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ABSTRACT

The purpose of this article is to fill some part of the gap existing between the mathematical theory of the Navier-Stokes Equations and the conventional theory of Turbulence and to provide a rigorous connection between these theories.

The number of degrees of freedom of a turbulent flow which was estimated on physical assumptions by Kolmogorov-Landau-Lifschitz is interpreted here as the fractal dimension of the corresponding attractor and the estimate is reobtained as a consequence of the (deterministic) Navier-Stokes equations.

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INTRODUCTION

Since the pioneering work of J. Leray [28] [29] on the equations of fluid mechanics, the difficult question of the regularity of the solutions of these equations remains open, namely we do not know yet if, the data being smooth, the solutions to the three dimensional Navier-Stokes Equations (N.S.E.) remain smooth for all time or not ; for the most recent results in this direction the reader is refered to V. Scheffer [38], L. Caffarelli, R. Kohn and L. Nirenberg [5]. Whether singularities do develop spontaneously or not, the question of the description of a turbulent flow remains open since the actual solution of the equations is expected to be highly oscillating and therefore to contain more information than needed. In order to overcome this difficulty it will be necessary in the future to develop appropriate mathematical tools and in a preliminary step to obtain, with the help of the new powerful computers, a better qualitative description of a turbulent flow.

A first result in this direction is the idea that a turbulent flow is finite dimensional, i.e. depends on a finite number of parameters (a finite number of degrees of freedom in the language of physics). This idea is familiar in the conventional theory of turbulence and follows from the Kolmogorov theory : cf. L. Landau and I.M. Lifschitz [27] where one can find an estimate of the number of degrees of freedom. On the mathematical side, this idea was investigated by E. Hopf [22] in the case of a simplified model equation. More recently, the authors of the present article have already, alone or in collaboration, derived in a rigorous way a set of results showing that under some circumstances a three dimensional flow depends indeed on a finite number of parameters : see C. Foias-G. Prodi [12] , C. Foias-R. Temam [14]-[16], C. Foias-O. Manley-R. Temam-Y. Trève [11][15] , R. Temam [41] .

One of the tasks of this article is to make more precise the conditions under which these results were proved, namely we show that all the above mentionned and related results are true under the only condition that <u>singularities do not develop in three dimensional flows</u> (see chapter 1 and below). Beside the development of mathematical tools which, in our opinion, could be helpful, another major task of this article is to give a rigorous proof of the result already mentionned of Kolmogorov-Landau-Lifschitz (see [27] p. 32-33) concerning the "number of degrees of freedom" of a turbulent flow, i.e. the number of parameters controling a turbulent flow. In [27] , it is shown using physical arguments pertaining to the conventional theory of turbulence, that the number N of degrees of freedom of a turbulent flow is of the order of

(0.1) $$N \sim \left(\frac{\ell_o}{\ell_d}\right)^3 ,$$

where ℓ_o is the large scale typical length and ℓ_d is the Kolmogorov dissipation length

(0.2) $$\ell_d = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} ,$$

given in terms of the dissipation ϵ of the energy per mass and time and of the kinematic viscosity ν. In Chapter 4, after a precise and appropriate definition of ℓ_o and ℓ_d, we give a rigorous proof of this result of Landau and Lifschitz (under again the assumption that no singularities develop in the flow). For this purpose the number of degrees of freedom is identified with the dimension of the attractor representing the flow.

We now describe how this article is organized. Chapter 1 gives the relation between the boundedness assumption used here and in the references [14][11][15][41] quoted above and the assumption that singularities do not develop spontaneously in 3-D flows ; they are shown to be equivalent (the boundedness assumption mentionned above is that the H^1 norm of the velocity of the considered solution of the NSE remain uniformly bounded for $0 < t < \infty$). Chapter 2 deals with the squeezing property for the trajectories. This is another form of the finite dimensionality of a flow which was first proved in C. Foias-R. Temam [14] : we provide here a much simpler proof and an improved form of this result which is optimal in some sense. Chapter 3 which is independent of the previous ones gives abstract results concerning the Hausdorff and the fractal dimensions of functional invariant sets (attractors in particular) ; these general results extend previous results of A. Douady-J. Oesterlé [8] and P. Constantin-C. Foias [6]. Finally Chapter 4, after some preliminary technical results, provides various estimates on the fractal (and thus Hausdorff) dimension of the attractor associated to a three dimensional turbulent flow : one of these estimates precisely corresponds to (0.1), the Kolmogorov-Landau-Lifschitz estimate. The other estimates are made in term of various Reynolds number that one can associate to the attractor : Reynolds numbers based on the time average of the supremum of the modulus of the velocity vector (Re), or on the absolute maximum in space and time of the modulus of the velocity vector ($\overline{Re} \geq Re$), or a Reynolds number based on the supremum of the enstrophy (directly related to the H^1-norm). Chapter 4 is concluded, for the sake of completness, with a brief reminder of some other aspects of finite dimensionality of flows which have been investigated elsewhere. The finite dimensionality of the attractors for the Navier-Stokes and related equations has been recently investigated by O.A. Ladyzhenskaya [26] , A.V. Babin-M.I. Vishik [2]-[4], who, however, do not investigate the physical significance of the bounds on the dimension which are obtained ; concerning the magnetohydrodynamic equations see [39] and for other related situations in fluid mechanics (thermo-hydraulic, N.S.E. with non homogeneous boundary conditions,...) see the general framework of J.M. Ghidaglia [19]. Most of the questions addressed here where already investigated in C. Foias-R. Temam [14] and in some way this article is intended as a continuation of [14].

During the realization of the present work we have benefited of stimulating discussions with O.P.Manley and part of the work is the result of a fruitful collaboration with him (see Sec. 4.3.a and [7]).

After this work was completed three related articles were bringed to our attention, D. Ruelle [37], E. Lieb [30] and very recently D. Ruelle [43]. In [37] D. Ruelle determines a bound of the Hausdorff dimension of the attractors associated to the three dimensional Navier-Stokes equations and expresses this bound in term of physical quantities ; his proof however relies on an assumption on the eigenvalues of the Schroedinger operators. In the article [30] , E. Lieb completes and simplifies the proof in [37] by utilization of a remarkable inequality of E. Lieb and W. Thirring [31] which improves the classical Sobolev inequality. The result of [37][30] related to the Hausdorff dimension of the attractors is better than ours as far as the Hausdorff dimension is concerned, however [37][30] do not cover our results since the fractal dimension is not considered in these articles. We have thus added afterwards the Sec. 4.5 incidating the improvements to our results which follow from a slight modification of our proof based on the use of the Lieb-Thirring's inequality instead of the usual Sobolev's inequality. Concerning the two dimensional Navier-Stokes equations, the best estimates presently available for the fractal and Hausdorff dimensions of the universal attractor are given (using Lieb-Thirring's inequality) in R. Temam [42] and, for the Hausdorff dimension , in D. Ruelle [43].

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TABLE OF CONTENTS

CHAPTER 1

ON THE APPEARANCE OF SINGULARITIES IN A THREE DIMENSIONAL FLOW

1.1. THE FUNCTIONAL SETTING OF THE NAVIER-STOKES EQUATIONS

The Navier-Stokes equations can be written as a nonlinear evolution equation in a Hilbert space H of the form

$$\frac{du}{dt} + \nu Au + B(u) = f \tag{1.1}$$

$$u(0) = u_0 , \tag{1.2}$$

where $\nu > 0$ is given and f is given say in $L^\infty(0,\infty;H)$. The operator A is a linear unbounded positive self-adjoint operator in H with domain $D(A)$; we denote by (u,v) and $|u|$ the scalar product and the norm in H , and clearly $D(A)$ is a Hilbert space for the scalar product and the norm (Au,Av) , $|Au|$. One can define the powers A^α of A , $\alpha \in \mathbb{R}$, with domain $D(A^\alpha)$. For $\alpha = 1/2$, we set $V = D(A^{1/2})$, which is a Hilbert space of dual $V' = D(A^{-1/2})$, and we endow V with the Hilbert scalar product and norm,

$$((u,v)) = (A^{1/2}u, A^{1/2}v) , \quad \|u\| = |A^{1/2}u| .$$

We recall also that A possesses an orthonormal family of eigenvectors w_j , $j \geq 1$, which is complete in H ,

$$Aw_j = \lambda_j w_j , \quad j \geq 1 , \quad 0 < \lambda_1 \leq \lambda_2 , \ldots, \tag{1.3}$$

$\lambda_j \longrightarrow +\infty$ as $j \longrightarrow \infty$.

For B we have $B(u) = B(u,u)$ where $B(.,.)$ is a bilinear continuous operator from $D(A) \times D(A)$ into H and from $V \times V$ into V' which enjoys several other continuity properties which will be recalled when needed.

The reader is referred for instance to R. Temam [41] for more details about the functional setting of the N.S.E., the definition and properties of the operators A and B and the concept of strong and weak solutions which will be recalled in Sec.2, Ch.1. Although the

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functional setting above applies to several situations in fluid mechanics, the main cases that we have in view are the flow in a bounded domain Ω of $\mathbb{R}^n$, $n = 3$ (or sometimes 2), with a homogeneous boundary condition ($u = 0$ on $\partial\Omega$), or the flow in $\mathbb{R}^n$ ($n = 3$ or 2) with space periodicity condition (the flow is periodic with period $L > 0$ in each direction $x_1,\dots,x_n$). In the first case we have (see [40] [41]) :

$$H = \{v \in L^2(\Omega)^n \,,\ \operatorname{div} v = 0 \,,\ v.\underset{\sim}{\nu} = 0 \text{ on } \partial\Omega\} \,,$$

$\underset{\sim}{\nu}$ the unit outward normal on $\partial\Omega$ = the boundary of Ω,

$$V = \{v \in H_0^1(\Omega)^n \,,\ \operatorname{div} v = 0\}$$

$$D(A) = H_0^1(\Omega)^n \cap H^2(\Omega)^n$$

$$Au = -P\Delta u \,,\ B(u,v) = P((u.\nabla)v) \,,$$

where P is the orthonogonal projector in $L^2(\Omega)^n$ onto H. We use the standard notation for the space $L^2(\Omega)$ and the Sobolev spaces $H_0^1(\Omega)$, $H^1(\Omega)$, $H^2(\Omega)$, ... In the case of the flow with space periodicity, we denote by Q the cube $(0,L)^n$ and by Γ_i and Γ_{i+n} its faces $x_i = 0$ and $x_i = L$; then (see [41] where this situation is emphasized) :

$$H = \{v \in L^2(Q)^n,\ \operatorname{div} v = 0,\ \int_Q v\,dx = 0,\ v_i|_{x_i=L} = v_i|_{x_i=0},\ i = 1,\dots,n\}$$

$$V = \{v \in H^1(Q)^n,\ \operatorname{div} v = 0,\ \int_Q v\,dx = 0,\ v|_{x_i=L} = v|_{x_i=0},\ i = 1,\dots,n\}$$

$$D(A) = \{v \in H^2(Q)^n,\ \operatorname{div} v = 0,\ \int_Q v\,dx = 0,\ v|_{x_i=0} = v|_{x_i=L},\ i = 1,\dots,n\}$$

$$Au = -P\Delta u = -\Delta u,\ B(u,v) = P((u.\nabla)v) \,,$$

P being the orthogonal projector in $L^2(Q)^n$ onto the space H.

In either case, we have

$$(u,v) = \int_{\mathcal{O}} u(x).v(x)\,dx \,,$$

$$((u,v)) = \sum_{i,j=1}^{n} \int_{\mathcal{O}} \frac{\partial u_i}{\partial x_j}(x)\,\frac{\partial v_i}{\partial x_j}(x)\,dx \,,$$

$\mathcal{O} = \Omega$ or Q. For $n = 3$, it is easy to check that for every $v \in V$,

$$\|v\|^2 = ((v,v)) = \int_{\mathcal{O}} |\operatorname{curl} v(x)|^2\,dx \,, \tag{1.4}$$

and $\frac{1}{2}\|v\|^2$ is called the enstrophy of the vector field v ; the same is true for $n = 2$ with $\operatorname{curl} v$ replaced by the scalar $\operatorname{curl} v = \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}$.

1.2. THE INITIAL VALUE PROBLEM

Given u_o in V and f as above

(1.5) $$u_o \in V \;,\; f \in L^\infty(0,\infty;H) \;,$$

a strong solution of the initial value problem (1.1)(1.2) defined on some interval $[0,T]$, $T > 0$, is a function u ,

(1.6) $$u \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$$

which satisfies (1.1) on $(0,T)$ and (1.2). A weak solution of these equations (the N.S.E.) on $(0,T)$ is a function u

(1.7) $$u \in L^\infty(0,T;H) \cap L^2(0,T;V) \;,$$

which satisfies (1.1) on $(0,T)$ and (1.2). We recall that given u_o and f satisfying (1.5), if the dimension of space is $n = 2$ ([1]), a strong (and therefore a weak) solution exists and is unique for all $T > 0$. If the dimension $n = 3$, then a strong solution is known to exist (and is unique) only on some interval $[0,T_1]$, where T_1 is of the form

(1.8) $$T_1 = T_1(\|u_o\|) = \frac{\kappa_1}{(1+\|u_o\|^2)^2} \;,$$

κ_1 depending only on $|f|_{K^\infty(0,\infty;H)}$, ν and Ω . A week solution exists for every $T > 0$, coincides with the strong solution on $[0,T_1]$, at least, but we do not know if this weak solution is unique ; for all these classical results see for instance [25][32][40][41] .

If we are interested in solutions defined for every $t > 0$, then if $n = 2$, according to a result of C. Foias and G. Prodi [12] the strong solution is uniformly bounded in the H^1-norm (see (1.3)), for $t > 0$, $u \in L^\infty(0,\infty;V)$ and

(1.9) $$\|u\|_{L^\infty(0,\infty;V)} \leq \kappa_2 < \infty$$

where κ_2 depends only on $|f|_{L^\infty(0,\infty;H)}$, ν and Ω (or Q) .

No such result is of course available if $n = 3$, since we are not even certain, in this case, that the strong solution exists on the whole interval $\mathbb{R}_+ = [0,\infty)$. All the results derived in [11][14][15][41] concerning three dimensional flows were made under the assumption that the

---

([1]) The difference between the dimensions n and $n = 3$ lies on the continuity properties of B which will be recalled below.

flow under consideration satisfies (1.9). Our aim in this Section is to investigate the significance of this assumption.

1.3. THE MAIN RESULT (of Chapter 1).

By lack of information on the three dimensional case, we must admit that it is conceivable that a strong solution exists for all time but does not satisfy (1.9) (i.e. no result analogous to that of C. Foias - G. Prodi [13]). Hence the assumption made in the references quoted above (and hereafter), that the solutions satisfy (1.9) seems stronger than the assumption that singularities do not develop in the flow, i.e. that

(1.10) $$\|u\|_{L^\infty(0,T;V)} \leq C(T) < \infty \ , \ \forall\, T > 0 \ ,$$

the quantiy $C(T)$ being perhaps allowed to be unbounded as $T \to \infty$.
We recall that it was the conjecture of J. Leray [30][31] and his motivation for the introduction of the concept of weak (or turbulent) solutions, that singularities do develop in a finite time, i.e. that

$$\|u(.,t)\|^2 = \int_\Omega |\mathrm{curl}\ u(x,t)|^2\, dx$$

becomes infinite at a finite time ; this assumption has not yet been proved nor disproved. Our aim here is to show, under a mild assumption on f , that the assumption (1.9) is not actually stronger than (1.10), i.e. the assumption that singularities do not develop in flows in a finite time.

The assumption that we make on f is that f is nonchaotic at infinity ; by that we mean the following :

(1.11) $f \in L^2_{loc}(0,\infty;H)$ is nonchaotic at infinity,

if there exists $\alpha > 0$ such that, for every sequence t_j converging to $+\infty$, the sequence of functions

$$f_j = \tau_{t_j} f|_{[t_j,t_j+\alpha]} \qquad (\tau_a f(s) = f(s-a)) \ ,$$

is relatively compact in $L^2(0,\alpha;H)$.

REMARK 1.1.

The assumption (1.11) is satisfied in the following cases

i) f is independent of t , $f(t) \equiv f \in H$, $\forall\, t > 0$;

ii) $f \in L^2_{loc}(0,\infty;H)$ is periodic with period T $(\alpha = T)$;

iii) $f \in L^2_{loc}(0,\infty;H \cap H^1(\Omega)^n)$, $f' = \frac{df}{dt} \in L^2_{loc}(0,\infty;V')$, and for some

$T > 0$ and for every $a > 0$,

$$|f|_{L^2(a,a+T;H^1(\Omega)^n)} + |f'|_{L^2(a,a+T;V')} \leq C(T) ,$$

where $C(T)$ may depend on T but is independent of a. In this case we can take $\alpha = T$ and (1.11) follows by compactness [1].

□

We have the

THEOREM 1.1.

If there exists $u_0 \in V$, $f \in L^\infty(0,\infty;H)$ *nonchaotic at infinity such that a solution* u *of (1.1) (1.2) does not satisfy (1.10), then we can find* $v_0 \in V$ *and* $g \in L(0,\infty;H)$ *such that the solution* v *of (1.1) (1.2) with* u_0, f, *replaced by* v_0, g, *becomes singular at a finite time* t_*.

Proof.

We assume that u satisfies (1.6) since otherwise the result is obvious. If (1.10) is not verified, there exists a sequence $s_j \to +\infty$, such that

$$\|u(s_j)\| \to +\infty \text{ as } j \to \infty . \tag{1.12}$$

. i) We first derive an a priori estimate on u. For that purpose we recall the classical energy equality

$$\frac{d}{dt}|u|^2 + 2\nu \|u\|^2 = 2(f,u) , \tag{1.13}$$

which is obtained by taking the scalar product in H of (1.1) with $2u$ and using the orthogonality property (see [41])

$$(B(\phi,\psi),\psi) = 0 , \ \forall \phi,\psi \in V . \tag{1.14}$$

From (1.13), and since

$$|\phi| \leq \lambda_1^{-1/2} \|\phi\| , \ \forall \phi \in V , \tag{1.15}$$

we obtain

$$\frac{d}{dt}|u|^2 + 2\nu \|u\|^2 \leq 2|f|_\infty \frac{\|u\|}{\sqrt{\lambda_1}} \leq \nu \|u\|^2 + \frac{|f|_\infty}{\nu\lambda_1}$$

(1) The space $\{g \in L^2(0,\alpha;H \cap H^1(\Omega)^n) , g' \in L^2(0,\alpha;V')\}$ is compactly imbedded in $L^2(0,\alpha;H)$; see for instance [40], Ch.3, sec. 2.

(1.16) $$\frac{d}{dt}|u|^2 + \nu\|u\|^2 \le \frac{|f|_\infty^2}{\nu\lambda_1},$$

where $|f|_\infty$ is the norm of f in $L^\infty(0,\infty;H)$. Hence with (1.15) and Gronwall Lemma

$$\frac{d}{dt}|u|^2 + \nu\lambda_1|u|^2 \le \frac{|f|_\infty^2}{\nu\lambda_1}$$

(1.17) $$|u(t)|^2 \le |u_o|^2\exp(-\nu\lambda_1 t) + \frac{(1-\exp(-\nu\lambda_1 t))}{\nu^2\lambda_1^2}|f|_\infty^2, \quad t>0,$$

(1.18) $$|u(t)|^2 \le |u_o|^2 + \frac{1}{\nu^2\lambda_1^2}|f|_\infty^2, \quad t>0.$$

For any $a>0$ and any $t>0$ we then conclude from (1.16) that

$$\nu\int_t^{t+a}\|u(s)\|^2\,ds \le |u(t)|^2 + \frac{a}{\nu\lambda_1}|f|_\infty^2$$

(1.19) $$\int_t^{t+a}\|u(s)\|^2\,ds \le \kappa_3, \quad t>0, \ a>0$$

where κ_3 depending on u_o, f, ν, λ_1, a, is equal to

(1.20) $$\kappa_3 = \frac{1}{\nu}|u_o|^2 + \left(\frac{1}{\nu^3\lambda_1^2} + \frac{a}{\nu^2\lambda_1}\right)|f|_\infty^2.$$

. ii) Let now $r = \sqrt{\frac{2\kappa_3}{a}}$; it follows from (1.19) that on any interval $[t,t+a]$ of length a, the set

$$M = \{s \in [t,t+a], \ \|u(s)\| > r\},$$

has its measure bounded by

$$\text{meas}(M) \le \frac{\kappa_3}{r^2} \le \frac{a}{2},$$

and therefore there exists points s in $[t,t+a]$ such that $\|u(s)\| \le r$ (1).

We set $a = \frac{\alpha}{4}$, α as in (1.11) and we conclude that for every j, there exists $t_j \in (s_j - \frac{\alpha}{4}, s_j)$, such that $\|u(t_j)\|^2 \le \frac{8\kappa_3}{\alpha}$. By translation, setting

$$u_j = \tau_{t_j} u|_{[t_j,t_j+\alpha]}, \quad f_j = \tau_{t_j} f|_{[t_j,t_j+\alpha]},$$

---

(1) This is an aspect of intermittency in turbulence. If $\|u(t)\|$ becomes very large at some time then $\|u(s)\|$ must become again smaller than some a priori bound at some other time s "close" to t.

we obtain sequences u_j, f_j, such that

$$u_j \in L^\infty(0,\alpha;V)$$

$f_j \in L^\infty(0,\alpha;W)$ and is relatively compact in $L^2(0,\alpha;H)$ (by (1.11))

$$\frac{du_j}{dt} + \nu A u_j + B(u_j) = f_j \quad \text{on} \quad (0,\alpha) , \tag{1.21}$$

$$\|u_j(0)\| \le \left(\frac{8\kappa_1}{\alpha}\right)^{1/2} \tag{1.22}$$

$$\|u_j(a_j)\| \longrightarrow +\infty \text{ as } j \longrightarrow \infty , \quad a_j = s_j - t_j \in (0,\alpha) . \tag{1.23}$$

By extracting a subsequence, we can assume that $a_j \longrightarrow a \in (0,\alpha)$ as $j \longrightarrow \infty$ and that $f_j \longrightarrow g$ in $L^2(0,\alpha;H)$ strongly and $L^\infty(0,\alpha;H)$ weak-star, as $j \longrightarrow \infty$. We can also assume that $u_j(0)$ converges weakly in V and strongly in H to some v_o such that

$$\|v_o\| \le \left(\frac{8\kappa_1}{\alpha}\right)^{1/2} . \tag{1.24}$$

It is then classical to pass to the limit $j \longrightarrow \infty$ in (1.21) (see for instance [40] for many similar situations) : we show that u_j is bounded in $L^\infty(0,\alpha;H)$ and $L^2(0,\alpha;V)$ and we extract a subsequence converging to some limit v , weakly in $L^2(0,\alpha;V)$ and weak-star in $L^\infty(0,\alpha;H)$. The passage to the limit in (1.21) gives then that

$$\frac{dv}{dt} + \nu A v + B(v) = g \quad \text{on} \quad (0,\alpha) . \tag{1.25}$$

By the result recalled in Sec.2, Ch.1, $v \in L^\infty(0,T_1;V)$, $T_1 = T_1(\|v_o\|)$. In fact we will see that $v \notin L^\infty(0,\alpha;V)$ because of (1.23)) (1).

. iii) We now assume that $v \in L^\infty(0,\alpha;V)$ and we will show that this leads to a contradiction. Let $w_j = u_j - v$; by substracting (1.25) from (1.21) we obtain easily

$$\frac{dw_j}{dt} + \nu A w_j + B(w_j) + B(v,w_j) + B(w_j,v) = f_j - g . \tag{1.26}$$

Taking the scalar product in H with $2w_j$ and using (1.14) we obtain as

---

(1) Since v remains smooth until (at least) the time $T_1(\|v_o\|)$, $v \notin L^\infty(0,\alpha;V)$ implies that the first interval of regularity for v is of the form $[0,t_*[$, with $T_1(\|v_o\|) < t_* \le \alpha$, and $\|v(t)\| \longrightarrow +\infty$ as $t \longrightarrow t_* - 0$.

in (1.13)

$$\frac{d}{dt}|w_j|^2 + 2\nu\|w_j\|^2 = -2(B(w_j,v),w_j) + 2(f_j-g,w_j) .$$

We have the following inequalities for B (see [41] Sec. 2) :

(1.27a) $|(B(\phi,\psi),\theta)| \leq c_1 \|\phi\| \, \|\psi\| \, |\theta|^{1/2} \|\theta\|^{1/2}$, $\forall\ \phi,\psi,\ \theta \in V$

(1.27b) $|(B(\phi,\psi),\theta)| \leq c_1 \|\phi\|^{1/2}|A\phi|^{1/2}\|\psi\| \, \|\theta\|$, $\forall \phi \in D(A),\ \psi,\ \theta \in V$

(1.27c) $|(B(\phi,\psi),\theta)| \leq c_1\|\phi\| \, \|\psi\|^{1/2} |A\psi|^{1/2}|\theta|$, $\forall\ \psi \in D(A),\ \phi \in V, \theta \in H.$

Thus with (1.19)

$$\frac{d}{dt}|w_j|^2 + 2\nu\|w_j\|^2 \leq 2c_1\|v\| \, \|w_j\|^{3/2}|w_j|^{1/2} + \frac{2}{\sqrt{\lambda_1}}|f_j-g| \, \|w_j\|$$

$\leq$ (with Schwarz and Young inequalities)

$$\leq \nu\|w_j\|^2 + c_1'\|v\|^4 |w_j|^2 + \frac{2}{\nu\lambda_1} |f_j-g|^2 ,$$

where c_i, c_i', c_i'', ..., denote positive constants.
We obtain

(1.28) $$\frac{d}{dt}|w_j|^2 + \nu\|w_j\|^2 \leq c_1'\|v\|^4|w_j|^2 + \frac{2}{\nu\lambda_1} |f_j-g|^2 .$$

We remove in a first step the term $\nu\|w_j\|^2$, and we apply Gronwall's lemma to obtain

$$|w_j(t)|^2 \leq |w_j(0)|^2 \exp(\int_0^t c_1'\|v(s)\|^4 ds) +$$
$$+ \frac{2}{\nu\lambda_1} (\int_0^t |f_j-g|^2 ds) \exp(\int_0^\alpha c_1' \|v(s)\|^4 ds) .$$

Since $v \in L^\infty(0,\alpha;V)$ by assumption and $w_j(0) \to 0$ in H strong as $j \to \infty$, we conclude that $w_j \to 0$ in $L^\infty(0,\alpha;H)$ strong as $j \to \infty$. Returning to (1.28) we then find also that $w_j = u_j - v \to 0$ in $L^2(0,\alpha;V)$ strong as $j \to \infty$.

Since $u_j - v \to 0$ strongly in $L^2(0,\alpha;V)$ we conclude by extracting a subsequence that, for almost every $t \in (0,\alpha)$

(1.29) $$u_j(t) \to v(t) \text{ in } V .$$

Let us consider a particular $t = t_1$, for which (1.29) is valid. The sequence $\|u_j(t_1)\|$ is bounded and for j sufficiently large

$$\|u_j(t_1)\| \leq r_1 = \|v\|_{L^\infty(0,\alpha;V)} + 1 .$$

Because of (1.8) (1), for j sufficiently large,

$$\|u_j(s)\| \leq 2 + 2r_1 , \text{ for } s \in [t_1, t_1+T_1(r_1)] . \tag{1.30}$$

Since $T(r_1)$ is actually independent of t_1, we can cover the interval $(0,\alpha)$ by a finite number of intervals $[t_k, t_k+T_1(r_1)]$, $k = 1,\dots,N$, such that $u_j(t_k) \longrightarrow v(t_k)$ for every k, and (1.30) holds for $s \in [t_k, t_k+T_1(r_1)]$. It follows that the norm of u_j in $L^\infty(0,\alpha;V)$ remains uniformly bounded as $j \longrightarrow \infty$. This contradicts (1.23) and the proof is complete.

REMARK 1.2.

The proof above shows that if (1.10) is satisfied but not (1.9), then we can find v_0 and g for which the corresponding solution of (1.1) (1.2) (u_0, f replaced by v_0, g), blows up in the V norm at a time t_* arbitrarily small, $t_* < a$, $\forall a$, $0 < a < \alpha$.

REMARK 1.3.

With a slight modification of the proof of Theorem 1.1, we can show the following : given Ω, $\nu > 0$, $T > 0$, $R > 0$ and $f \in L^\infty(0,T;V)$, if for every $u_0 \in V$ with $\|u_0\| \leq R$, all the solutions to the (3 dimensional) Navier-Stokes equations belong to $L^\infty(0,T;V)$ (i.e. are strong solutions) then there exists a number κ_4 depending on Ω, ν, T, R, f, such that (2)

$$\|u\|_{L^\infty(0,T;V)} \leq \kappa_4 , \tag{1.31}$$

for every solution u of (1.1) (1.2) with $\|u_0\| \leq R$.

By contradiction, if (1.31) were not true, then we could find a sequence $\{u_{0j}, u_j\}$, u_j solution of (1.1) on $(0,T)$ with $u_j(0) = u_{0j}$, such that $\|u_{0j}\| \leq R$ and

$$\|u_j\|_{L^\infty(0,T;V)} \longrightarrow +\infty , \text{ as } j \longrightarrow \infty .$$

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(1) We use a more precise form of (1.8) (see [41]) :

$$\|u(t)\| \leq 2(1+\|u_0\|) , \text{ for } t \in [0, T_1(\|u_0\|)] .$$

(2) The functional dependence of κ_4 on the data is not given by the following proof which provides only the existence of $\kappa_4 < \infty$.