Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1030

Ulrich Christian

Selberg's Zeta-, L-, and Eisensteinseries

Springer-Verlag
Berlin Heidelberg New York Tokyo

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1030

Ulrich Christian

Selberg's Zeta-, L-, and Eisensteinseries

Springer-Verlag
Berlin Heidelberg New York Tokyo 1983

Author

Ulrich Christian
Mathematisches Institut, Georg-August-Universität
Bunsenstr. 3–5, 3400 Göttingen, Federal Republic of Germany

CR Subject Classifications (1982): 3, 10

AMS Subject Classifications (1980): 10 D 05, 10 D 20, 10 D 24

ISBN 3-540-12701-1 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-12701-1 Springer-Verlag New York Heidelberg Berlin Tokyo

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

Lecture Notes in Mathematics

For information about Vols. 1–817, please contact your bookseller or Springer-Verlag.

Vol. 818: S. Montgomery, Fixed Rings of Finite Automorphism Groups of Associative Rings. VII, 126 pages. 1980.

Vol. 819: Global Theory of Dynamical Systems. Proceedings, 1979. Edited by Z. Nitecki and C. Robinson. IX, 499 pages. 1980.

Vol. 820: W. Abikoff, The Real Analytic Theory of Teichmüller Space. VII, 144 pages. 1980.

Vol. 821: Statistique non Paramétrique Asymptotique. Proceedings, 1979. Edited by J.-P. Raoult. VII, 175 pages. 1980.

Vol. 822: Séminaire Pierre Lelong–Henri Skoda, (Analyse) Années 1978/79. Proceedings. Edited by P. Lelong et H. Skoda. VIII, 356 pages, 1980.

Vol. 823: J. Král, Integral Operators in Potential Theory. III, 171 pages. 1980.

Vol. 824: D. Frank Hsu, Cyclic Neofields and Combinatorial Designs. VI, 230 pages. 1980.

Vol. 825: Ring Theory, Antwerp 1980. Proceedings. Edited by F. van Oystaeyen. VII, 209 pages. 1980.

Vol. 826: Ph. G. Ciarlet et P. Rabier, Les Equations de von Kármán. VI, 181 pages. 1980.

Vol. 827: Ordinary and Partial Differential Equations. Proceedings, 1978. Edited by W. N. Everitt. XVI, 271 pages. 1980.

Vol. 828: Probability Theory on Vector Spaces II. Proceedings, 1979. Edited by A. Weron. XIII, 324 pages. 1980.

Vol. 829: Combinatorial Mathematics VII. Proceedings, 1979. Edited by R. W. Robinson et al.. X, 256 pages. 1980.

Vol. 830: J. A. Green, Polynomial Representations of GL_n. VI, 118 pages. 1980.

Vol. 831: Representation Theory I. Proceedings, 1979. Edited by V. Dlab-and P. Gabriel. XIV, 373 pages. 1980.

Vol. 832: Representation Theory II. Proceedings, 1979. Edited by V. Dlab and P. Gabriel. XIV, 673 pages. 1980.

Vol. 833: Th. Jeulin, Semi-Martingales et Grossissement d'une Filtration. IX, 142 Seiten. 1980.

Vol. 834: Model Theory of Algebra and Arithmetic. Proceedings, 1979. Edited by L. Pacholski, J. Wierzejewski, and A. J. Wilkie. VI, 410 pages. 1980.

Vol. 835: H. Zieschang, E. Vogt and H.-D. Coldewey, Surfaces and Planar Discontinuous Groups. X, 334 pages. 1980.

Vol. 836: Differential Geometrical Methods in Mathematical Physics. Proceedings, 1979. Edited by P. L. García, A. Pérez-Rendón, and J. M. Souriau. XII, 538 pages. 1980.

Vol. 837: J. Meixner, F. W. Schäfke and G. Wolf, Mathieu Functions and Spheroidal Functions and their Mathematical Foundations Further Studies. VII, 126 pages. 1980.

Vol. 838: Global Differential Geometry and Global Analysis. Proceedings 1979. Edited by D. Ferus et al. XI, 299 pages. 1981.

Vol. 839: Cabal Seminar 77 – 79. Proceedings. Edited by A. S. Kechris, D. A. Martin and Y. N. Moschovakis. V, 274 pages. 1981.

Vol. 840: D. Henry, Geometric Theory of Semilinear Parabolic Equations. IV, 348 pages. 1981.

Vol. 841: A. Haraux, Nonlinear Evolution Equations- Global Behaviour of Solutions. XII, 313 pages. 1981.

Vol. 842: Séminaire Bourbaki vol. 1979/80. Exposés 543–560. IV, 317 pages. 1981.

Vol. 843: Functional Analysis, Holomorphy, and Approximation Theory. Proceedings. Edited by S. Machado. VI, 636 pages. 1981.

Vol. 844: Groupe de Brauer. Proceedings. Edited by M. Kervaire and M. Ojanguren. VII, 274 pages. 1981.

Vol. 845: A. Tannenbaum, Invariance and System Theory: Algebraic and Geometric Aspects. X, 161 pages. 1981.

Vol. 846: Ordinary and Partial Differential Equations, Proceedings. Edited by W. N. Everitt and B. D. Sleeman. XIV, 384 pages. 1981.

Vol. 847: U. Koschorke, Vector Fields and Other Vector Bundle Morphisms – A Singularity Approach. IV, 304 pages. 1981.

Vol. 848: Algebra, Carbondale 1980. Proceedings. Ed. by R. K. Amayo. VI, 298 pages. 1981.

Vol. 849: P. Major, Multiple Wiener-Itô Integrals. VII, 127 pages. 1981.

Vol. 850: Séminaire de Probabilités XV. 1979/80. Avec table générale des exposés de 1966/67 à 1978/79. Edited by J. Azéma and M. Yor. IV, 704 pages. 1981.

Vol. 851: Stochastic Integrals. Proceedings, 1980. Edited by D. Williams. IX, 540 pages. 1981.

Vol. 852: L. Schwartz, Geometry and Probability in Banach Spaces. X, 101 pages. 1981.

Vol. 853: N. Boboc, G. Bucur, A. Cornea, Order and Convexity in Potential Theory: H-Cones. IV, 286 pages. 1981.

Vol. 854: Algebraic K-Theory. Evanston 1980. Proceedings. Edited by E. M. Friedlander and M. R. Stein. V, 517 pages. 1981.

Vol. 855: Semigroups. Proceedings 1978. Edited by H. Jürgensen, M. Petrich and H. J. Weinert. V, 221 pages. 1981.

Vol. 856: R. Lascar, Propagation des Singularités des Solutions d'Equations Pseudo-Différentielles à Caractéristiques de Multiplicités Variables. VIII, 237 pages. 1981.

Vol. 857: M. Miyanishi. Non-complete Algebraic Surfaces. XVIII, 244 pages. 1981.

Vol. 858: E. A. Coddington, H. S. V. de Snoo: Regular Boundary Value Problems Associated with Pairs of Ordinary Differential Expressions. V, 225 pages. 1981.

Vol. 859: Logic Year 1979–80. Proceedings. Edited by M. Lerman, J. Schmerl and R. Soare. VIII, 326 pages. 1981.

Vol. 860: Probability in Banach Spaces III. Proceedings, 1980. Edited by A. Beck. VI, 329 pages. 1981.

Vol. 861: Analytical Methods in Probability Theory. Proceedings 1980. Edited by D. Dugué, E. Lukacs, V. K. Rohatgi. X, 183 pages. 1981.

Vol. 862: Algebraic Geometry. Proceedings 1980. Edited by A. Libgober and P. Wagreich. V, 281 pages. 1981.

Vol. 863: Processus Aléatoires à Deux Indices. Proceedings, 1980. Edited by H. Korezlioglu, G. Mazziotto and J. Szpirglas. V, 274 pages. 1981.

Vol. 864: Complex Analysis and Spectral Theory. Proceedings, 1979/80. Edited by V. P. Havin and N. K. Nikol'skii. VI, 480 pages. 1981.

Vol. 865: R. W. Bruggeman, Fourier Coefficients of Automorphic Forms. III, 201 pages. 1981.

Vol. 866: J.-M. Bismut, Mécanique Aléatoire. XVI, 563 pages. 1981.

Vol. 867: Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin. Proceedings, 1980. Edited by M.-P. Malliavin. V, 476 pages. 1981.

Vol. 868: Surfaces Algébriques. Proceedings 1976–78. Edited by J. Giraud, L. Illusie et M. Raynaud. V, 314 pages. 1981.

Vol. 869: A. V. Zelevinsky, Representations of Finite Classical Groups. IV, 184 pages. 1981.

Vol. 870: Shape Theory and Geometric Topology. Proceedings, 1981. Edited by S. Mardešić and J. Segal. V, 265 pages. 1981.

Vol. 871: Continuous Lattices. Proceedings, 1979. Edited by B. Banaschewski and R.-E. Hoffmann. X, 413 pages. 1981.

Vol. 872: Set Theory and Model Theory. Proceedings, 1979. Edited by R. B. Jensen and A. Prestel. V, 174 pages. 1981.

continuation on page 199

PREFACE

This course of lectures was given at the University of Göttingen in the summer-semester 1983.

I thank Mrs. Christiane Gieseking for her careful typing of the troublesome manuscript.

Ulrich Christian

INTRODUCTION

In these lecture notes we prove analytic continuation and functional equations for Selberg's Eisensteinseries, Selberg's zetafunctions, Selberg's L-series, and Siegel's Eisensteinseries.

We start with Epstein's zetafunction for a binary quadratic form and Epstein's L-functions which are connected with Epstein's zetafunction like Dirichlet's L-series are connected with Riemann's zetafunction. Then we consider Eisensteinseries for the elliptic modular group which are also closely related to Epstein's zetafunction.

In the next chapters we come to Selberg's zetafunction (see Maaβ [33], § 17, Selberg [41], and Terras [45], [46]). Furthermore we consider Selberg's L-series which are connected with Selberg's zetafunctions like Dirichlet's L-series are connected with Riemann's zetafunction.

These functions may be described as follows. Let Y be a real symmetric, positive $n\times n$ matrix and $\overset{\nu}{Y}$ positive matrices ($\nu=1,\dots,n$) which are connected by

(a) $$\overset{n}{Y} = Y;\quad \overset{\nu}{Y} = G'_{\nu}\,\overset{\nu+1}{Y}\,G_{\nu} \qquad (\nu = 1,\dots,n-1)$$

with integral $(\nu+1)\times\nu$ matrices G_ν. Here ' denotes the transposed matrix. The above mentioned authors then consider the zetafunction

(b) $$\zeta(Y;z_1,\dots,z_{n-1}) = \sum_{\langle \overset{n}{Y},\dots,\overset{1}{Y}\rangle} \prod_{\nu=1}^{n-1} (\operatorname{Det} \overset{\nu}{Y})^{-z_\nu}\ ,$$

where the summation is taken over all possible $\overset{n-1}{Y},\dots,\overset{1}{Y}$ for which (a) holds.

Let $\Omega(n) = GL(n,\mathbb{Z})$ be the group of unimodular $n\times n$ matrices and $\Delta(n)$ the subgroup of upper triangular matrices. The above mentioned authors then show, that the function (b) is closely connected to the function

(c) $$\zeta^{*}(Y;z_1,\dots,z_{n-1}) = \sum_{U\in\Omega(n)/\Delta(n)} \Big(\prod_{\nu=1}^{n-1}(\operatorname{Det}(U'YU)_{\nu})^{-z_\nu}\Big),$$

here A_ν means generally the left upper $\nu\times\nu$ submatrix of a matrix A. The summation is over all cosets $\Omega(n)/\Delta(n)$.

By computing residues Maaß [33], pages 279-299 furthermore obtaines analytic continuation of zetafunctions which are more general than (b) and (c).

In the present lecture notes we generalize the functions (b), (c) as follows. Let q be a natural number and $\chi_1,\dots,\chi_{n-1}$ even Dirichletcharacters mod q. In (a) we assume that the elements of G_ν below the main-diagonal are divisible by q. Furthermore, we put inside the sum (b) the Dirichletcharacters $\chi_1,\dots,\chi_{n-1}$. It is difficult to describe in the introduction how this has to be done. It is simpler for the function (c) for which we replace $\Omega(n)$ by the subgroup $\Psi(n)$ consisting of all unimodular matrices

$$U \equiv \begin{pmatrix} u_1 & * \\ 0 & u_n \end{pmatrix} \mod q .$$

Then instead of (c) we consider the function

$$\text{(d)} \quad \zeta^*(\chi_1,\dots,\chi_{n-1};Y;z_1,\dots,z_{n-1}) = \sum_{\Psi(n)/\Delta(n)} \prod_{\nu=1}^{n-1} (\chi_\nu(u_\nu)(\mathrm{Det}(U'YU)_\nu)^{-z_\nu}) .$$

Under the assumption that all the products $\chi_\mu \cdots \chi_\nu$ $(1 \le \mu \le \nu \le \le n-1)$ are primitive characters mod q we derive analytic continuation and functional equations for our functions.

For these functions we prove results that may be described as follows. Choose even Dirichletcharacters $\psi_1,\dots,\psi_n$ with $\psi_{\nu+1}^{-1}\psi_\nu = \chi_\nu$ $(\nu = 1,\dots,n-1)$; introduce new variables $s_1,\dots,s_n$ by $z_\nu = s_{\nu+1} - s_\nu + \frac{1}{2}$ $(\nu = 1,\dots,n-1)$ and put $\psi = (\psi_1,\dots,\psi_n)$; $s = (s_1,\dots,s_n)$. Let $L(\chi,s)$ be Dirichlet's L-series and put

$$\xi(\chi,s) = \left(\frac{\pi}{q}\right)^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(\chi,s).$$

If then all characters $\psi_\nu^{-1}\psi_\mu$ $(1 \le \mu < \nu \le n)$ are primitive, the function

$$\lambda(\psi,Y,s) = \Big(\prod_{1\le\mu<\nu\le n} \xi(\psi_\nu^{-1}\psi_\mu, 2(s_\nu-s_\mu)+1)\Big) \times$$

$$(\operatorname{Det} Y)^{s_n-\frac{1}{n}(s_1+\dots+s_n)+\frac{n-1}{4}} \zeta^*(\chi,Y,z)$$

can be holomorphically continued to all $s \in \mathbb{C}^n$. Furthermore $\lambda(\psi,Y,s)$ satisfies certain functional equations which we shall now describe.

Let

$\tilde{s} = (-s_n,\dots,-s_1)$; $\tilde{\psi} = (\psi_n^{-1},\dots,\psi_1^{-1})$; $\check{s} = (-s_{n-1},\dots,-s_1,-s_n)$;

$\check{\psi} = (\psi_{n-1}^{-1},\dots,\psi_1^{-1},\psi_n^{-1})$; $\mathfrak{z}s = (s_n,s_1,\dots,s_{n-1})$;

$\mathfrak{z}\psi = (\psi_n,\psi_1,\dots,\psi_{n-1})$; $\hat{s} = (s_{n-1},s_1,\dots,s_{n-2},s_n)$;

$\hat{\psi} = (\psi_{n-1},\psi_1,\dots,\psi_{n-2},\psi_n)$.

Form the Gaussian sum

$$G(\chi) = q^{-\frac{1}{2}} \sum_{d \bmod q} \chi(d)\exp\left(\frac{2\pi i d}{q}\right)$$

and put

$$\eta(\psi) = \prod_{\mu=1}^{n-1} G(\chi_\mu) .$$

Form the $n\times n$ matrices

$$W(n) = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}, \quad Q(n) = \begin{pmatrix} W(n-1) & 0 \\ 0 & q \end{pmatrix}, \quad P(n) = q^{\frac{1}{n}} W(n)Q^{-1}$$

and the

$$r = q^{n-2}$$

matrices

$$K_\rho = \begin{pmatrix} P^{-1}(n-1) & 0 \\ 0 & q^{1-\frac{1}{n}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & E^{(n-2)} & 0 \\ 0 & L & 1 \end{pmatrix} \qquad (\rho=1,\dots,r)$$

where $L = (l_2,\dots,l_{n-1})$ runs over all r residue classes mod q. Set

$$\tilde{Y} = W(n)Y^{-1}W(n), \check{Y} = (Y[Q(n)])^{-1} .$$

Then the following functional equations hold:

$$\lambda(\tilde{\psi},\tilde{Y},\tilde{s}) = \lambda(\psi,Y,s) ,$$

$$\lambda(\psi,Y,s) = \eta(\psi) q^{2(\frac{1}{n}\sum_{\nu=1}^{n} s_\nu - s_n)} \lambda(\check{\psi},\check{Y},\check{s}) ,$$

$$\lambda(\psi,Y,s) = \eta(\psi) q^{2(s_1 - \frac{1}{n}\sum_{\nu=1}^{n} s_\nu)} \lambda(\mathfrak{z}\psi, P(n)'^{-1} Y P(n)^{-1}, \mathfrak{z} s),$$

$$\lambda(\psi, Y,s) = \eta^{-1}(\psi) q^{\frac{4}{n}\sum_{\nu=1}^{n} s_\nu - 2s_{n-1} - Zs_n + 1 - \frac{n}{2}} \sum_{\rho=1}^{r} \lambda(\hat{\psi}, K_\rho' Y K_\rho, \hat{s}).$$

The transformations $\mathfrak{z}$ and $\wedge$ generate the symmetric group $\mathfrak{S}_n$.

As in the case of Riemann's zetafunction and Dirichlet's L-series however the L-series have less poles than the zetafunctions. Therefore in the case $q > 1$ poles and residues of type Maaβ [33], pages 279-299 do not exist. So it is impossible to get any results about more general series by computing residues. For this reason we start already with more general series and derive the analytic continuation and functional equations for them.

Let $\mathfrak{z}(n) = \{Z = Z' = X + iY,\ Y > 0\}$ Siegel's upper halfplane of degree n and $\Gamma(n) = Sp(n, \mathbb{Z})$ Siegel's modular group of degree n. Set

(e) $$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(n)$$

with $n \times n$ matrices A, B, C, D and

(f) $$M\langle Z\rangle = (AZ + B)(CZ + D)^{-1} = X_M + iY_M .$$

Set

(g) $$M\{Z\} = CZ + D .$$

Let $s = (s_1,\dots,s_n)$ a complex variablerow and $Z \in \mathfrak{z}(n)$. We consider Selberg's Eisensteinseries

(h) $$\mathcal{E}(n,r,Z,s) = \sum_{M \in \Gamma_B\backslash\Gamma(n)} (\mathrm{Det}\ M\{\overline{Z}\})^{2r}(\mathrm{Det}\ Y_M)^{s_n+\frac{n}{2}+r} \prod_{\nu=1}^{n-1}(\mathrm{Det}(Y_M)_\nu)^{s_\nu - s_{\nu+1} - \frac{1}{2}} .$$

Here $\Gamma_B(n)$ is the Borel subgroup of $\Gamma(n)$ consisting of all 2nx2n matrices

(i) $$M = \begin{pmatrix} U' & SU^{-1} \\ 0 & U^{-1} \end{pmatrix}$$

with integral nxn matrices U, S. Here $S = S'$ and U is an upper triangular matrix. For the functions (h) we prove again analytic continuation and functional equations by applying a method of Diehl [11]. Since the functional equations are very similar to those of Selberg's zetafunctions I do not write them down here.

Finally consider Siegel's Eisensteinseries

(j) $$E(n,r,Z,\omega) = \sum_{M \in \Gamma_n(n)\backslash\Gamma(n)} (\mathrm{Det}\ M\{Z\})^{-2r}(\mathrm{Det}\ Y_M)^{\omega-r} .$$

Here ω is a complex variable and $\Gamma_n(n)$ the group of matrices (i) where now U is arbitrarily unimodular. We show that Siegel's Eisensteinseries may be obtained by computing residues of Selberg's Eisensteinseries. Hence the analytic continuation and the functional equations of Selberg's Eisensteinseries give us analytic continuation and a functional equation for Siegel's Eisensteinseries.

Especially we get the following results

A) It is $E(n,r,Z,\omega)$ holomorphic at $\omega = r$ for

(k) $$r = 1,2, [\tfrac{n-1}{2}], [\tfrac{n+1}{2}] ,$$

so for these values of r the Eisensteinseries $E(n,r,Z,\omega)$ has Hecke summation.

B) It is $E(n,1,Z,1) = 0$ $(n \geq 3)$.

C) Let

$$(1) \qquad S(r) = \left\{ \begin{array}{ll} r-2 & (3 \leq r < \frac{n+2}{4}) \\ [\frac{n-1}{2}]-r & (\frac{n+2}{4} \leq r \leq [\frac{n-3}{2}]) \end{array} \right\} .$$

If $3 \leq r \leq [\frac{n-3}{2}]$, the Eisensteinseries $E(n,r,Z,\omega)$ has at $\omega = r$ a pole of order $S(r)$ at the most.

All functions considered in this lecture play an important rôle in the theory of Siegel's modular functions but it seems to me that they are also interesting for themselves. They are eigenfunctions of invariant differential operators (see Selberg [39] till [42] and Maaß [33]) and they may be used to describe the continuous spectrum of those differential operators. Furthermore, as we have seen they may be used to get analytic continuation of Siegel's Eisenstein-series. It is an important open question if $E(n,r,Z,\omega)$ is holomorphic at $\omega = r$ also for $3 \leq r \leq [\frac{n-3}{2}]$.

CONTENTS

CHAPTER I. EPSTEIN'S ZETAFUNCTIONS OF A BINARY QUADRATIC FORM

In the first chapter we consider Epstein's zetafunction for binary quadratic forms. To this zetafunction we associate L-series in the same way as Dirichlet's L-series are associated to Riemann's zetafunction. Furthermore we consider Eisensteinseries for the elliptic modular group which are closely related to Epstein's zetafunction. For all these functions we prove analytic continuation and functional equations with the aid of thetafunctions.

---

§ 1. PRELIMINARIES

§ 1 contains some preliminary definitions and results on thetafunctions for binary quadratic forms.

A matrix $K = (k_{\iota\varkappa})$ of ρ rows and σ columns is called a $\rho\times\sigma$ matrix; Rk K is the rank, K' the transposed and $\overline{K}$ the conjugate complex matrix. Occasionally we write Dg K = $= [k_1,\ldots,k_{\min(\rho,\sigma)}]$ for the diagonalmatrix formed of the diagonalelements $k_\iota = k_{\iota\iota}$ $(\iota = 1,\ldots,\min(\rho,\sigma))$ from K. In case $\rho = \sigma$ let Tr K be the trace, Det K the determinante and abs K the absolute value of Det K. Let K_ν be the upper left $\nu\times\nu$ submatrix of K. With a $\rho\times\tau$ matrix L define $K[L] = L'KL$. Let $0,E$ be zero- and identity-matrix. The number of rows and columns will either be seen from the connection or it will be written as upper indices in brackets. A real symmetric $\rho\times\rho$ matrix Y is called positive $(Y > 0)$ respectively semipositive $(Y \geq 0)$, if the quadratic form $Y[x]$ is positive respectively nonnegative for all real columns $x \neq 0$ with ρ elements. Let $\mathcal{Y}(n)$ denote the space of all positive symmetric $n\times n$ matrices Y. Then $\mathcal{Y}(n)$ is real and has

$$(1) \qquad d(n) = \frac{n(n+1)}{2}$$

dimensions. $Y_1 > Y_2$ is defined by $Y_1 - Y_2 > 0$ and $Y_1 \geq Y_2$ by $Y_1 - Y_2 \geq 0$. A matrix is called "integral", "rational", "real" or "complex" if all elements are in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. Brackets of type $\langle\ \rangle$ denote the greatest common devisor of integers. "exp" is the exponential function.

Let $Y \in \mathcal{Y}(n)$. By an "isotropic vector" of Y we mean a complex column w satisfiying

$$(2) \qquad Y[w] = 0 .$$

If w is an isotropc vector of Y, obviously Yw is an isotropc vector of Y^{-1}.

THEOREM 1: Let $Y \in \mathcal{Y}(n)$; $g \in \mathbb{Z}$, $g \geq 0$; u and v be two arbitrary n-rowed complex columns, w an isotropic vector of Y. Then

$$(3) \qquad \sum_{m \in \mathbb{Z}^n} ((m+v)'Yw)^g \exp(-\pi Y[m+v] + 2\pi i m'u) =$$

$$\frac{\exp(-2\pi i u'v)}{i^g}(\operatorname{Det} Y)^{-\frac{1}{2}} \sum_{m \in \mathbb{Z}^n}((m-u)'w)^g \exp(-\pi Y^{-1}[m-u]+ 2\pi i m'v).$$

Here m runs over all integral columns with n elements.

PROOF: Use Siegel [44], Page 65, formula (57).

Let $q \in \mathbb{N}$. The concept of a character mod q, a primitive character mod q and an even character mod q is defined like in Hasse [15], § 13 or Landau [22], Kapitel 22. With an even character χ mod q and $a \in \mathbb{Z}$ form the Gaussian sum

$$(4) \qquad G(\chi,a) = q^{-\frac{1}{2}} \sum_{b \bmod q} \chi(b)\exp\left(2\pi i\frac{ab}{q}\right) .$$

Then if χ is even, an easy computation shows

$$(5) \qquad \overline{G(\chi,a)} = G(\overline{\chi},a)$$

THEOREM 2: Let χ be a primitive character mod q. Then

$$(6) \qquad G(\chi,a) = \overline{\chi}(a)G(\chi) \qquad (a \in \mathbb{Z}).$$

Here

$$(7) \qquad G(\chi) = G(\chi,1).$$

PROOF: See Landau [22], § 126.

Let χ_1 be an even character mod q, form the row $\mathfrak{l} = (1,1)$, let

$$(8) \qquad Y \in \mathcal{Y}(2);\ t \in \mathbb{R},\ t > 0$$

and define the thetafunction

(9) $$\check{\theta}(q,l,\chi_1,Y,t) = (\text{Det } Y)^{\frac{1}{4}}\, t^{\frac{1}{2}} \sum_{\substack{a=(a_1,a_2)' \\ a_2 \equiv 0 \bmod q}} \chi_1(a_1)\exp(-\tfrac{\pi}{q}\, Y[a]t).$$

Put

(10) $$Q(l) = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix},$$

(11) $$\check{Y} = (Y[Q(l)])^{-1}.$$

Then

(12) $$\text{Det } \check{Y} = q^{-2}\text{Det } Y^{-1}.$$

THEOREM 3: Let χ_1 be an even primitive character mod q. Then

(13) $$\check{\theta}(q,l,\chi_1,Y,t) = G(\chi_1)\check{\theta}(q,l,\overline{\chi_1},\check{Y},t^{-1})$$

and

(14) $$\text{abs } G(\chi_1) = 1.$$

PROOF: In (9) set $a = b+qc$ with $b = (b_1 0)'$ and c an integral column. Then (9) gives

(15) $$\check{\theta}(q,l,\chi_1,Y,t) = (\text{Det}(tY))^{\frac{1}{4}} \sum_{b_1 \bmod q} \chi_1(b_1) \sum_{c \in \mathbb{Z}^2} \exp(-\pi(qYt)[c+\tfrac{b}{q}]).$$

Apply (3) with $n=2$, $g=0$, $u=0$, $v=\frac{b}{q}$, qYt instead of Y. Then

(16) $$\check{\theta}(q,l,\chi_1,Y,t) = (\mathrm{Det}(tY))^{\frac{1}{4}}(\mathrm{Det}(qYt))^{-\frac{1}{2}} \times$$
$$\sum_{c \in \mathbb{Z}^2} \Big(\sum_{b_1 \bmod q} \chi_1(b_1)\exp(2\pi i \frac{c_1 b_1}{q})\Big)\exp(-\frac{\pi}{q} Y^{-1}[c]t^{-1}).$$

From (4), (6), (12), (16) we get

(17) $$\check{\theta}(q,l,\chi_1,Y,t) = G(\chi_1)(\mathrm{Det}\ \check{Y})^{\frac{1}{4}}\ t^{-\frac{1}{2}} \sum_{c \in \mathbb{Z}^2} \overline{\chi}(c_1)\exp(-\frac{\pi}{q}\check{Y}[Q(l)c]t^{-1}) =$$
$$G(\chi_1)\check{\theta}(q,l,\overline{\chi}_1,\check{Y},t^{-1}) .$$

Herewith one has formula (13). Inserting $\overline{\chi}_1$ $,\check{Y},t^{-1}$ in the left hand side of (13) and applying (13) once more one gets

(18) $$G(\chi_1)G(\overline{\chi}_1) = 1 .$$

From (5), (7), (18) we deduce (14). Theorem 3 is proved.

According to Maaß [33], pages 210, 267 form the differential operator

(19) $$D^*(t) = t^{\frac{1}{2}} \frac{d}{dt} t^2 \frac{d}{dt} t^{-\frac{1}{2}} = -\frac{1}{4} + t\frac{d}{dt} + t^2 \frac{d^2}{dt^2} .$$

Using

(20) $$\frac{d}{dt} = -t^{-2} \frac{d}{dt^{-1}}$$

one see's

(21) $$D^*(t) = D^*(t^{-1}),$$

Put

(22) $$D(q,t) = \begin{cases} -D^*(t) & (q = 1) \\ 1 & (q > 1 \end{cases} .$$