MATHEMATICS IN SCIENCE AND ENGINEERING

Volume 167

Volterra Integral and Differential Equations

T. A. Burton

Volterra Integral and Differential Equations

T. A. BURTON

Department of Mathematics
Southern Illinois University at Carbondale
Carbondale, Illinois

ACADEMIC PRESS
A Subsidiary of Harcourt Brace Jovanovich, Publishers
New York London
Paris San Diego San Francisco São Paulo Sydney Tokyo Toronto

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1 7DX

Library of Congress Cataloging in Publication Data

Burton, T. A. (Theodore Allen), Date
Volterra integral and differential equations.

(Mathematics in science and engineering ;)
Bibliography: p.
Includes indexes.
1. Volterra equations. 2. Integro-differential equations. I. Title. II. Series.
QA431.B87 1982 515.3'8 82-18932
ISBN 0-12-147380-5

PRINTED IN THE UNITED STATES OF AMERICA

83 84 85 86 9 8 7 6 5 4 3 2 1

Preface

This book provides an introduction to the structure and stability properties of solutions of Volterra integral and integro-differential equations. It is primarily an exposition of Liapunov's direct method. Chapter 0 gives a detailed account of the subjects treated.

To most seasoned investigators in the theory of Volterra equations, the study centers in large measure on operator theory, measure and integration, and general functional analysis. This book, however, is aimed at a different audience. There are today hundreds of mathematicians, physicists, engineers, and other scientists who are well versed in stability theory of ordinary differential equations on the real line using elementary differentiation and Riemann integration. The purpose of this book is to enable such investigators to parlay their existing expertise into a knowledge of theory and application of Volterra equations and to introduce them to the great range of physical applications of the subject.

Stability theory of Volterra equations is an area in which there is great activity among a moderate number of investigators. Basic knowledge is advancing rapidly, and it appears that this area will be an excellent field of research for some time to come. There are elementary theorems on Liapunov's direct method waiting to be proved; really usable results concerning the resolvent in nonconvolution cases are scarce; much remains to be done concerning the existence of periodic solutions; good Liapunov functionals have abounded for 10 years and await development of general theory to permit really effective applications; and there is a great need for careful analysis of specific simple Volterra equations as a guide to the development of the general theory.

I am indebted to many for assistance with the book: to the editors at Academic Press for their interest; to Professor Ronald Grimmer for reading Chapters 1 and 2; to the graduate students who took formal courses from Chapters 1–6 and offered suggestions and corrections; to Professor John Haddock for reading Chapters 3–8; to Professor L. Hatvani for reading Chapters 5 and 6; to Mr. M. Islam for carefully working through Chapters 3 and 5; to Professor Wadi Mahfoud for reading Chapters 1–6; to my wife, Freddä, for drawing the figures; and to Shelley Castellano for typing the manuscript. A special thanks is due Professor Qichang Huang for reading and discussing the entire manuscript.

Contents

3. Existence Properties

4. History, Examples, and Motivation

5. Instability, Stability, and Perturbations

6. Stability and Boundedness

7. Perturbations

8. Functional Differential Equations

0

Introduction and Overview

0.1. Statement of Purpose

Although the theory of Volterra integral and integro-differential equations is old, well developed, and dense in the literature and in applications, we have been unable to find a systematic treatment of the theory's basic structure and stability properties. This book is a modest attempt to fill that void.

There are, of course, numerous treatments of the subject, but none seem to present a coherent set of results parallel to the standard treatments of stability theory given ordinary differential equations. Indeed, the student of the subject is hard put to find in the literature that the solution spaces of certain Volterra equations are identical to those for certain ordinary differential equations. Even the outstanding investigators have tended to deny such connections. For example, Miller (1971a, p. 9) states: "While it is true that all initial value problems for ordinary differential equations can be considered as Volterra integral equations, this fact is of limited importance." It is our view that this fact is of fundamental importance, and consequently, it is our goal to develop the theory of Volterra equations in such a manner that the investigator in the area of ordinary differential equations may parlay his expertise into a comprehension of Volterra equations. We hasten to add that there are indeed areas of Volterra equations that do not parallel the standard theory for ordinary differential equations. For a study of such areas, we heartily recommend the excellent treatment by Miller (1971a).

0.2. An Overview

It is assumed that the reader has some background in ordinary differential equations. Thus, Chapter 1 deals with numerous examples of Volterra equations reducible to ordinary differential equations. It also introduces the concept of initial functions and presents elementary boundedness results.

In Chapter 2 we point out that the structure of the solution space for the vector system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \int_0^t C(t,s)\mathbf{x}(s)\,ds + \mathbf{f}(t) \tag{0.2.1}$$

is indistinguishable from that of the ordinary differential system

$$\mathbf{x}'(t) = B(t)\mathbf{x}(t) + \mathbf{g}(t). \tag{0.2.2}$$

In fact, if $Z(t)$ is the $n \times n$ matrix satisfying

$$Z'(t) = A(t)Z(t) + \int_0^t C(t,s)Z(s)\,ds, \qquad Z(0) = I, \tag{0.2.3}$$

and if $\mathbf{x}_p(t)$ is any solution of (0.2.1), then any solution $\mathbf{x}(t)$ of (0.2.1) on $[0, \infty)$ may be written as

$$\mathbf{x}(t) = Z(t)[\mathbf{x}(0) - \mathbf{x}_p(0)] + \mathbf{x}_p(t). \tag{0.2.4}$$

Moreover, when A is a constant matrix and C is of convolution type, the solution of (0.2.1) on $[0, \infty)$ is expressed by the variation of parameters formula

$$\mathbf{x}(t) = Z(t)\mathbf{x}(0) + \int_0^t Z(t-s)\mathbf{f}(s)\,ds,$$

which is familiar to the college sophomore.

Chapter 2 also covers various types of stability, primarily using Liapunov's direct method. That material is presented with little background explanation, so substantial stability results are quickly obtained. Thus, by the end of Chapter 2 the reader has related Volterra equations to ordinary differential equations, has thoroughly examined the structure of the solution space, and has acquired tools for investigating boundedness and stability properties. The remainder of the book is devoted to consolidating these gains, bringing the reader to the frontiers in several areas, and suggesting certain research problems urgently in need of solution.

Chapter 3 outlines the basic existence, uniqueness, and continuation results for nonlinear ordinary differential equations. Those results and techniques are then extended to Volterra equations, making as few changes as are practical.

Chapter 4 is an in-depth account of some of the more interesting historical problems encountered in the development of Volterra equations. We trace biological growth problems from the simple Malthusian model, through the logistic equation, the predator–prey system of Lotka and Volterra, and on to Volterra's own formulation of integral equations regarding age distribution in populations. Feller's work with the renewal equation is briefly described. We then present many models of physical problems using integral equations. These problems range from electrical circuits to nuclear reactors.

Chapters 5–8 deal exclusively with Liapunov's direct method. Indeed, this book is mainly concerned with the study of stability properties of solutions of integral and integro-differential equations by means of Liapunov functionals or Liapunov–Razumikhin functions.

Chapter 5 deals with very specific Liapunov functionals yielding necessary and sufficient conditions for stability.

Chapter 6 is a basic introduction to stability theory for both ordinary differential equations and Volterra equations. Having shown the reader in Chapters 2 and 5 the power and versatility of Liapunov's direct method, we endeavor in Chapter 6 to promote a fundamental understanding of the subject. The basic theorems of ordinary differential equations are presented, proved, and discussed in terms of their history and their faults. Numerous examples of construction of Liapunov functions are given. We then show how Liapunov functionals for Volterra equations can be constructed in terms of extensions of the idea of a first integral. Theorems are proved, and examples are given concerning stability, uniform stability, asymptotic stability, uniform asymptotic stability, and perturbations.

Chapter 7 deals with perturbations, the construction of collections of Liapunov functionals, and it contains a converse theorem of Miller on the existence of Liapunov functionals.

Chapter 8 is a brief treatment of general functional differential equations involving both bounded and unbounded delays. A main feature is the existence and stability theory synthesized and improved by Driver for functional differential equations with unbounded delay. It also contains a brief account of stability and limit sets for the equations

$$x' = F(t, x_t) \tag{0.2.5}$$

and

$$x' = f(x_t). \tag{0.2.6}$$

Much effort is devoted to certain recurring problems in earlier chapters. These may be briefly described as follows:

(i) If $V(t, \mathbf{x})$ is a scalar function whose derivative along solutions of

$$\mathbf{x}' = \mathbf{F}(t, \mathbf{x}) \tag{0.2.7}$$

is negative for $|\mathbf{x}|$ large, then it is frequently possible to conclude that solutions are bounded. Such results are of great importance in proving the existence of periodic solutions. We survey literature that tends to extend such results to Volterra and functional differential equations.

(ii) If $V(t, \mathbf{x})$ is a scalar function whose derivative along solutions of (0.2.7) is negative in certain sets, then knowledge about limit sets of solutions of (0.2.7) may be obtained, provided that $\mathbf{F}(t, \mathbf{x})$ is bounded for $\mathbf{x}$ bounded. This boundedness hypothesis is sometimes reasonable for (0.2.7), but it is ludicrous for a general functional differential equation. Yet, authors have required it for decades. We explore three alternatives to asking $\mathbf{F}(t, \mathbf{x})$ bounded for $\mathbf{x}$ bounded in the corresponding treatment of functional differential equations.

1

The General Problems

1.1. Introduction

We are concerned with the boundedness and stability properties of the integral equation

$$\mathbf{x}(t) = \mathbf{f}(t) + \int_0^t \mathbf{g}(t, s, \mathbf{x}(s))\,ds \tag{1.1.1}$$

in which $\mathbf{x}$ is an n vector, $\mathbf{f}$: $[0, \infty) \to R^n$ is continuous, and $\mathbf{g}: \pi \times R^n \to R^n$ is continuous, where $\pi = \{(t, s): 0 \le s \le t < \infty\}$.

It is unusual to ask that $\mathbf{g}$ be continuous. With considerable additional effort, one may obtain many of the results obtained here with weaker assumptions. For some such work, see Miller (1971a). The techniques we use to show boundedness will frequently require that (1.1.1) be differentiated to obtain an *integro-differential equation*

$$\mathbf{x}'(t) = \mathbf{f}'(t) + \mathbf{g}(t, t, \mathbf{x}(t)) + \int_0^t \mathbf{g}_1(t, s, \mathbf{x}(s))\,ds,$$

where $\mathbf{g}_1$ denotes $\partial \mathbf{g}/\partial t$ or, more generally,

$$\mathbf{x}'(t) = \mathbf{h}(t, \mathbf{x}(t)) + \int_0^t \mathbf{F}(t, s, \mathbf{x}(s))\,ds. \tag{1.1.2}$$

Notation For a vector $\mathbf{x}$ and an $n \times n$ matrix A, the norm of $\mathbf{x}$ will usually be $|\mathbf{x}| = \max_i |x_i|$, whereas $|A|$ will mean $\sup_{|\mathbf{x}| \le 1} |A\mathbf{x}|$.

Convention It will greatly simplify notation if it is understood that a function written without its argument means that the function is evaluated at t. Thus (1.1.2) is

$$\mathbf{x}' = \mathbf{h}(t, \mathbf{x}) + \int_0^t \mathbf{F}(t, s, \mathbf{x}(s))\, ds.$$

We notice that if $\mathbf{f}$ is differentiable and $\mathbf{g}$ is independent of t, in (1.1.1), then differentiation yields an *ordinary differential equation*

$$\mathbf{x}'(t) = \mathbf{G}(t, \mathbf{x}(t)). \tag{1.1.3}$$

The process of going from (1.1.1) to (1.1.3) is easily reversed, as we simply write

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^t \mathbf{G}(s, \mathbf{x}(s))\, ds.$$

To pass from (1.1.2) to (1.1.1), integrate (1.1.2) and then change the order of integration.

It is assumed that the reader has some familiarity with (1.1.3). Our procedure will generally be to state, but usually not prove, the standard result for (1.1.3) and then develop the parallel result for (1.1.1) or (1.1.2).

While investigating (1.1.1) we shall occasionally be led to examine

$$\mathbf{x}' = \mathbf{h}(t, \mathbf{x}) + \int_{t-T}^t \mathbf{F}(t, s, \mathbf{x}(s))\, ds \tag{1.1.4}$$

and

$$\mathbf{x}(t) = \mathbf{f}(t) + \int_{-\infty}^t \mathbf{g}(t, s, \mathbf{x}(s))\, ds. \tag{1.1.5}$$

It will turn out that results proved for (1.1.4) may be applied to the general functional differential equation with bounded delay

$$\mathbf{x}'(t) = \mathbf{H}(t, \mathbf{x}_t), \tag{1.1.6}$$

where $\mathbf{x}_t$ is that segment of $\mathbf{x}(s)$ on the interval $t - h \le s \le t$ shifted back to $[-h, 0]$.

In the same way, we shall frequently see that results for (1.1.2) and (1.1.5) apply to a general functional differential equation

$$\mathbf{x}'(t) = \mathbf{K}(t, \mathbf{x}(s);\ \alpha \le s \le t), \tag{1.1.7}$$

where $\alpha = -\infty$ is allowed, including

$$\mathbf{x}'(t) = \mathbf{L}(t, \mathbf{x}(t), \mathbf{x}(t - r(t))), \tag{1.1.8}$$

with $r(t) \geq 0$.

One may note that Eqs. (1.1.1)–(1.1.3) are given in their order of generality.

1.2. Relations between Differential and Integral Equations

Most ordinary differential equations can be expressed as integral equations, but the reverse is not true. A given nth-order equation

$$x^{(n)}(t) = f(t, x, x', \ldots, x^{(n-1)})$$

may be expressed as a system of n first-order equations and then formally integrated. For example, if $x'' = f(t, x, x')$, then write $x = x_1$ and $x' = x_1' = x_2$, so that $x'' = x_2' = f(t, x_1, x_2)$, and the system of two first-order equations

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} x_2 \\ f(t, x_1, x_2) \end{bmatrix}$$

results.

And, in general, if $\mathbf{x} \in R^n$, then

$$\mathbf{x}' = \mathbf{G}(t, \mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1.2.1}$$

is a system of n first-order equations with initial condition (called an *initial-value problem*), written as

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{G}(s, \mathbf{x}(s))\, ds, \tag{1.2.2}$$

a system of n integral equations.

Thus, it is trivial to express such differential equations as integral equations. It is mainly a matter of renaming variables.

It may, however, be a surprise to find that when n is a positive integer, $f \in C^{n+1}$ on $[t_0, T)$, and g continuous, then

$$x(t) = f(t) + \int_{t_0}^{t} (t - s)^n g(s, x(s))\, ds \tag{1.2.3}$$

represents an $(n + 1)$st-order differential equation.

For example, when $n = 2$, we have

$$x'(t) = f'(t) + \int_{t_0}^{t} 2(t - s)g(s, x(s))\, ds,$$

$$x''(t) = f''(t) + \int_{t_0}^{t} 2g(s, x(s))\, ds,$$

and, finally,

$$x'''(t) = f'''(t) + 2g(t, x(t)),$$

a third-order differential equation.

Note that $x(t_0) = f(t_0)$, $x'(t_0) = f'(t_0)$, and $x''(t_0) = f''(t_0)$, so (1.2.3) actually represents an initial-value problem and, if g is locally Lipschitz in x, we would expect a unique solution.

For a general positive integer n, we see that (1.2.3) represents an initial-value problem of order $n + 1$. Before we discuss the reverse process, let us consider a simple example in some detail. We emphasize that the form of (1.2.3) is not the only one possible for the reduction.

Example 1.2.1 Consider the scalar equation

$$x(t) = 1 + \int_0^t [-4 + e^{-(t-s)}]x(s)\, ds. \tag{a}$$

Differentiation yields the integro-differential equation

$$x' = -3x - \int_0^t e^{-(t-s)}x(s)\, ds. \tag{b}$$

Now multiply by e^t and differentiate to obtain

$$x'' + 4x' + 4x = 0 \tag{c}$$

whose general solution is

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}. \tag{d}$$

Thus, (a) gives rise to (c) with two linearly independent solutions. In (b) we have $x'(0) = -3x(0)$, which, when combined with (d), yields

$$x'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t},$$

$$x'(0) = -2c_1 + c_2,$$

$$-3x(0) = -3c_1,$$

hence,

$$-2c_1 + c_2 = -3c_1$$

or

$$c_2 = -c_1.$$

Thus

$$x(t) = c_1 e^{-2t} - c_1 t e^{-2t} \tag{e}$$

is the solution of (b), and, as c_1 is arbitrary, (b) has one linearly independent solution. Finally, in (a) we have $x(0) = 1$, which, when applied to (e), yields

$$x(t) = e^{-2t} - te^{-2t} \tag{f}$$

as the unique solution of (a).

We consider now the inverse problem for linear equations. It is worthwhile to consider $n = 2$ separately.

Let $a(t)$, $b(t)$, and $f(t)$ be continuous on an interval $[0, T)$, and consider

$$x'' + a(t)x' + b(t)x = f(t), \qquad x(0) = x_0, \qquad x'(0) = x_1. \tag{1.2.4}$$

A Liouville transformation will transform (1.2.4) to

$$u'' = -c(t)u + h(t), \qquad u(0) = u_0, \quad u'(0) = u_1, \tag{1.2.5}$$

for $c(t)$ and $h(t)$ continuous. Integrate (1.2.5) from 0 to $t > 0$ twice obtaining successively

$$u'(t) = u_1 - \int_0^t c(s)u(s)\,ds + \int_0^t h(s)\,ds$$

and

$$u(t) = u_0 + u_1 t - \int_0^t \int_0^v c(s)u(s)\,ds\,dv + \int_0^t \int_0^v h(s)\,ds\,dv.$$

The integral

$$J = \int_0^t \int_0^v c(s)u(s)\,ds\,dv$$

is taken over the triangle in Fig. 1.1. We interchange the order of integration and obtain

$$J = \int_0^t \int_s^t c(s)u(s)\,dv\,ds = \int_0^t (t-s)c(s)u(s)\,ds,$$

so that if we set

$$H(t) = \int_0^t \int_0^v h(s)\,ds\,dv,$$

then (1.2.5) becomes

$$u(t) = u_0 + u_1 t + H(t) - \int_0^t (t-s)c(s)u(s)\,ds. \tag{1.2.6}$$

Incidentally, the same process allows us to pass from an integro-differential equation

$$\mathbf{x}'(t) = \mathbf{h}(t, \mathbf{x}(t)) + \int_0^t \mathbf{F}(t, s, \mathbf{x}(s))\,ds \tag{1.1.2}$$