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# Value Distribution Theory

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**Editors**

Ilpo Laine
University of Joensuu, Department of Mathematics
P.O. Box 111, SF-80101 Joensuu 10, Finland

Seppo Rickman
University of Helsinki, Department of Mathematics
Hallituskatu 15, SF-00100 Helsinki 10, Finland

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Dedicated to the memory of

ROLF NEVANLINNA

# PREFACE

This volume contains most of the invited lecture series presented at the Nordic Summer School in Mathematics held at the University of Joensuu, June 1 - 12, 1981. The Summer School was devoted to the value distribution theory, the main emphasis being in several variables. The invited speakers were, in alphabetical order, W. K. Hayman, O. Lehto, S. Rickman, B. Shiffman and W. Stoll. The lecture series given by O. Lehto has been published elsewhere; therefore it is replaced here by an introduction to Nevanlinna theory by S. Toppila.

The volume is dedicated to the memory of Rolf Nevanlinna, by the consent with all authors. Let it be remarked here that Joensuu is the city of birth of Rolf Nevanlinna.

We wish to thank the Nordiska Forskarkurser whose financial support has been decisive to continue the tradition of Nordic Summer School in Mathematics. We also wish to thank the staff of the Department of Mathematics and Physics in the University of Joensuu for their co-operation in organizing this meeting and preparing this volume. Finally, our gratitude is directed to Springer-Verlag for their willingness to publish the main lectures of the Summer School.

Joensuu and Helsinki, November 1982,

Ilpo Laine　　　　　　　　　　Seppo Rickman

## OTHER LECTURES GIVEN AT THE SUMMER SCHOOL

Essén, Matts: On the value distribution and $L \log L$.

Iwaniec, Tadeusz: Regularity theorems and Liouville theorem.

Iwaniec, Tadeusz: On systems of partial differential equations in the theory of quasiconformal mappings.

Kopiecki, Ryszard: Stability in the differential equations for quasiregular mappings.

Kuusalo, Tapani: Conformal moduli for compact manifolds.

Laine, Ilpo: Some applications of the value distribution theory into complex differential equations.

Lehto, Olli: Introduction into classical value distribution theory. (For this invited lecture series, see e.g. Ann. Acad. Sci. Fenn. A I 7 (1982), 5 - 23.)

Molzon, Robert: Transfinite diameter, Tchebycheff constant and capacity in several variables - application to Nevanlinna theory.

Riihentaus, Juhani: A remark concerning Radó's theorem in several complex variables.

Vuorinen, Matti: On the uniqueness of sequential limits of quasiconformal mappings.

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# AN INTRODUCTION TO NEVANLINNA THEORY

Sakari Toppila

## 1. Introduction

Let $f$ be meromorphic in $|z| < R$, $0 < R \leq \infty$. For any complex value and any $r$, $0 < r < R$, we denote by $n(r,a) = n(r,a,f)$ the number of the a-points of $f$ lying in $|z| \leq r$ when the multiple roots of the equation $f(z) = a$ are counted according to their multiplicity. We write

$$N(r,a) = N(r,a,f) = \int_0^r \frac{n(t,a) - n(0,a)}{t}\, dt + n(0,a)\log r$$

for $0 < r < R$. We set $\log^+ x = \max(0, \log x)$ for $x \geq 0$,

$$m(r,a) = m(r,a,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\varphi}) - a|^{-1} d\varphi$$

if $a \neq \infty$, and

$$m(r,f) = m(r,\infty) = m(r,\infty,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\varphi})|\, d\varphi$$

for $0 < r < R$. The characteristic function $T$ of $f$ is defined by

$$T(r) = T(r,f) = m(r,\infty,f) + N(r,\infty,f)$$

for $0 < r < R$.

The Laurent expansion

$$f(z) - a = c(a)z^p + c_{p+1}z^{p+1} + \cdots$$

where $c(a) \neq 0$, defines $c(a)$ for any finite complex value $a$, if $f$ is nonconstant.

## 2. The first main theorem

Let $f$ be a nonconstant meromorphic function in $|z| < R$, $0 < R \leq \infty$. Let $b_q$ be the poles and $a_p$ the zeros of $f$. Let $r$, $0 < r < R$, be chosen such that $f$ has no poles or zeros on $|z| = r$. We write

$$F(z) = f(z)\left(\prod_{|b_q|<r} \frac{r(z - b_q)}{r^2 - \bar{b}_q z}\right)\prod_{|a_p|<r} \frac{r^2 - \bar{a}_p z}{r(z - a_p)}\,.$$

Then $\log|F(z)|$ is harmonic in $|z| \leq r$, and we deduce from the Gauss mean value theorem that

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\varphi})|\,d\varphi. \tag{2.1}$$

Since

$$\log|F(0)| = \log|c(0)| + (n(0,0) - n(0,\infty))\log r$$

$$+ \sum_{0<|a_p|<r} \log|r/a_p| - \sum_{0<|b_q|<r} \log|r/b_q|$$

$$= \log|c(0)| + N(r,0) - N(r,\infty)$$

and $|f(z)| = |F(z)|$ on $|z| = r$, we get from (2.1)

$$\log|c(0)| + N(r,0) - N(r,\infty)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\log^+|f(re^{i\varphi})| - \log^+|f(re^{i\varphi})|^{-1})\,d\varphi = m(r,\infty) - m(r,0) \tag{2.2}$$

From the continuity of the terms in (2.2) we deduce that (2.2) holds also in the case that $f$ has a finite number of zeros or poles on $|z| = r$.

Let $a$ be a finite complex value. Applying (2.2) to the function $f(z) - a$ we get

$$\log|c(a)| + N(r,a,f) - N(r,\infty,f) = m(r,\infty,f - a) - m(r,a,f). \tag{2.3}$$

Since

$$\log^+|f - a| \leqq \log^+|f| + \log^+|a| + \log 2$$

and

$$\log^+|f| \leqq \log^+|f - a| + \log^+|a| + \log 2,$$

we deduce after integration that

$$|m(r,\infty,f) - m(r,\infty,f - a)| \leqq \log^+|a| + \log 2.$$

This combined with (2.3) yields

$$N(r,a) + m(r,a) = N(r,\infty) + m(r,\infty) + \varphi(r,a) \tag{2.4}$$

where

$$|\varphi(r,a)| \leqq \log^+|a| + \log 2 + |\log|c(a)||.$$

We have proved the following result.

First main theorem. Let $f$ be a nonconstant meromorphic function in $|z| < R$ where $0 < R \leq \infty$. Then for any $r$, $0 < r < R$, and for any finite complex value $a$,

$$m(r,a) + N(r,a) = T(r,f) + \varphi(r,a)$$

where

$$|\varphi(r,a)| \leq \log^+|a| + |\log|c(a)|| + \log 2.$$

## 3. Some applications of the lemma on the logarithmic derivative

For the logarithmic derivative of a function meromorphic in the plane Nevanlinna [3] has proved the following

Lemma A. Let $f$ be a nonconstant meromorphic function in the plane. Then

$$m(r,f'/f) = O(\log(rT(r,f))) \tag{3.1}$$

as $r \to \infty$ outside an exceptional set $E$ of finite linear measure. If $f$ has finite order then (3.1) holds as $r \to \infty$ through all positive real values.

Let $f$ be a transcendental meromorphic function in the plane and let $a_1, a_2, \ldots, a_q$ be $q \geq 2$ different finite complex values. We choose $d$, $0 < d < 1/2$, such that $|a_k - a_p| \geq 2d$ if $1 \leq k < p \leq q$. We set

$$F(z) = \prod_{k=1}^{q} (f(z) - a_k).$$

Then

$$\sum_{k=1}^{q} \frac{1}{f(z) - a_k} = (1/f'(z))(F'(z)/F(z))$$

and we deduce that

$$m(r, \sum_{k=1}^{q} \frac{1}{f - a_k}) \leq m(r,0,f') + m(r,F'/F) \tag{3.2}$$

for all $r > 0$.

We have

$$\sum_{k=1}^{q} \log^+ \frac{1}{|f(z) - a_k|} \leq \sum_{k=1}^{q} \log^+ \frac{d/q}{|f(z) - a_k|} + q \log(q/d). \tag{3.3}$$

For any fixed $z$, there exists at most one $a_k$ such that

$$|f(z) - a_k| < d/q, \tag{3.4}$$

and if (3.4) holds for $k$ then

$$|f(z) - a_p| \geqq |a_k - a_p| - d/q \geqq d$$

for $p \neq k$, and

$$\sum_{s=1}^{q} \log^+ \frac{d/q}{|f(z) - a_s|} \leqq \log^+ \frac{1}{|f(z) - a_k|}$$

$$\leqq \log^+ \left| \sum_{s=1}^{q} \frac{1}{f(z) - a_s} - \sum_{s \neq k} \frac{1}{f(z) - a_s} \right|$$

$$\leqq \log^+ \left| \sum_{s=1}^{q} \frac{1}{f(z) - a_s} \right| + \log 2 + \log^+ \frac{q - 1}{d}.$$

Combining this estimate with (3.3), we deduce that

$$\sum_{k=1}^{q} \log^+ \frac{1}{|f(z) - a_k|} \leqq \log^+ \left| \sum_{k=1}^{q} \frac{1}{f(z) - a_k} \right| + (q + 2)\log(q/d). \quad (3.5)$$

Integrating the estimate (3.5), we get

$$\sum_{k=1}^{q} m(r,a_k,f) \leqq m(r, \sum_{k=1}^{q} \frac{1}{f - a_k}) + (q + 2)\log(q/d)$$

for all $r > 0$, and we deduce from (3.2) that

$$\sum_{k=1}^{q} m(r,a_k,f) \leqq m(r,0,f') + m(r,F'/F) + (q + 2)\log(q/d) \quad (3.6)$$

for all positive $r$.

We write

$$N_1(r,f) = 2N(r,\infty,f) - N(r,\infty,f') + N(r,0,f'). \quad (3.7)$$

Since

$$m(r,f') \leqq m(r,f) + m(r,f'/f),$$

we get from the first main theorem

$$m(r,0,f') = N(r,\infty,f') + m(r,f') - N(r,0,f') + 0(1)$$

$$\leqq N(r,\infty,f') + m(r,f) - N(r,0,f') - 2N(r,\infty,f)$$

$$+ 2N(r,\infty,f) + m(r,f'/f) + 0(1)$$

$$\leqq 2T(r,f) - m(r,f) - N_1(r,f) + m(r,f'/f) + 0(1). \quad (3.8)$$

We denote by $S(r,f)$ any function which satisfies

$$S(r,f) = 0(\log(rT(r,f)))$$

as $r \to \infty$ without restriction if the order of $f$ is finite, and as $r \to \infty$ outside an exceptional set of finite linear measure if the order of $f$ is infinite. Since

$$T(r,F) = O(T(r,f)) \quad \text{as} \quad r \to \infty,$$

we deduce from Lemma A that

$$m(r,F'/F) + m(r,f'/f) = S(r,f). \tag{3.9}$$

This implies that if we write $a_{q+1} = \infty$, we get from (3.6) and (3.8)

$$\sum_{k=1}^{q+1} m(r,a_k,f) \leqq 2T(r,f) - N_1(r,f) + S(r,f). \tag{3.10}$$

Here $N_1(r,f) \geqq 0$ for $r \geqq 1$. The formula (3.10) is the second main theorem for functions meromorphic in the plane.

For any complex value $a$, the Nevanlinna deficiency is defined by

$$\delta(a) = \delta(a,f) = \liminf_{r \to \infty} \frac{m(r,a)}{T(r,f)} = 1 - \limsup_{r \to \infty} \frac{N(r,a)}{T(r,f)},$$

and the Valiron deficiency is defined by

$$\Delta(a) = \Delta(a,f) = \limsup_{r \to \infty} \frac{m(r,a)}{T(r,f)} = 1 - \liminf_{r \to \infty} \frac{N(r,a)}{T(r,f)}.$$

It follows from (3.10) that

$$\sum_{k=1}^{q+1} \delta(a_k) \leqq (1 + o(1)) \sum_{k=1}^{q+1} \frac{m(r,a_k,f)}{T(r,f)} \leqq 2 + \frac{S(r,f)}{T(r,f)} + o(1) \quad (r \to \infty),$$

and since

$$\liminf_{r \to \infty} \frac{S(r,f)}{T(r,f)} = 0, \tag{3.11}$$

we deduce that

$$\sum_{k=1}^{q+1} \delta(a_k) \leqq 2. \tag{3.12}$$

This holds for any choice of $q + 1$ different values $a_k$, and we deduce that the set of values $a$ for which $\delta(a) > 0$ is a countable set and that

$$\sum_{\delta(a)>0} \delta(a) \leqq 2. \tag{3.13}$$

It follows from Lemma A that

$$m(r,f') \leqq m(r,f) + m(r,f'/f) \leqq m(r,f) + S(r,f), \tag{3.14}$$

and since $N(r,\infty,f') \leqq 2N(r,\infty,f)$ for $r \geqq 1$, we deduce that

$$T(r,f') \leqq 2T(r,f) + S(r,f).$$

This implies together with (3.6) and (3.9) that for any $q$ different finite values $a_k$ we have

$$\sum_{k=1}^{q} \frac{m(r,a_k,f)}{T(r,f)} \leqq \frac{m(r,0,f') + S(r,f)}{T(r,f)} \leqq \frac{T(r,f')m(r,0,f')}{T(r,f)T(r,f')} + \frac{S(r,f)}{T(r,f)}$$

$$\leqq \left(2 + \frac{S(r,f)}{T(r,f)}\right)\frac{m(r,0,f')}{T(r,f')} + \frac{S(r,f)}{T(r,f)}. \qquad (3.15)$$

If the order of $f$ is finite then

$$\frac{S(r,f)}{T(r,f)} \to 0 \quad \text{as} \quad r \to \infty,$$

and we get from (3.15)

$$\sum_{k=1}^{q} \delta(a_k,f) \leqq \liminf_{r \to \infty} \sum_{k=1}^{q} \frac{m(r,a_k,f)}{T(r,f)}$$

$$\leqq 2 \liminf_{r \to \infty} \frac{m(r,0,f')}{T(r,f')} = 2\delta(0,f'). \qquad (3.16)$$

and

$$\Delta(a_1,f) \leqq 2 \limsup_{r \to \infty} \frac{m(r,0,f')}{T(r,f')} = 2\Delta(0,f'). \qquad (3.17)$$

If the order of $f$ is finite or infinite, we get from (3.15) and (3.11)

$$\sum_{k=1}^{q} \delta(a_k,f) \leqq 2 \limsup_{r \to \infty} \frac{m(r,0,f')}{T(r,f)} = 2\Delta(0,f'). \qquad (3.18)$$

Since $N(r,f) \leqq N(r,f')$ for $r \geqq 1$, we deduce from (3.14) that

$$\frac{m(r,f')}{T(r,f')} \leqq \frac{m(r,f')}{N(r,f') + m(r,f')} \leqq \frac{m(r,f')}{N(r,f) + m(r,f')} \leqq \frac{m(r,f) + S(r,f)}{T(r,f) + S(r,f)}. \qquad (3.19)$$

This implies that if the order of $f$ is finite then

$$\delta(\infty,f') \leqq \delta(\infty,f) \qquad (3.20)$$

and

$$\Delta(\infty,f') \leqq \Delta(\infty,f), \qquad (3.21)$$

and if the order of $f$ is finite or infinite then we still have

$$\delta(\infty,f') \leqq \Delta(\infty,f). \qquad (3.22)$$

Combining the estimates (3.16) - (3.22), we get the following result for the connection between the deficiences of $f$ and $f'$.

Theorem 1. Let $f$ be a transcendental meromorphic function in the plane. If the order of $f$ is finite, then

$$\delta(\infty,f') \leqq \delta(\infty,f), \qquad (3.23)$$

$$\Delta(\infty,f') \leqq \Delta(\infty,f), \tag{3.24}$$

$$\sum_{a\neq\infty} \delta(a,f) \leqq 2\delta(0,f') \tag{3.25}$$

and for any finite $a$

$$\Delta(a,f) \leqq 2\Delta(0,f'). \tag{3.26}$$

If $f$ has infinite or finite order, then

$$\delta(\infty,f') \leqq \Delta(\infty,f) \tag{3.27}$$

and

$$\sum_{a\neq\infty} \delta(a,f) \leqq 2\Delta(0,f'). \tag{3.28}$$

## 4. Some examples

We set $r_o = s_o = 8$, and for $n \geqq 1$, we choose $r_n$ and $s_n$, $s_n$ being an integer, such that

$$r_n = \exp(s_{n-1}r_{n-1}) \tag{4.1}$$

and

$$\log(s_n/r_n) \geqq nr_n^{s_{n-1}} > \log((s_n - 1)/r_n). \tag{4.2}$$

We set

$$f(z) = \sum_{n=1}^{\infty} (z/r_n)^{s_n}$$

and

$$F(z) = \int_0^z f(\xi)d\xi = z\sum_{n=1}^{\infty} (1 + s_n)^{-1}(z/r_n)^{s_n}.$$

We have

$$f'(z) = (1/z)\sum_{n=1}^{\infty} s_n(z/r_n)^{s_n}.$$

For large values of $n$, we choose $t_n < t_n' < R_n < R_n'$ such that

$$100s_n(t_n/r_n)^{s_n} = s_{n-1}(t_n/r_{n-1})^{s_{n-1}},$$

$$s_n(t_n'/r_n)^{s_n} = 100s_{n-1}(t_n'/r_{n-1})^{s_{n-1}},$$

$$(100/(1 + s_n))(R_n/r_n)^{s_n} = (1/(1 + s_{n-1}))(R_n/r_{n-1})^{s_{n-1}}$$

and

$$(1/(1 + s_n))(R'_n/r_n)^{s_n} = (100/(1 + s_{n-1}))(R'_n/r_{n-1})^{s_{n-1}}.$$

We denote by $d_k(z)$, $k = 1,2,\ldots$, functions which satisfy $|d_k(z)| < 1/50$ for all finite $z$. If $R'_n \leq |z| \leq R_{n+1}$, we have

$$F(z) = (1 + d_1(z))(z/(1 + s_n))(z/r_n)^{s_n}. \quad (4.3)$$

Similarly, we have

$$f(z) = (1 + d_2(z))(z/r_n)^{s_n} \quad (4.4)$$

for $R_n \leq |z| \leq t'_{n+1}$, and

$$f'(z) = (1 + d_3(z))(s_n/z)(z/r_n)^{s_n} \quad (4.5)$$

for $t'_n \leq |z| \leq t_{n+1}$.

In Lemma 1 we give a function which does not satisfy (3.26).

Lemma 1. Let $g(z) = F^2(z)/f(z)$. Then $\Delta(0,g) = 1$ but $\Delta(0,g') = 0$.

Proof. We write

$$h(z) = \frac{F(z)f'(z)}{f^2(z)}.$$

Then $g'(z) = F(z)(2 - h(z))$.

From the choice of $t'_{n+1}$ and (4.5) we deduce that

$$|f'(z)| \geq (49/50)((100s_n)/t'_{n+1})(t'_{n+1}/r_n)^{s_n}$$

on $|z| = t'_{n+1}$, and we get from (4.3) and (4.4)

$$|h(z)| \geq 98(49/50)(s_n/(1 + s_n))(50/51)^2 > 10 \quad (4.6)$$

for $|z| = t'_{n+1}$. From the choice of $R_n$ and (4.3) we deduce that

$$|F(z)| \geq 98(R_n/(1 + s_n))(R_n/r_n)^{s_n}$$

on $|z| = R_n$, and we get from (4.4) and (4.5)

$$|h(z)| \geq 10 \quad (4.7)$$

for $|z| = R_n$. Since $h$ has no zeros in $t'_n < |z| < R_n$, it follows from (4.6), (4.7) and the minimum principle that (4.7) holds for all $z$ lying on $t'_n \leq |z| \leq R_n$. For $R'_n \leq |z| \leq t_{n+1}$ we get from (4.3) - (4.5)

$$|h(z)| \leq (51/50)^2(s_n/(1 + s_n))(50/49)^2 < 11/10. \quad (4.8)$$

Since $g' = F(2-h)$, it follows from (4.3), (4.7) and (4.8) that

$$m(r,0,g') = 0 \tag{4.9}$$

for $R_n' \leq r \leq t_{n+1}$ and for $t_n' \leq r \leq R_n$.

From the choices of $t_n$ and $t_n'$ we get

$$(s_n - s_{n-1})\log(t_n'/t_n) = 2\log 100$$

which implies together with (4.1) and (4.2) that

$$s_n \log(t_n'/t_n) < 20. \tag{4.10}$$

Similarly, we get

$$s_n \log(R_n'/R_n) < 20. \tag{4.11}$$

It follows from (4.3) - (4.5) that

$$T(4r_n,g') = O(s_n) \quad \text{as} \quad n \to \infty. \tag{4.12}$$

From the first main theorem we deduce that if $t_n \leq r \leq t_n'$, then

$$\begin{aligned} m(r,0,g') &= T(r,g') - N(r,0,g') + O(1) \\ &\leq T(t_n',g') - N(t_n,0,g') + O(1) \\ &\leq T(t_n',g') - T(t_n,g') + m(t_n,0,g') + O(1), \end{aligned}$$

and since $m(t_n,0,g') = 0$ and $T$ is an increasing and convex function of $\log r$, we get from (4.10) and (4.12)

$$m(r,0,g') \leq O(1) + \frac{T(4r_n,g')\log(t_n'/t_n)}{\log(4r_n) - \log t_n} = O(1) \tag{4.13}$$

for $t_n \leq r \leq t_n'$. Similarly, we see from (4.11) that (4.13) holds for $R_n \leq r \leq R_n'$. Combining the estimates on $m(r,0,g')$, we deduce that

$$m(r,0,g') = O(1) \quad \text{as} \quad r \to \infty, \tag{4.14}$$

and we get $\Delta(0,g') = 0$.

Suppose that $|z| = R_n$. From (4.3) we deduce that

$$\log M(R_n,F) \leq (1 + s_{n-1})\log R_n \leq (1 + o(1))s_{n-1}\log r_n, \tag{4.15}$$

and from (4.4) and the choice of $R_n$ we get

$$\log|f(z)| = (1 + o(1))\log(R_n/r_n)^{s_n} \geq (1 + o(1))\log(s_n/s_{n-1}). \tag{4.16}$$

These estimates imply together with (4.1) and (4.2) that

$$|g(z)| < 1 \tag{4.17}$$

on $|z| = R_n$ and that

$$T(R_n,F) = o(T(R_n,f)) \quad \text{as} \quad n \to \infty. \tag{4.18}$$

Since

$$T(r,f) = m(r,f) = m(r,F^2(f/F^2))$$

$$\leqq m(r,1/g) + 2m(r,F) \leqq m(r,0,g) + 2T(r,F),$$

we get from (4.18)

$$m(R_n,0,g) \geqq T(R_n,f) - 2T(R_n,F) \geqq (1 + o(1))T(R_n,f)$$

$$= (1 + o(1))T(R_n,g) \quad \text{as} \quad n \to \infty, \tag{4.19}$$

and we deduce that $\Delta(0,g) = 1$. This completes the proof of Lemma 1.

The following lemma gives a function which does not satisfy (3.25).

Lemma 2. Let $g_1(z) = 1/F(z)$. Then $\delta(0,g_1) = 1$ but $\delta(0,g_1') = 0$.

Proof. Since $g_1'(z) = -(1/g(z))$, we see from (4.17) that $\delta(0,g_1') = 0$. Since $n(r,0,g_1) \equiv 0$, we have $\delta(0,g_1) = 1$. Lemma 2 is proved.

The following lemma gives a function which does not satisfy (3.24).

Lemma 3. If a finite value $a$ is chosen such that $\Delta(a,F) = 0$, then the function $g_2(z) = 1/(a - F(z))$ satisfies $\Delta(\infty,g_2) = 0$ and $\Delta(\infty,g_2') = 1$.

Proof. Since $\Delta(a,F) = 0$, we have $\Delta(\infty,g_2) = 0$. Since

$$g_2'(z) = f(z)(a - F(z))^{-2}$$

and

$$m(r,f) \leqq m(r,g_2') + 2m(r,a - F),$$

we deduce from (4.18) that

$$m(R_n,g_2') \geqq m(R_n,f) - 2m(R_n,a - F) \geqq (1 + o(1))T(R_n,f)$$

$$= (1 + o(1))T(R_n,g_2') \quad \text{as} \quad n \to \infty.$$

This shows that $\Delta(\infty,g_2') = 1$ which completes the proof of Lemma 3.

Now we shall construct a function $g_3$ which does not satisfy (3.23).

From the choice of $t_p$ and $t_p'$ we deduce that $|f'(z)| \geqq 1$ on $t_{p-1}' \leqq |z| \leqq t_p$, and we deduce from (4.1) and the choice of $t_p$ that

$$s_{p-1} \log r_p = (1 + o(1))T(t_p, f') = (1 + o(1))N(t_p, 0, f')$$

$$= (1 + o(1))n(t_{p-1}', 0, f')\log r_p \quad \text{as} \quad p \to \infty.$$

This implies that for all large $p$, say for $p \geqq p_o$,

$$n(t_{p-1}', 0, f') = n(t_p, 0, f') = (1 + o(1))s_{p-1} \quad \text{as} \quad p \to \infty. \tag{4.20}$$

We choose a sequence $k_p$ of positive integers such that

$$s_p/p \leqq k_p < 1 + s_p/p \tag{4.21}$$

for any $p$, and a sequence $\varepsilon_p$, $\varepsilon_p > 0$ for any $p$, $\varepsilon_p \to 0$ as $p \to \infty$, is chosen such that the function

$$h_1(z) = \sum_{p=p_o}^{\infty} (\varepsilon_p^{-1}(z - t_p'))^{-k_p}$$

satisfies

$$T(r, h_1) = (1 + o(1))N(r, \infty, h_1) \quad \text{as} \quad r \to \infty \tag{4.22}$$

and

$$T(r, h_1') = (1 + o(1))N(r, \infty, h_1') \quad \text{as} \quad r \to \infty. \tag{4.23}$$

From (4.20) and (4.21) we deduce that

$$\frac{n(r, \infty, h_1')}{n(r, 0, f')} \to 0 \quad \text{as} \quad r \to \infty,$$

which together with (4.23) implies that

$$T(r, h_1') = O(N(r, \infty, h_1')) = o(N(r, 0, f')) = o(T(r, f')) \quad \text{as} \quad r \to \infty. \tag{4.24}$$

We set $g_3 = f + h_1$. Since

$$T(r, f') = m(r, f') = m(r, g_3' - h_1') \leqq m(r, g_3') + m(r, h_1') + \log 2,$$

we get from (4.24)

$$m(r, g_3') \geqq (1 + o(1))T(r, f') = (1 + o(1))T(r, g_3')$$

as $r \to \infty$, which implies that $\delta(\infty, g_3') = 1$.

From (4.21) and the choice of $t_p'$ we get for $p > p_o$

$$T(r_p, h_1) \geqq N(r_p, \infty, h_1) \geqq (s_p/p)\log(r_p/t_p')$$