(\lambda)} \Big| = 0. \tag{5}$$

Then

$$\sum_i |\alpha_i^{(\lambda)}| \leq \sum_i \Big| \alpha_i^{(\lambda)} + \frac{1}{15} \alpha_{ii+1}^{(\lambda)} + \sum_{j>i} \alpha_{ij}^{(\lambda)} \Big| + \frac{1}{15} \sum_i |\alpha_{ii+1}^{(\lambda)}| + \sum_i \sum_{j>i} |\alpha_{ij}^{(\lambda)}|$$

$$\leq \sum_i \Big| \alpha_i^{(\lambda)} + \frac{1}{15} \alpha_{ii+1}^{(\lambda)} + \sum_{j>i} \alpha_{ij}^{(\lambda)} \Big| + \frac{16}{15} \sum_i \sum_{j>i} |\alpha_{ij}^{(\lambda)}|$$

$$\leq \sum_i \Big| \alpha_i^{(\lambda)} + \frac{1}{15} \alpha_{ii+1}^{(\lambda)} + \sum_{j>i} \alpha_{ij}^{(\lambda)} \Big| + \frac{16}{15} \sum_i \sum_{j>i} |\alpha_{ij}^{(\lambda)}| 2^{j-i}$$

which tends to zero, by (4) and (5); contradicting (3). Thus A has no t.a.i. □

(2.3) <u>Proposition</u>. Let A be a normed algebra. If A has a bounded l.a.i. and a r.a.i., then it has a t.a.i.