Differential Geometry of Curves and Surfaces

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Preface

This book is an introduction to the differential geometry of curves and surfaces, both in its local and global aspects. The presentation differs from the traditional ones by a more extensive use of elementary linear algebra and by a certain emphasis placed on basic geometrical facts, rather than on machinery or random details.

We have tried to build each chapter of the book around some simple and fundamental idea. Thus, Chapter 2 develops around the concept of a regular surface in R^3; when this concept is properly developed, it is probably the best model for differentiable manifolds. Chapter 3 is built on the Gauss normal map and contains a large amount of the local geometry of surfaces in R^3. Chapter 4 unifies the intrinsic geometry of surfaces around the concept of covariant derivative; again, our purpose was to prepare the reader for the basic notion of connection in Riemannian geometry. Finally, in Chapter 5, we use the first and second variations of arc length to derive some global properties of surfaces. Near the end of Chapter 5 (Sec. 5-10), we show how questions on surface theory, and the experience of Chapters 2 and 4, lead naturally to the consideration of differentiable manifolds and Riemannian metrics.

To maintain the proper balance between ideas and facts, we have presented a large number of examples that are computed in detail. Furthermore, a reasonable supply of exercises is provided. Some factual material of classical differential geometry found its place in these exercises. Hints or answers are given for the exercises that are starred.

The prerequisites for reading this book are linear algebra and calculus. From linear algebra, only the most basic concepts are needed, and a standard undergraduate course on the subject should suffice. From calculus, a certain familiarity with calculus of several variables (including the statement of the implicit function theorem) is expected. For the reader's convenience, we have tried to restrict our references to R. C. Buck, *Advancd Calculus*, New York: McGraw-Hill, 1965 (quoted as Buck, *Advanced Calculus*). A certain knowledge of differential equations will be useful but it is not required.

This book is a free translation, with additional material, of a book and a set of notes, both published originally in Portuguese. Were it not for the enthusiasm and enormous help of Blaine Lawson, this book would not have come into English. A large part of the translation was done by Leny Cavalcante. I am also indebted to my colleagues and students at IMPA for their comments and support. In particular, Elon Lima read part of the Portuguese version and made valuable comments.

Robert Gardner, Jürgen Kern, Blaine Lawson, and Nolan Wallach read critically the English manuscript and helped me to avoid several mistakes, both in English and Mathematics. Roy Ogawa prepared the computer programs for some beautiful drawings that appear in the book (Figs. 1-3, 1-8, 1-9, 1-10, 1-11, 3-45 and 4-4). Jerry Kazdan devoted his time generously and literally offered hundreds of suggestions for the improvement of the manuscript. This final form of the book has benefited greatly from his advice. To all these people—and to Arthur Wester, Editor of Mathematics at Prentice-Hall, and Wilson Góes at IMPA—I extend my sincere thanks.

Rio de Janeiro Manfredo P. do Carmo

Some Remarks on Using This Book

We tried to prepare this book so it could be used in more than one type of differential geometry course. Each chapter starts with an introduction that describes the material in the chapter and explains how this material will be used later in the book. For the reader's convenience, we have used footnotes to point out the sections (or parts thereof) that can be omitted on a first reading.

Although there is enough material in the book for a full-year course (or a topics course), we tried to make the book suitable for a first course on differential geometry for students with some background in linear algebra and advanced calculus.

For a short one-quarter course (10 weeks), we suggest the use of the following material: Chapter 1: Secs. 1-2, 1-3, 1-4, 1-5 and one topic of Sec. 1-7—2 weeks. Chapter 2: Secs. 2-2 and 2-3 (omit the proofs), Secs. 2-4 and 2-5—3 weeks. Chapter 3: Secs. 3-2 and 3-3—2 weeks. Chapter 4: Secs. 4-2 (omit conformal maps and Exercises 4, 13-18, 20), 4-3 (up to Gauss theorema egregium), 4-4 (up to Prop. 4; omit Exercises 12, 13, 16, 18-21), 4-5 (up to the local Gauss-Bonnet theorem; include applications (b) and (f))—3 weeks.

The 10-week program above is on a pretty tight schedule. A more relaxed alternative is to allow more time for the first three chapters and to present survey lectures, on the last week of the course, on geodesics, the Gauss theorema egregium, and the Gauss-Bonnet theorem (geodesics can then be defined as curves whose osculating planes contain the normals to the surface).

In a one-semester course the first alternative could be taught more leisurely and the instructor could probably include additional material (for instance, Secs. 5-2 and 5-10 (partially), or Secs. 4-6, 5-3 and 5-4).

Please also note that an asterisk attached to an exercise does not mean the exercise is either easy or hard. It only means that a solution or hint is provided at the end of the book. Second, we have used for parametrization a bold-faced **x** and that might become clumsy when writing on the blackboard. Thus we have reserved the capital X as a suggested replacement.

Where letter symbols that would normally be italic appear in italic context, the letter symbols are set in roman. This has been done to distinguish these symbols from the surrounding text.

Contents

1 Curves

1-1. Introduction

The differential geometry of curves and surfaces has two aspects. One, which may be called classical differential geometry, started with the beginnings of calculus. Roughly speaking, classical differential geometry is the study of local properties of curves and surfaces. By local properties we mean those properties which depend only on the behavior of the curve or surface in the neighborhood of a point. The methods which have shown themselves to be adequate in the study of such properties are the methods of differential calculus. Because of this, the curves and surfaces considered in differential geometry will be defined by functions which can be differentiated a certain number of times.

The other aspect is the so-called global differential geometry. Here one studies the influence of the local properties on the behavior of the entire curve or surface. We shall come back to this aspect of differential geometry later in the book.

Perhaps the most interesting and representative part of classical differential geometry is the study of surfaces. However, some local properties of curves appear naturally while studying surfaces. We shall therefore use this first chapter for a brief treatment of curves.

The chapter has been organized in such a way that a reader interested mostly in surfaces can read only Secs. 1-2 through 1-5. Sections 1-2 through 1-4 contain essentially introductory material (parametrized curves, arc length, vector product), which will probably be known from other courses and is included here for completeness. Section 1-5 is the heart of the chapter and contains the material of curves needed for the study of surfaces. For those wishing to go a bit further on the subject of curves, we have included Secs. 1-6 and 1-7.

1-2. Parametrized Curves

We denote by R^3 the set of triples (x, y, z) of real numbers. Our goal is to characterize certain subsets of R^3 (to be called curves) that are, in a certain sense, one-dimensional and to which the methods of differential calculus can be applied. A natural way of defining such subsets is through differentiable functions. We say that a real function of a real variable is *differentiable* (or *smooth*) if it has, at all points, derivatives of all orders (which are automatically continuous). A first definition of curve, not entirely satisfactory but sufficient for the purposes of this chapter, is the following.

DEFINITION. *A* parametrized differentiable curve *is a differentiable map* α*:* $\mathrm{I} \longrightarrow \mathrm{R}^3$ *of an open interval* $\mathrm{I} = (\mathrm{a}, \mathrm{b})$ *of the real line* R *into* R^3.†

The word *differentiable* in this definition means that α is a correspondence which maps each $t \in I$ into a point $\alpha(t) = (x(t), y(t), z(t)) \in R^3$ in such a way that the functions $x(t), y(t), z(t)$ are differentiable. The variable t is called the *parameter* of the curve. The word *interval* is taken in a generalized sense, so that we do not exclude the cases $a = -\infty$, $b = +\infty$.

If we denote by $x'(t)$ the first derivative of x at the point t and use similar notations for the functions y and z, the vector $(x'(t), y'(t), z'(t)) = \alpha'(t) \in R^3$ is called the *tangent vector* (or *velocity vector*) of the curve α at t. The image set $\alpha(I) \subset R^3$ is called the *trace* of α. As illustrated by Example 5 below, one should carefully distinguish a parametrized curve, which is a map, from its trace, which is a subset of R^3.

A warning about terminology. Many people use the term "infinitely differentiable" for functions which have derivatives of all orders and reserve the word "differentiable" to mean that only the existence of the first derivative is required. We shall not follow this usage.

Example 1. The parametrized differentiable curve given by

$$\alpha(t) = (a \cos t, a \sin t, bt), \qquad t \in R,$$

has as its trace in R^3 a helix of pitch $2\pi b$ on the cylinder $x^2 + y^2 = a^2$. The parameter t here measures the angle which the x axis makes with the line joining the origin 0 to the projection of the point $\alpha(t)$ over the xy plane (see Fig. 1-1).

†In italic context, letter symbols will not be italicized so they will be clearly distinguished from the surrounding text.

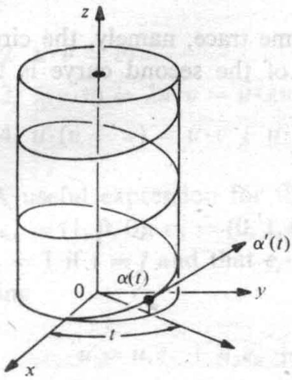

Figure 1-1

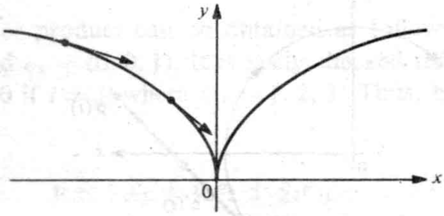

Figure 1-2

Example 2. The map $\alpha: R \longrightarrow R^2$ given by $\alpha(t) = (t^3, t^2)$, $t \in R$, is a parametrized differentiable curve which has Fig. 1-2 as its trace. Notice that $\alpha'(0) = (0, 0)$; that is, the velocity vector is zero for $t = 0$.

Example 3. The map $\alpha: R \longrightarrow R^2$ given by $\alpha(t) = (t^3 - 4t, t^2 - 4)$, $t \in R$, is a parametrized differentiable curve (see Fig. 1-3). Notice that $\alpha(2) = \alpha(-2) = (0, 0)$; that is, the map α is not one-to-one.

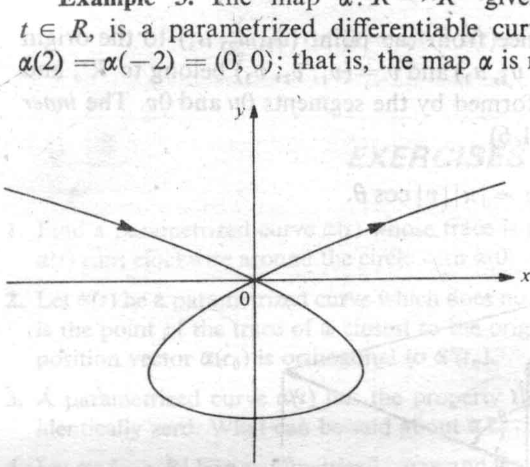

Figure 1-3

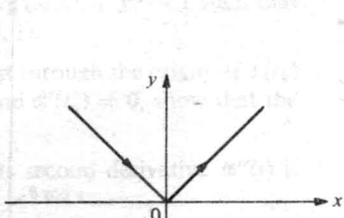

Figure 1-4

Example 4. The map $\alpha: R \longrightarrow R^2$ given by $\alpha(t) = (t, |t|)$, $t \in R$, is *not* a parametrized differentiable curve, since $|t|$ is not differentiable at $t = 0$ (Fig. 1-4).

Example 5. The two distinct parametrized curves

$$\alpha(t) = (\cos t, \sin t),$$
$$\beta(t) = (\cos 2t, \sin 2t),$$

where $t \in (0 - \epsilon, 2\pi + \epsilon)$, $\epsilon > 0$, have the same trace, namely, the circle $x^2 + y^2 = 1$. Notice that the velocity vector of the second curve is the double of the first one (Fig. 1-5).

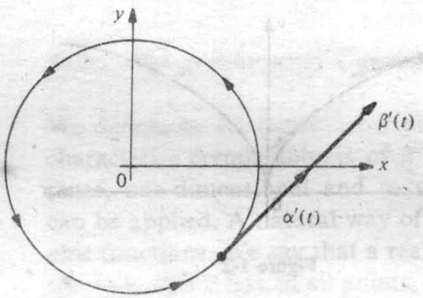

Figure 1-5

We shall now recall briefly some properties of the inner (or dot) product of vectors in R^3. Let $u = (u_1, u_2, u_3) \in R^3$ and define its *norm* (or *length*) by

$$|u| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

Geometrically, $|u|$ is the distance from the point (u_1, u_2, u_3) to the origin $0 = (0, 0, 0)$. Now, let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ belong to R^3, and let θ, $0 \leq \theta \leq \pi$, be the angle formed by the segments $0u$ and $0v$. The *inner product* $u \cdot v$ is defined by (Fig. 1-6)

$$u \cdot v = |u||v| \cos \theta.$$

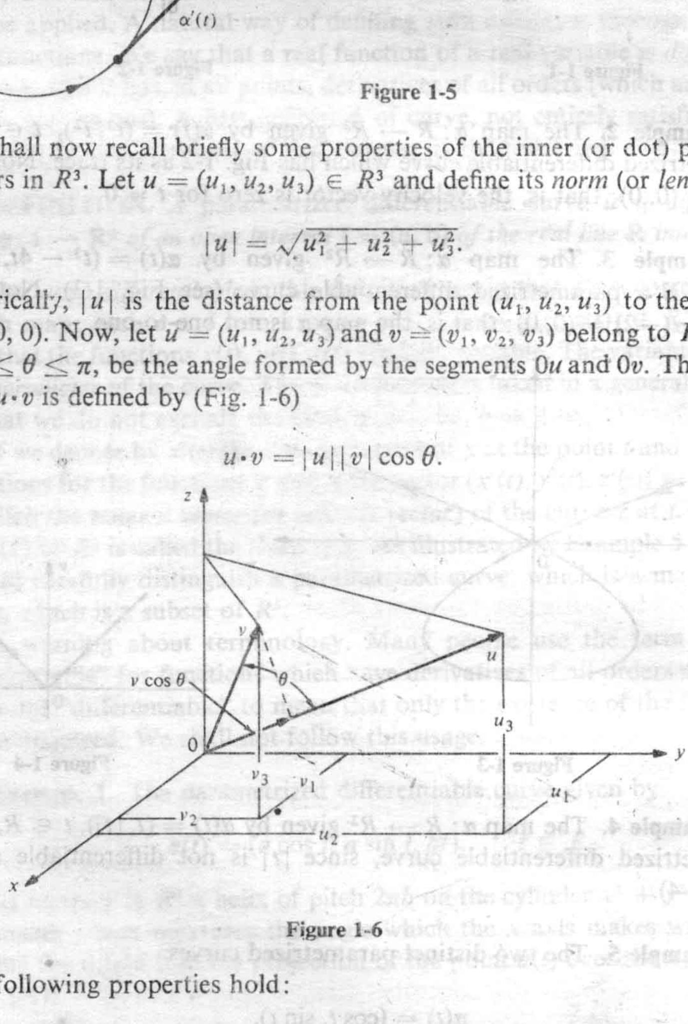

Figure 1-6

The following properties hold:

1. Assume that u and v are nonzero vectors. Then $u \cdot v = 0$ if and only if u is orthogonal to v.

2. $u \cdot v = v \cdot u$.
3. $\lambda(u \cdot v) = \lambda u \cdot v = u \cdot \lambda v$.
4. $u \cdot (v + w) = u \cdot v + u \cdot w$.

A useful expression for the inner product can be obtained as follows. Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. It is easily checked that $e_i \cdot e_j = 1$ if $i = j$ and that $e_i \cdot e_j = 0$ if $i \neq j$, where $i, j = 1, 2, 3$. Thus, by writing

$$u = u_1 e_1 + u_2 e_2 + u_3 e_3, \qquad v = v_1 e_1 + v_2 e_2 + v_3 e_3,$$

and using properties 3 and 4, we obtain

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

From the above expression it follows that if $u(t)$ and $v(t)$, $t \in I$, are differentiable curves, then $u(t) \cdot v(t)$ is a differentiable function, and

$$\frac{d}{dt}(u(t) \cdot v(t)) = u'(t) \cdot v(t) + u(t) \cdot v'(t).$$

EXERCISES

1. Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.
2. Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.
3. A parametrized curve $\alpha(t)$ has the property that its second derivative $\alpha''(t)$ is identically zero. What can be said about α?
4. Let $\alpha: I \longrightarrow R^3$ be a parametrized curve and let $v \in R^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.
5. Let $\alpha: I \longrightarrow R^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

1-3. Regular Curves; Arc Length

Let $\alpha: I \longrightarrow R^3$ be a parametrized differentiable curve. For each $t \in I$ where $\alpha'(t) \neq 0$, there is a well-defined straight line, which contains the point $\alpha(t)$ and the vector $\alpha'(t)$. This line is called the *tangent line* to α at t. For the study of the differential geometry of a curve it is essential that there exists such a tangent line at every point. Therefore, we call any point t where $\alpha'(t) = 0$ a *singular point* of α and restrict our attention to curves without singular points. Notice that the point $t = 0$ in Example 2 of Sec. 1-2 is a singular point.

DEFINITION. *A parametrized differentiable curve* $\alpha: I \longrightarrow R^3$ *is said to be* regular *if* $\alpha'(t) \neq 0$ *for all* $t \in I$.

From now on we shall consider only regular parametrized differentiable curves (and, for convenience, shall usually omit the word differentiable).

Given $t \in I$, the *arc length* of a regular parametrized curve $\alpha: I \longrightarrow R^3$, from the point t_0, is by definition

$$s(t) = \int_{t_0}^{t} |\alpha'(t)|\, dt,$$

where

$$|\alpha'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

is the length of the vector $\alpha'(t)$. Since $\alpha'(t) \neq 0$, the arc length s is a differentiable function of t and $ds/dt = |\alpha'(t)|$.

In Exercise 8 we shall present a geometric justification for the above definition of arc length.

It can happen that the parameter t is already the arc length measured from some point. In this case, $ds/dt = 1 = |\alpha'(t)|$; that is, the velocity vector has constant length equal to 1. Conversely, if $|\alpha'(t)| \equiv 1$, then

$$s = \int_{t_0}^{t} dt = t - t_0;$$

i.e., t is the arc length of α measured from some point.

To simplify our exposition, we shall restrict ourselves to curves parametrized by arc length; we shall see later (see Sec. 1-5) that this restriction is not essential. In general, it is not necessary to mention the origin of the arc length s, since most concepts are defined only in terms of the derivatives of $\alpha(s)$.

It is convenient to set still another convention. Given the curve α parametrized by arc length $s \in (a, b)$, we may consider the curve β defined in $(-b, -a)$ by $\beta(-s) = \alpha(s)$, which has the same trace as the first one but is described in the opposite direction. We say, then, that these two curves differ by a *change of orientation*.

EXERCISES

1. Show that the tangent lines to the regular parametrized curve $\alpha(t) = (3t, 3t^2, 2t^3)$ make a constant angle with the line $y = 0$, $z = x$.

2. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a *cycloid* (Fig. 1-7).

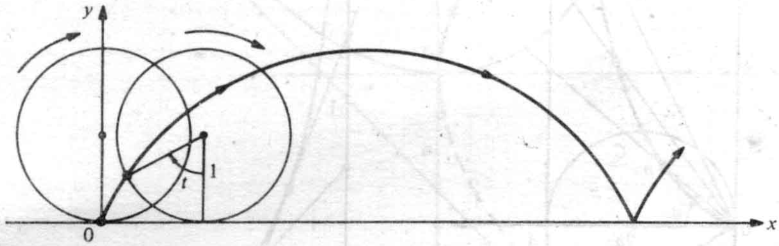

Figure 1-7. The cycloid.

***a.** Obtain a parametrized curve $\alpha: R \longrightarrow R^2$ the trace of which is the cycloid, and determine its singular points.

b. Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

3. Let $0A = 2a$ be the diameter of a circle S^1 and $0Y$ and AV be the tangents to S^1 at 0 and A, respectively. A half-line r is drawn from 0 which meets the circle S^1 at C and the line AV at B. On $0B$ mark off the segment $0p = CB$. If we rotate r about 0, the point p will describe a curve called the *cissoid of Diocles*. By taking $0A$ as the x axis and $0Y$ as the y axis, prove that

a. The trace of

$$\alpha(t) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right), \qquad t \in R,$$

is the cissoid of Diocles ($t = \tan\theta$; see Fig. 1-8).

b. The origin (0, 0) is a singular point of the cissoid.

c. As $t \longrightarrow \infty$, $\alpha(t)$ approaches the line $x = 2a$, and $\alpha'(t) \longrightarrow (0, 2a)$. Thus, as $t \longrightarrow \infty$, the curve and its tangent approach the line $x = 2a$; we say that $x = 2a$ is an *asymptote* to the cissoid.

4. Let $\alpha: (0, \pi) \longrightarrow R^2$ be given by

$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2}\right),$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the *tractrix* (Fig. 1-9). Show that

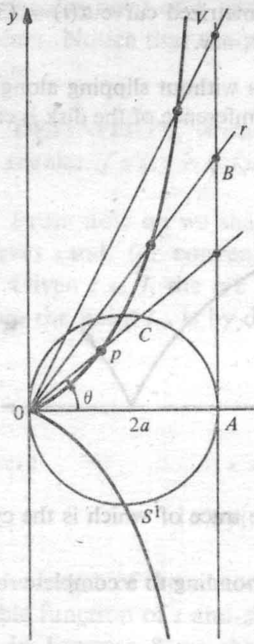

Figure 1-8. The cissoid of Diocles.

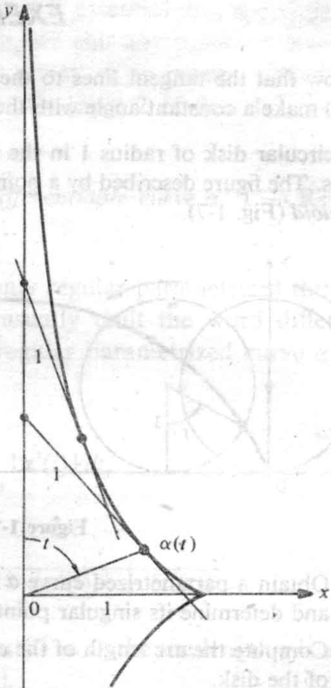

Figure 1-9. The tractrix.

a. α is a differentiable parametrized curve, regular except at $t = \pi/2$.

b. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

5. Let $\alpha: (-1, +\infty) \longrightarrow R^2$ be given by

$$\alpha(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right).$$

Prove that:

a. For $t = 0$, α is tangent to the x axis.

b. As $t \longrightarrow +\infty$, $\alpha(t) \longrightarrow (0, 0)$ and $\alpha'(t) \longrightarrow (0, 0)$.

c. Take the curve with the opposite orientation. Now, as $t \longrightarrow -1$, the curve and its tangent approach the line $x + y + a = 0$.

The figure obtained by completing the trace of α in such a way that it becomes symmetric relative to the line $y = x$ is called the *folium of Descartes* (see Fig. 1-10).

6. Let $\alpha(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$, $t \in R$, a and b constants, $a > 0$, $b < 0$, be a parametrized curve.