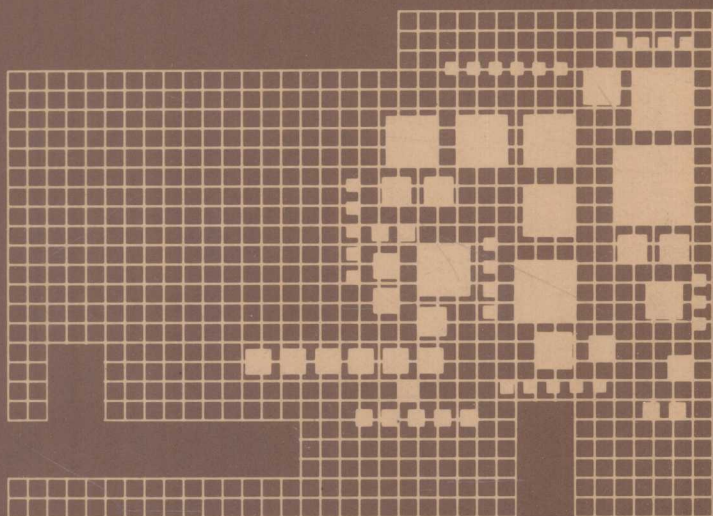


REGIONAL POPULATION PROJECTION MODELS



**Andrei
ROGERS**

Volume 4

**SCIENTIFIC GEOGRAPHY
Series**

**Editor:
Grant Ian THRALL**

REGIONAL POPULATION PROJECTION MODELS

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INTRODUCTION TO THE SCIENTIFIC GEOGRAPHY SERIES

Scientific geography is one of the great traditions of contemporary geography. The scientific approach in geography, as elsewhere, involves the precise definition of variables and theoretical relationships that can be shown to be logically consistent. The theories are judged on the clarity of specification of their hypotheses and on their ability to be verified through statistical empirical analysis.

The study of scientific geography provides as much enjoyment and intellectual stimulation as does any subject in the university curriculum. Furthermore, scientific geography is also concerned with the demonstrated usefulness of the topic toward explanation, prediction, and prescription.

Although the empirical tradition in geography is centuries old, scientific geography could not mature until society came to appreciate the potential of the discipline and until computational methodology became commonplace. Today, there is widespread acceptance of computers, and people have become interested in space exploration, satellite technology, and general technological approaches to problems on our planet. With these prerequisites fulfilled, the infrastructure needed for the development of scientific geography is in place.

Scientific geography has demonstrated its capabilities in providing tools for analyzing and understanding geographic processes in both human and physical realms. It has also proven to be of interest to our sister disciplines, and is becoming increasingly recognized for its value to professionals in business and government.

The Scientific Geography Series will present the contributions of scientific geography in a unique manner. Each topic will be explained in a small book, or module. The introductory books are designed to reduce the barriers of learning; successive books at a more advanced level will follow the introductory modules to prepare the reader for contemporary developments in the field. The Scientific Geography Series begins with several important topics in human geography, followed by studies in other branches of scientific geography. The modules are intended to be used as classroom texts and as reference books for researchers and professionals. Wherever possible, the series will emphasize practical utility and include real-world examples.

We are proud of the contributions of geography, and are proud in particular of the heritage of scientific geography. All branches of geography should have the opportunity to learn from one another; in the past, however, access to the contributions and the literature of scientific geography has been very limited. I believe that those who have contributed significant research to topics in the field are best able to bring its contributions into focus. Thus, I would like to express my appreciation to the authors for their dedication in lending both their time and expertise, knowing that the benefits will by and large accrue not only to themselves but to the discipline as a whole.

—*Grant Ian Thrall*
Series Editor

SERIES EDITOR'S INTRODUCTION

Professor Andrei Rogers, a recognized world leading authority in population futures, turns his attention here to regional population projection. By reading this book you can learn how to do subnational population projections. Follow the presentation with pencil and paper at hand; trace with Andrei Rogers through the mathematical analysis step by step, made clear and relevant by demonstrations of actual projections of population futures at the regional level. The examples show how to calculate numerically regional population growth rates, age compositions, and spatial distributions using data from several developed and less developed countries.

Andrei Rogers demonstrates that projecting population futures at the regional level is both a rewarding intellectual exercise and a powerful technique. Public and private institutions, organizations, and firms require information on potential demographic futures. Public organizations must anticipate future needs and thereby judge whether or not efforts should be launched to alter current population processes and trends. Private firms maximize possible profits by adjusting product lines and shifting distribution networks using information obtained from regional demographic projections.

By disaggregating national populations spatially, Andrei Rogers can analyze the evolution of multiple regional populations, each interconnected by migration flows. The dynamics of the evolution of every subnational human population is governed by the interaction of births, death, and migration. Individuals are born into a population, with the passage of time they age and reproduce, and because of death or outmigration they ultimately leave the population. These events and flows enter into an accounting relationship in which the growth of the regional population is determined by the combined effects of natural increase (births minus deaths) and net migration (inmigrants minus outmigrants).

Andrei Rogers adopts a geographical perspective by considering how fertility, mortality, and migration combine to determine the growth, age composition, and spatial distribution of a national multiregional population; his analysis considers simultaneously: several interdependent subnational population *stocks*; the *events* that alter the levels of such stocks; the aggregate directions *flows* that connect these stocks to form a system of interacting subnational populations.

This module should be of use to those responsible for carrying out regional population projections in public and private organizations such as national, state, and local governments, business firms, foundations, universities, labor unions, social service organizations, and various public interest groups. Students will find that this work by Andrei Rogers contributes a new and significant dimension to human geography and anthropology, sociology and demography, business marketing, regional economics, environmental studies, and city planning.

—*Grant Ian Thrall*
Series Editor

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REGIONAL POPULATION PROJECTION MODELS

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1. INTRODUCTION

Population projections are numerical estimates of future demographic totals, usually obtained by the extrapolation of past and current trends. Such calculations are fundamental inputs to social and economic planning. They identify potential demographic futures, anticipate the needs that such futures are likely to create, and provide a basis for judging whether or not efforts should be launched to alter current population processes and trends.

The principal uses of population projections arise in connection with the planning activities of governmental organizations and private firms. National, state, and local governments, business firms, universities, foundations, labor unions, social service organizations, and various public interest groups all use population projections. In addition to various expected population totals, their planning efforts may also

require more specialized types of projections, such as expected future numbers of teachers, classrooms, housing units, medical personnel, hospital beds, nursery schools, day care centers, highways, dams—the list goes on and on.

Besides their uses in public and private planning activities, population projections also contribute to the development of a better understanding of demographic phenomena. In particular, they “permit experiments out of which we obtain causal knowledge; they explain data; they focus research by identifying theoretical and practical issues; they systematize comparative study across space and time; they reveal formal analogies between problems that on their surface are quite different; they even help assemble data” (Keyfitz, 1971, p. 573).

Subnational Population Projection: A Geographical Perspective

The subject of this book is subnational population projection. As the modifier subnational suggests, the focus is on the regional populations that collectively make up the national total. National populations are spatially disaggregated, and attention is directed at the evolution of multiple regional populations interconnected by interregional migration flows. It is this orientation that brings forth the particular contribution of geography to demographic analysis.

In the absence of data on births, deaths, and migration, early efforts at population projection necessarily relied on crude methods of extrapolating past observations, usually by fitting simple curves to the data. But a curve may fit observed data for over a century with considerable accuracy and yet fail to predict the situation for the next few years. The generally unsatisfactory results of curve-fitting efforts led to the development of an approach that introduces the behavior of the principal components of population change into the projection exercise—an approach in which trends in fertility, mortality, and migration are taken into account in the projection of population totals. Chapter 1 examines both methods of population projection and also presents the method used by the United Nations to generate urbanization projections.

Chapter 2 considers the consequences for subnational population projection of dividing an aggregate population into spatially distinct, interacting regional populations that exchange migrants in both directions. This permits one to associate gross migration flows with the regional populations that are exposed to the possibility of experiencing them. Gone is the statistical fiction of the net migrant; an outmigrant from one region becomes an immigrant to another, creating a link between the two regional populations. As a result, instead of considering the dynamics of population redistribution one region at a time, the

analyst examines the evolution of a complete system of interacting regional populations simultaneously in a single operation.

Although a number of useful results can be derived without introducing the age dimension into the analysis, serious systematic attention normally is accorded only to demographic projections that disaggregate population totals by age. Such a disaggregation allows one to study the diverse demographic behavior of heterogeneous subpopulations exhibiting differing propensities to experience events and movements. The incorporation of such differences in a formal analysis further illuminates aggregate patterns of demographic behavior. Chapter 3 describes the demographer's classical age-disaggregated single-region approach to population projection. It sets out both the life table and the cohort-survival model and illustrates their application to subnational population projection.

Finally, Chapter 4 integrates the age dimension of the demographer with the locational dimension of the geographer. Populations disaggregated by age and region of residence are advanced over time and across space. Age dynamics are linked with spatial dynamics. Then a reinterpretation of migration between regions as a transition between states of existence generalizes the multiregional projection model into a more general class of models called *multistate* models. The mathematical apparatus for dealing with people moving from one region to another turns out to be formally identical with one dealing with a wide class of transitions that individuals experience during their lifetime: for example, transitions from healthy to ill, from single to married, from employed to unemployed, and from being in school to having graduated. An example of the multiregional/multistate model's generality is illustrated in Chapter 4 below with a population projection that is disaggregated by four different marital statuses and two regions of residence.

Age is usually the most important characteristic of a population in demographic calculations, and to ignore its influence is to invite potentially erroneous findings. Nevertheless, a number of crude but useful results can be derived even if age is disregarded in population projections—as it is in this chapter and the next. Fundamental to these results is the notion of a rate of increase, a subject to which we now turn.

Rates of Increase and Exponential Growth

A population numbering $P(t)$ at a time t and $P(t + 1)$ a year later has exhibited an annual rate of increase of

$$r = \frac{P(t + 1) - P(t)}{P(t)} \quad [1.1]$$

and is said to have been growing at the rate of $100r\%$ per annum. Continued growth at this rate for a decade would mean a total population of

$$P(t+10) = (1+r)P(t+9) = (1+r)^2 P(t+8) = \dots = (1+r)^{10} P(t) \quad [1.2]$$

or $(1+r)^{10}$ times the present population.

A convenient means of illuminating the consequences of an unchanging rate of increase is offered by the concept of doubling time. If at the end of ten years a population is $(1+r)^{10}$ as great as it was a decade before and $(1+r)^n$ times as large at the end of n years, then the population's doubling time is given by the value of n that satisfies the equation

$$(1+r)^n = 2 \quad [1.3]$$

Taking natural logarithms and dividing both sides by $\ln(1+r)$ gives

$$n = \frac{\ln 2}{\ln(1+r)} \quad [1.4]$$

And recalling that the Taylor series expansion of $\ln(1+r)$ is $\ln(1+r) = r - r^2/2 + r^3/3 - \dots$, we obtain the approximate solution

$$n = \frac{\ln 2}{r} \doteq \frac{0.693}{r} \quad [1.4']$$

when r is small enough for all terms beyond the first to be disregarded in the expansion. Keyfitz and Beekman (1984) show that for commonly observed values of r among human populations (i.e., between 0 and 0.04), a slightly more precise approximation is offered by $n = 0.70/r$, when r is compounded annually. We conclude, therefore, that a population increasing at 2% per annum ($r = 0.02$) doubles in 35 years, at 3% in 23.3 years, and at 4% in 17.5 years.

Interest is said to be compounded annually when a sum invested at the beginning of a year increases to $(1+r)$ times its value by the end of the year. If the rate of increase r is compounded k times during the year, then the sum at the end will have grown to $[1+r/k]^k$ times its value at the start of the year. When the rate is compounded continuously, we have

$$\lim_{k \rightarrow \infty} \left[1 + \frac{r}{k} \right]^k = e^r \quad [1.5]$$

where e is the base of natural logarithms and is equal to approximately 2.718. Thus a population, $P(t)$, growing at 100r% a year compounded continuously (that is, the k becomes infinitely large), would total

$$P(t + 1) = e^r P(t) \quad [1.6]$$

by the end of a year, and over n years would grow to

$$P(t + n) = e^{rn} P(t) \quad [1.6']$$

To reach this same total with annual compounding would require a rate of increase, r^* say, that is slightly larger than the corresponding continuous rate r . Specifically, recalling equations 1.2 and 1.6, we observe that

$$(1 + r^*) P(t) = e^r P(t) \quad [1.7]$$

whence

$$r^* = e^r - 1 \quad [1.8]$$

Thus, for example, the population of Kenya, which has been increasing at about 4.0% per year compounded annually, has been growing at the rate of 3.92% per year compounded continuously. Its doubling time, n , under the latter model is defined by the relation

$$e^{rn} = 2 \quad [1.9]$$

which satisfies equation 1.4' exactly and yields a value of 17.7 years.

So far we have considered only fixed rates of increase, r . We now turn to time dependent variable rates of increase $r(t)$ and examine the population $P(t)$ that evolves after T years of exposure to such a rate.

Imagine that the time interval of T years is divided into short sub-intervals dt in length, and suppose that $r(t)$ over the first subinterval is r_0 , over the second is r_1 , and so on. If $P(0)$ is the initial population total, then $P(0) e^{r_0 dt}$ is the total at time dt , and, writing $e^{r_i dt}$ as $\exp [r_i dt]$, gives

$$\begin{aligned} P(T) &= P(0) \exp [r_0 dt] \cdot \exp [r_1 dt] \cdot \exp [r_2 dt] \cdots \exp [r_{T-1} dt] \quad [1.10] \\ &= P(0) \exp [\sum r_i dt] \end{aligned}$$

As dt tends to zero, one obtains, in the limit, an integral in place of the summation:

$$P(T) = P(0) \exp \left[\int_0^T r(t) dt \right] \quad [1.11]$$

Taking logarithms of both sides of 1.11 and then differentiating with respect to t yields the following definition of the variable rate of increase:

$$r(t) = \frac{1}{P(t)} \frac{dP(t)}{dt} \quad [1.12]$$

As Keyfitz (1977) points out, a convenient way of assessing the numerical effect of a variable rate of increase on a population total is to apply the average rate across the entire time interval from zero to T :

$$P(T) = P(0)e^{\bar{r}T} \quad [1.13]$$

where

$$\bar{r} = \frac{\int_0^T r(t) dt}{T} \quad [1.14]$$

Alternative Projections of National Population Growth and Urbanization

In 1980 the United Nations projected a population of 238 million for Indonesia in the year 2000 (United Nations, 1980). Underlying that projection were the following assumed average annual rates of increase: 2.59% for the period 1975-80, 2.38% for 1980-90, and 1.89% for 1990-2000. This projection adds 16 million more people to the year 2000 total projected three years earlier by the same agency, and it exceeds the corresponding projection carried out by the World Bank in 1979 by some 31 million people. How are we to judge whether the projection is reasonable or not?

The UN estimate of Indonesia's population in 1975 is 136 million. During the preceding five years this population was growing at an

average annual rate of 2.6%. Were it to continue to grow at that rate, its total by the year 2000 would be (equation 1.6):

$$P(2000) = 136 e^{0.026(25)} = 261 \text{ million}$$

a total that exceeds the 1980 UN projection by some 23 million people.

One can argue that the 261-million figure is undoubtedly too high because the 1970-75 growth rate is likely to decline as national development proceeds. The UN projection assumed a nonlinear pattern of decline to 1.89% by the 1990-2000 decade; alternatively, an assumed linear decline to 2.0% implies instead an average growth rate of 2.3% during the 25-year period and projects a total population of

$$P(2000) = 136 e^{0.023(25)} = 242 \text{ million}$$

Extrapolating this latter pace of decline into the twenty-first century drops the rate to zero by the year 2083. If one assumes that this rate will remain fixed at zero forever thereafter, then 556 million is the corresponding ultimate zero population growth (ZPG) total. Table 1.1 sets out these totals for Indonesia and compares them with corresponding results for four other Southeast Asian nations.

Moving from national projections to regional ones, we let $P(t)$, $P_u(t)$, and $P_v(t)$ denote, respectively, the total, urban, and rural populations of a country at time t , and let m be the net outmigration rate from rural areas. (We use the letter v as a subscript for rural area variables to avoid having the letter r denote two different attributes: a rate of increase and a rural location.) Assume that the rate of natural increase (birth rate minus death rate) of the urban population is equal to that of the rural population, both of them therefore being equal to the national rate of increase r . (We assume a national population that is undisturbed by international migration.) It follows, then, that

$$P(t) = P(0)e^{rt} \quad [1.15]$$

Because the growth rate of the rural population will be the rate of natural increase, r , less the rate of net outmigration to urban areas, m :

$$P_v(t) = P_v(0)e^{(r-m)t} \quad [1.16]$$

and, because the total population is the sum of its rural and urban subpopulations,

$$P_u(t) = P(t) - P_v(t) \quad [1.17]$$

TABLE 1.1 Historical Population Data and Alternative National Projections (in millions) to the Year 2000 and Beyond

Nation	Historical Data		Alternative Projections				Transparent Models*		
			Published Projections						
	1950	1975	UN80	UN77	World Bank	Const. r	Decl. r	ZPG Yr.	ZPG Pop.
Cambodia	4	8	16	13	16	16	15	2065	28
Indonesia	75	136	238	222	207	261	242	2083	556
Malaysia	6	12	22	20	20	25	22	2056	39
Philippines	21	44	90	83	76	103	87	2037	125
Thailand	20	42	86	76	69	95	81	2039	121

SOURCES: United Nations (1980), World Bank (1979), and Rogers (1981).

*Constant r means the 1970-1975 value given for the country in United Nations (1980); declining r means a linear decline to r = 20 per thousand by the year 2000; continuing this linear decline to zero gives, in the final two columns, the year at which zero population growth (ZPG) first occurs and the ZPG total, respectively.

TABLE 1.2 Average Annual Rural Net Outmigration Rates (per thousand) Implied by Current United Nations Estimates and Projections, 1950-2000

Nation	Historical Data			UN80 Projection		
	1950-60	1960-70	1970-75	1975-80	1980-90	1990-2000
Cambodia	0.55	1.12	2.14	2.94	4.52	7.55
Indonesia	2.52	2.95	3.31	4.39	6.42	9.95
Malaysia	6.28	2.38	2.49	4.16	7.09	11.92
Philippines	4.44	3.76	4.32	5.88	8.90	13.55
Thailand	2.30	0.81	0.84	1.83	3.66	7.20

SOURCE: Ledent and Rogers (1979).

Such a simple model of population dynamics was adopted by Keyfitz (1980) to illuminate a number of demographic aspects of the urbanization process. It can also be used to generate projections of urbanization.

Table 1.2 presents rough estimates of the past and future rural net outmigration rates that are implied by the 1980 United Nations urban and rural population projections for the five Southeast Asian nations included in Table 1.1 (Ledent and Rogers, 1979). It shows, for example, that during the 1970-75 period rural areas in Indonesia were losing population at a net annual rate of 3.3 per thousand. Adopting the simplifying assumption that the urban and rural populations were then both exhibiting an annual rate of natural increase that was equal to the national growth rate of 26.0 per thousand, and assuming fixed rates of natural increase and migration over the projection interval, gives (equation 1.16):

$$P_u(2000) = 0.8157 (136) e^{(0.0260-0.0033)25} = 196 \text{ million} \quad [1.18]$$

where 0.8157 is the fraction of the national population in 1975 that was rural. Earlier we projected the corresponding national total to be

$$P(2000) = 261 \text{ million}$$

thus, using equation 1.17, we have that

$$P_u(2000) = 261 - 196 = 65 \text{ million}$$

a projection that yields an urbanization level of 25.0%.

Relaxing the assumption of fixed rates by allowing r to follow the nonlinear trajectory assumed in the UN projections, while keeping m fixed at 3.3 per thousand, gives Indonesia an urbanization level of 24.9%

TABLE 1.3 Alternative National Urbanization Projections (in percentages)
to the Year 2000 and Beyond

<i>Nation</i>	<i>Historical Data</i>		<i>Alternative Projections</i>			
	<i>1950</i>	<i>1975</i>	<i>Published Projections</i>		<i>Transparent Models*</i>	
			<i>UN80</i>	<i>UN76</i>	<i>Constant m</i>	<i>Inc. m</i>
Cambodia	10.21	12.64	23.70	40.00	17.19	34.57
Indonesia	12.41	18.43	32.26	31.44	24.92	39.48
Malaysia	20.37	27.88	41.59	45.08	32.22	46.13
Philippines	27.13	34.30	49.04	50.82	41.03	51.65
Thailand	10.47	13.58	23.18	27.36	15.39	34.61

SOURCES: United Nations (1976, 1980) and Rogers (1981).

*Constant *m* means the 1970-75 value given for the country in Table 1.2; increasing *m* means a linear increase to *m* = 16 by the year 2000. The nonlinear trajectory of *r* is kept the same as in the United Nations (1980) projections.

by the year 2000 (again using equations 1.16 and 1.17 as above). The 1980 UN projection gives 32.3% for this figure, a consequence of the assumed gradual increase in net rural outmigration to an annual rate of 9.95 per thousand. To bracket this UN projection, we also show in Table 1.3 the corresponding projection with the rural net outmigration rate increasing linearly from its 1970-75 value to 16 per thousand by 1990-2000. This assumption, of course, produces a higher urbanization level than is envisioned in the UN projections—a level of 39.5% to the UN's 32.3%. Analogous findings are obtained for the other four Southeast Asian nations.

The Demographic Sources of Urban Growth

Do cities grow mostly by the surplus of urban births over urban deaths (urban natural increase) or do they grow mostly as a consequence of net immigration from rural areas? A recent study by the United Nations concluded that urban population growth in the less developed nations results primarily from the natural increase of their urban populations:

Considering only the most recent observation for a country, an average of 60.7 per cent of growth is attributable to this source, compared with only 39.3 per cent for migration. These figures are nearly reversed for the more developed countries (40.2 and 59.8 percent)[United Nations, 1980, p. 23].

The United Nations' decomposition strives to disentangle the immediate contributions of natural increase and migration to urban population growth by estimating the fraction of today's growth that would be eliminated if rates either of natural increase or of migration were

suddenly to drop to zero. But this is a static cross-sectional view, one that ignores the evolution of the changing contributions of migration and of natural increase to urban growth over time. The long-run impacts of current patterns of natural increase and migration on urban population growth and urbanization levels can be conveniently assessed by population projection.

Without a city population there obviously cannot be any urban natural increase; and for some time after the establishment of a city, when its population is still relatively small in size, the contribution of urban net immigration is likely to exceed that of urban natural increase. At the other extreme, when a nation is mostly urbanized, the out-migration of its rural population can contribute little to urban growth. Between these two extremes comes a time at which the contribution of natural increase begins to dominate that of net immigration.

Imagine a hypothetical population, initially entirely rural, that experiences an annual rate of natural increase of r and a net rural out-migration rate of m . Recalling equations 1.12 and 1.15-1.17, one may establish that the rate of growth of the urban population is the sum of the contribution of urban natural increase and the contribution of urban net immigration (Keyfitz, 1980):

$$r_u = \frac{1}{P_u(t)} \frac{dP_u(t)}{dt} = \frac{d}{dt} \ln P_u(t) = r + \frac{m}{e^{mt} - 1} \quad [1.19]$$

The first term, r , is the assumed rate of natural increase of the urban population; thus the second term, $m/(e^{mt} - 1)$, must be its rate of increase through migration. Dividing the second by the first gives

$$R(t) = \frac{m}{r(e^{mt} - 1)} \quad [1.20]$$

the ratio of the contribution of migration to natural increase.

At what point does the contribution of natural increase to urban growth first begin to exceed that of urban net immigration? The former overtakes the latter immediately after $R(t)$ reaches unity, an event that takes place when

$$\frac{m}{r} = e^{mt} - 1 \quad [1.21]$$