Tensor Analysis

SECOND EDITION

Theory and Applications to Geometry and Mechanics of Continua

I. S. Sokolnikoff

TENSOR ANALYSIS

THEORY AND APPLICATIONS TO GEOMETRY AND MECHANICS OF CONTINUA

Second Edition

I. S. SOKOLNIKOFF

PROFESSOR OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
LOS ANGELES

JOHN WILEY & SONS, INC.

NEW YORK · LONDON · SYDNEY

Library of Congress Catalog Card Number: 64-13223
PRINTED IN THE UNITED STATES OF AMERICA

PREFACE TO THE SECOND EDITION

In preparing the Second Edition of this book I have been guided by suggestions kindly made to me by users of the First Edition. There appeared to be no compelling reasons for making major changes in the introductory chapter concerned with linear transformations and matrices, or in the second chapter, devoted to algebra and calculus of tensors.

In Chapter 3 some sections concerned with the uses of calculus of variations in geometry have been expanded, some new illustrative material introduced, and two new sections, on parallel surfaces and the Gauss-Bonnet theorem, have been added. Chapters 2 and 3 in the present edition contain adequate material for an introductory course on metric differential geometry at the beginning graduate level or, for that matter, at the upper-division undergraduate level.

Chapter 4, dealing with analytical mechanics, has been expanded. It contains a distillation of the essentials of classical analytical mechanics and potential theory, which, together with Chapter 5 on relativistic mechanics, should be, but often is not, a part of the equipment of every student of mathematics. A number of illustrative examples that further illuminate the theory have been introduced, and the discussion of nonholonomic dynamical systems, of Hamilton's canonical equations, and of potential theory has been made more detailed.

The concluding chapter, devoted to mechanics of continua, was entirely rewritten. It presents from a unified point of view and, it is hoped, with sufficient clarity, the essentials of the nonlinear theory of mechanics of deformable media. This chapter provides a common basis for a careful development of the mathematical theories of elasticity, plasticity, hydrodynamics, and gas dynamics.

I. S. SOKOLNIKOFF

Pacific Palisades, California
January 1964

PREFACE TO THE FIRST EDITION

This book is an outgrowth of a course of lectures I gave over a period of years at the University of Wisconsin, Brown University, and the University of California. My audience consisted, for the most part, of graduate students interested in applications of mathematics, and this fact shaped both the content and the character of exposition.

Because of the importance of linear transformations in motivating the development of tensor theory, the first chapter in this book is given to a discussion of linear transformations and matrices, in which stress is placed on the geometry and physics of the situation. Although a large part of the subject matter treated in this chapter is normally covered in courses on matrix algebra, only a few of my listeners have had the sort of appreciation of matrix transformations that an applied mathematician should have.

The second chapter is concerned with algebra and calculus of tensors. The treatment in it is self-contained and is not made to depend on some special field of mathematics as a vehicle for the development of tensor analysis. This is a departure from the customary practice of making geometry or relativity a medium for the unfolding of tensor analysis. Although this latter practice has a great deal to commend it because it provides a simple means for motivating the study of tensors, it often leaves an erroneous impression that the formulation of tensor analysis depends somehow on geometry or relativity.

The remaining four chapters in this volume deal with the applications of tensor calculus to geometry, analytical mechanics, relativistic mechanics, and mechanics of deformable media. Thus, Chapter 3 contains a selection of those geometrical topics that are important in the study of analytical dynamics and in such portions of elasticity and plasticity as deal with the deformation of plates and shells. This chapter provides a substantial introduction to the subject of metric differential geometry. In Chapter 4, the essential concepts of analytical mechanics are presented adequately and concisely. An introduction to relativistic mechanics is contained in Chapter 5. The treatment there was intentionally made very brief because some excellent books on relativity have appeared recently and there seems little point in duplicating their contents.

The final chapter of the book is concerned with a formulation of the essential ideas of nonlinear mechanics of continuous media in the most general tensor form. The classical linearized equations of elasticity and fluid mechanics appear as special cases of the general treatment.

Perhaps the best evidence of the remarkable effectiveness of the tensor apparatus in the study of Nature is in the fact that it was possible to include, between the covers of one small volume, a large amount of material that is of interest to mathematicians, physicists, and engineers.

A survey of applied mathematics as broad as that in this book must inevitably reflect contributions of so many scholars that it is futile to attempt to assign proper credit for original ideas or methods of attack. However, in the treatment of geometry, the influence of T. Levi-Civita and A. J. McConnell, whose books (especially McConnell's *Applications of the Absolute Differential Calculus*) I used in my classes for many years as required reading, is clearly discernible. Specific acknowledgments to these and other authors are made in the appropriate places in the text. However, my greatest debt is to my listeners, who have made the job of writing this book seem both enjoyable and worth while.

It is a particular pleasure to single out among my listeners Mr. William R. Seugling, Research Assistant at the University of California at Los Angeles, who gave unstintingly of his time in following this book through press.

I. S. SOKOLNIKOFF

Los Angeles, California
November 1951

CONTENTS

1 LINEAR VECTOR SPACES. MATRICES

2 TENSOR THEORY

5 RELATIVISTIC MECHANICS

6 MECHANICS OF CONTINUOUS MEDIA

1

LINEAR VECTOR SPACES. MATRICES

1. Coordinate Systems

In order to locate a geometrical configuration a reference frame is needed. Among the simplest reference frames used in mathematics are the cartesian coordinate systems. Although the construction of such coordinate systems is familiar to the reader from courses in analytic geometry, we review it here in order to set in relief certain basic notions that underlie the concept of coordinates covering the space of our physical intuition. This review will pave the ground for some far-reaching generalizations of the concept of physical space, which we formulate in Sec. 4.

The cardinal idea responsible for the invention of coordinate systems by Descartes is the identification of the set of points composing a straight line with the totality of real numbers. It consists of the assumption that to each real number there corresponds a unique point on a straight line, and conversely.[1]

We choose a straight line X and a point O on it (Fig. 1). This point O, which we call the origin, divides the line into two half-rays. We

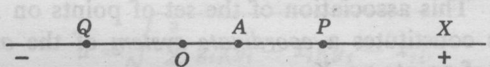

Fig. 1.

[1] Although the idea of one-to-one reciprocal correspondence between the set of points composing a line and the totality of real numbers had it roots in the Eudoxus theory of incommensurables, dating back to the fourth century B.C., the invention of coordinate systems did not come until the first part of the seventeenth century. It should be also noted that a rigorous analysis of the relation between linear sets of points and real numbers was made only during the closing years of the last century, chiefly through the efforts of Dedekind and Cantor. The concept of rigor depends entirely on conventions dictated by prevailing tastes indicative of the degree of mathematical sophistication in a given chronological period. Fruitful intuitive concepts are usually made rigorous by (*a*) making explicit agreements as to which ideas fall into a category of definable concepts and which do not, and (*b*) introducing into mathematical theories new modes of reasoning which (one hopes) are free of contradiction.

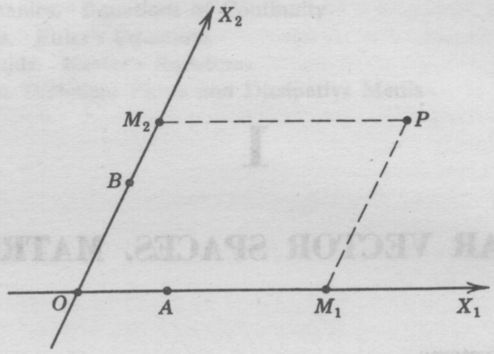

Fig. 2.

designate one of these as the *positive* and the other as the *negative* half-ray. On the positive half-ray we choose a point A and call the length of the line segment OA the *unit length.* We next *coordinate* points on X with a set of real numbers in the following way: If P is any point on the positive half-ray, we define a number x associated with P by the formula

$$x = \frac{\overline{OP}}{\overline{OA}},$$

where $\overline{OP}$ and $\overline{OA}$ are lengths of the line segments OP and OA. The number x is the *coordinate* of P. The coordinate x of the point Q on the negative half-ray is defined by the ratio

$$x = -\frac{\overline{OQ}}{\overline{OA}}.$$

We also assume that each real number x corresponds to one and only one point on X. This association of the set of points on X with the set of real numbers constitutes a *coordinate system* of the *one-dimensional space* consisting of points on X.

The coordination of the set of points lying in the plane with sets of real numbers is accomplished by taking two straight lines X_1 and X_2 intersecting at a single point O (Fig. 2). On each line a coordinate system is constructed as above, but the units on each line need not be equal. A pair of such lines with unit points A and B marked on them form the *coordinate axes* X_1, X_2. With each point P in the plane of coordinate axes we associate an *ordered pair* of real numbers (x_1, x_2) determined as follows. The line through P drawn parallel to the X_2-axis intersects the X_1-axis in a point M_1 with coordinate x_1, and the line through P parallel to the X_1-axis cuts X_2 in a point M_2 with coordinate x_2. The ordered pair of numbers (x_1, x_2) are the coordinates of P in the plane, and the one-to-one correspondence of ordered pairs of numbers with the set of points in the plane X_1X_2 is the *coordinate system* of the two-dimensional space consisting of points in the plane.

The extension of this representation to points in a three-dimensional space is obvious. We take three noncoplanar lines X_1, X_2, X_3 intersecting at the common point O. On each of these lines we establish coordinate systems, and we associate with each point P an ordered triplet of numbers (x_1, x_2, x_3) determined by the intersection with the axes of three planes drawn through P parallel to the *coordinate planes* X_1X_2, X_2X_3, and X_1X_3.

The coordinate systems just described are called *oblique cartesian systems*. Their construction makes use of the notions of length and parallelism of ordinary Euclidean geometry, and the essential feature of it is the concept of one-to-one correspondence of points with ordered sets of numbers. In the event the coordinate axes X_1, X_2, X_3 intersect at right angles, the coordinate system is said to be *orthogonal cartesian*, or *rectangular cartesian*. In applications, orthogonal coordinate systems are generally used because the expression for the length d of the line segment $\overline{AB}$ joining a pair of points with coordinates $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ has the simple form

$$d = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}. \tag{1.1}$$

This is the familiar formula of Pythagoras. If the coordinate system is oblique, the formula for the distance d is somewhat more complicated. We will learn in Sec. 9 that one can pass from an orthogonal system of coordinates to an oblique system by making a linear transformation of coordinates. From this fact and from the structure of formula 1.1, it would follow that the length of the line segment joining the points with oblique coordinates (x_1, x_2, x_3) and (y_1, y_2, y_3) is

$$d = \sqrt{\sum_{i,j=1}^{3} g_{ij}(y_i - x_i)(y_j - x_j)}, \tag{1.2}$$

where the g_{ij}'s are constants that depend on the coefficients in the above-mentioned linear transformation of coordinates. We will be concerned in the sequel with a detailed study of quadratic forms appearing under the radical in formula 1.2 and with their bearing on metric properties of space.

2. The Geometric Concept of a Vector

In the preceding section we recalled the construction of coordinate systems in the familiar three-dimensional space where the formula of Pythagoras is used to measure distances between pairs of points. Spaces

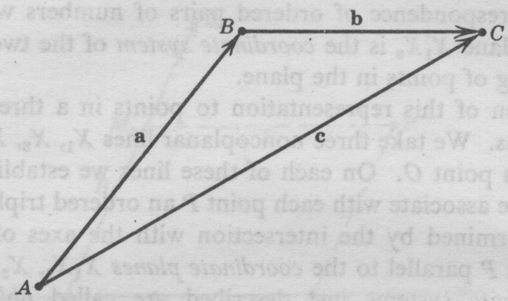

Fig. 3.

where it is possible to construct a coordinate system such that the length of a line segment is given by the formula of Pythagoras are called *Euclidean spaces*. In these spaces the notion of displacement is fundamental. Thus, if a point A is moved to a new position B, the displacement from A to B can be visualized as *directed line segment* $\overrightarrow{AB}$ (Fig. 3). If B is displaced to a new position C, the resultant displacement can be achieved by moving the point A to the position C. These operations can be denoted symbolically by the equation

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

In the elementary treatment of vector analysis, directed line segments are termed *vectors*, and they are usually denoted by a single letter printed in boldface type. Thus the foregoing formula can be written

$$\mathbf{a} + \mathbf{b} = \mathbf{c}, \tag{2.1}$$

where $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$, $\overrightarrow{AC} = \mathbf{c}$.

The rule for the composition of vectors indicated in Fig. 3 was first formulated by Stevinus in 1586 in connection with the experimental study of laws governing the composition of forces. It is known as the *parallelogram law of addition*. The fact that many entities occurring in physics can be represented by directed line segments, whose law of composition is symbolized by formula 2.1, is responsible for the usefulness of vector analysis in applications. We have here an instance of geometrization of physics which had no less important influence on the evolution of this subject than the arithmetization of geometry had on the growth of mathematical analysis.

From the idea of a vector as displacement determined by a pair of points, we are led to conclude that two vectors are equal if the line segments representing them are equal in length and their directions parallel. We shall denote the length of the vector $\mathbf{a}$ by the symbol $|\mathbf{a}|$. We will assume that the concept of length is independent of the chosen reference frame, so that the length $|\mathbf{a}|$ can be calculated (by Pythagorean formula) from the coordinates of the initial and terminal points of **a**.

The negative of the vector **a** (written $-\mathbf{a}$) is the vector whose length is the same as that of **a** but whose direction is opposite. We define the vector zero (written **0**) corresponding to a zero displacement by the formula

$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}.$$

From the geometrical properties of directed line segments we deduce that

(I) $$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

(II) $$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}).$$

(III) If **a** and **b** are vectors, there exists a unique vector **x** such that

$$\mathbf{a} = \mathbf{b} + \mathbf{x}.$$

We next define the operation of multiplication of vectors by real numbers. If α is a real number the symbol $\alpha\mathbf{a} \equiv \mathbf{a}\alpha$ is a vector whose length is $|\alpha|\,|\mathbf{a}|$ and whose direction is the same as that of **a** if $\alpha > 0$, opposite to **a** if $\alpha < 0$. If $\alpha = 0$, then $\alpha\mathbf{a} = \mathbf{0}$.

From this definition and from properties of real numbers we conclude that

(IV) $$(\alpha_1 + \alpha_2)\mathbf{a} = \alpha_1\mathbf{a} + \alpha_2\mathbf{a}$$

(V) $$\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$$

(VI) $$\alpha_1(\alpha_2\mathbf{a}) = (\alpha_1\alpha_2)\mathbf{a}, \qquad 1 \cdot \mathbf{a} = \mathbf{a},$$

for any real numbers α_1 and α_2.

We introduce next the definition of *scalar product* of two vectors, which will provide us with a new notation for the length of a vector.

DEFINITION. *The scalar product of two vectors* **a** *and* **b**, *written* $\mathbf{a} \cdot \mathbf{b}$, *is a real number* $|\mathbf{a}|\,|\mathbf{b}| \cos(\mathbf{a}, \mathbf{b})$, *where* $\cos(\mathbf{a}, \mathbf{b})$ *is the cosine of the angle between* **a** *and* **b**.

Stated in the language of geometry, $\mathbf{a} \cdot \mathbf{b}$ is equal to the product of the projection of **a** on **b** multiplied by the length of **b**. Thus the length of the vector **a** is given by the positive square root of $\mathbf{a} \cdot \mathbf{a}$. We also note that **a** and **b** are orthogonal if, and only if, $\mathbf{a} \cdot \mathbf{b} = 0$.

From this definition and the properties of real numbers we can easily deduce the following theorems.

(VII) $$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 > 0, \quad \text{unless } \mathbf{a} = \mathbf{0}.$$

(VIII) $$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$$

(IX) $$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$

(X) $$\alpha(\mathbf{a} \cdot \mathbf{b}) = (\alpha\mathbf{a} \cdot \mathbf{b}), \quad \text{where } \alpha \text{ is a real number.}$$

3. Linear Vector Spaces. Dimensionality of Space

We formulate next the definition of *linear dependence* of a set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, which will have an important connection with the concept of dimensionality of space.

Linear Dependence. A set of n vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *is called linearly dependent if there exist numbers* $\alpha_1, \alpha_2, \ldots, \alpha_n$, *not all of which are zero, such that*

$$\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \cdots + \alpha_n\mathbf{a}_n = \mathbf{0}.$$

If no such numbers exist, the vectors are said to be linearly independent.

Consider two vectors $\mathbf{a}$ and $\mathbf{b}$ which are like, or oppositely, directed (Fig. 4). Then there exists a number $k \neq 0$ such that

$$\mathbf{b} = k\mathbf{a}. \tag{3.1}$$

If we set $k = -\alpha/\beta$, we can write this equation as

$$\alpha\mathbf{a} + \beta\mathbf{b} = \mathbf{0},$$

and hence two collinear (or parallel) vectors are linearly dependent since neither α nor β is zero. We will say that the totality of vectors $k\mathbf{a}$ for an arbitrary real k and $\mathbf{a} \neq \mathbf{0}$ forms a one-dimensional real *linear vector space*. The reason for this terminology is clear since every point on the line can be represented by some *position vector* $k\mathbf{a}$.

If $\mathbf{a}$ and $\mathbf{b}$ are two noncollinear vectors, represented by directed line segments with common origin O (Fig. 5), any vector $\mathbf{c}$ lying in the plane of $\mathbf{a}$ and $\mathbf{b}$ can be represented in the form

$$\mathbf{c} = m\mathbf{a} + n\mathbf{b}. \tag{3.2}$$

Formula 3.2 follows at once from the rule for addition of vectors and from the definition of multiplication of vectors by scalars. Equation 3.2 can be rewritten in symmetric form to read

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0},$$

which is the condition for linear dependence of the set of three vectors, since not all constants in this formula vanish. The formula $m\mathbf{a} + n\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are two linearly independent vectors and m and n are arbitrary real numbers, defines a *two-dimensional real linear vector space*. We see that in a two-dimensional linear vector space a set of three vectors is always linearly dependent.

Fig. 4.

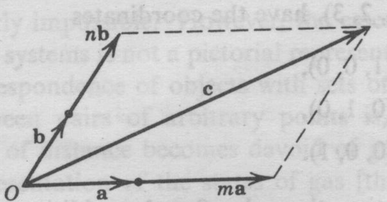

Fig. 5.

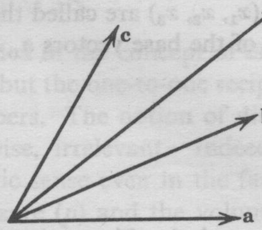

Fig. 6.

If we start with three noncoplanar vectors **a**, **b**, **c** issuing from the common origin O (Fig. 6), we can clearly represent every vector **d** in the form

$$\mathbf{d} = m\mathbf{a} + n\mathbf{b} + p\mathbf{c}, \tag{3.3}$$

from which it follows that among four vectors **a**, **b**, **c**, **d** there always exists a nontrivial relation of the form

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} + \delta\mathbf{d} = \mathbf{0}.$$

Formula 3.3, for an arbitrary choice of real numbers m, n, p, defines a *three-dimensional real linear vector space*. The terminal points of position vectors **d** sweep out a three-dimensional space of points if m, n, and p are allowed to range over the entire set of real numbers. In a three-dimensional linear vector space every set of four vectors is linearly dependent. We will make use of the connection of the number of linearly independent vectors with the dimensionality of space to formulate the concept of dimensionality of a linear vector space of n dimensions.

The vectors **a**, **b**, and **c** in (3.3) are called *base* or *coordinate vectors*, and the numbers m, n, and p are the *measure numbers* or *components* of the vector **d**. Once a set of base vectors is specified, every vector is determined uniquely by a triplet of measure numbers.

A set of three mutually orthogonal vectors in a three-dimensional space is obviously linearly independent, and if we choose as our coordinate vectors three mutually orthogonal vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, each of length 1, the resulting set of base vectors is said to be *orthonormal*.

We can visualize a set of orthonormal vectors directed along the axes of a suitable rectangular cartesian reference frame; in this case every vector **x** has the representation

$$\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3,$$