Lecture Notes in Mathematics

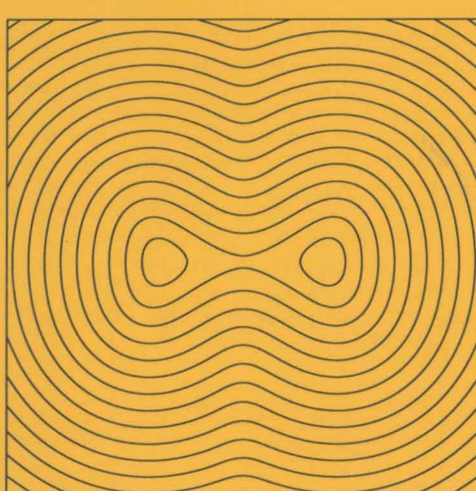
L. Ambrosio K. Deckelnick
G. Dziuk M. Mimura
V. A. Solonnikov H. M. Soner

Mathematical Aspects of Evolving Interfaces

1812

Madeira, Funchal, Portugal 2000

Editors: P. Colli, J.F. Rodrigues



Springer

Fondazione
C.I.M.E.

L. Ambrosio K. Deckelnick G. Dziuk
M. Mimura V.A. Solonnikov H.M. Soner

Mathematical Aspects of Evolving Interfaces

Lectures given at the C.I.M.-C.I.M.E. joint Euro-Summer School held in Madeira, Funchal, Portugal, July 3–9, 2000

Editors: P. Colli
J.F. Rodrigues

Fondazione
C.I.M.E.

Springer

Lecture Notes in Mathematics

Edited by J.-M. Morel, F. Takens and B. Teissier

Editorial Policy for Multi-Author Publications: Summer Schools / Intensive Courses

1. GENERAL. Lecture Notes aim to report new developments in all areas of mathematics and their applications - quickly, informally and at a high level. Manuscripts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. They should provide sufficient motivation, examples and applications. There should also be an introduction making the text comprehensible to a wider audience. This clearly distinguishes Lecture Notes from journal articles or technical reports which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this "lecture notes" character.

2. In general SUMMER SCHOOL and other similar INTENSIVE COURSES are held to present mathematical topics that are close to the frontiers of recent research to an audience at the beginning or intermediate graduate level, who may want to continue with this area of work, for a thesis or later. This makes demands on the didactic aspects of the presentation. Because the subjects of such schools are advanced, there often exists no textbook, and so ideally, the publication resulting from such a school could be a first approximation to such a textbook. Usually several authors are involved in the writing, so it is not always simple to obtain a unified approach to the presentation.

   For prospective publication in LNM, the resulting manuscript should not be just a collection of course notes, each of which has been developed by an individual author with little or no co-ordination with the others, and with little or no common concept. The subject matter should dictate the structure of the book, and the authorship of each part or chapter should take secondary importance. Of course the choice of authors is crucial to the quality of the material at the school and in the book, and the intention here is not to belittle their impact, but simply to say that the book should be planned to be written by these authors jointly, and not just assembled as a result of what these authors happen to submit.

   This represents considerable preparatory work (as it is imperative to ensure that the authors know these criteria before they invest work on a manuscript), and also considerable editing work afterwards, to get the book into final shape. Still it is the form that holds the most promise of a successful book that will be used by its intended audience, rather than yet another volume of proceedings for the library shelf.

3. Manuscripts should be submitted (preferably in duplicate) either to one of the series editors or to Springer-Verlag, Heidelberg. Volume editors are expected to arrange for the refereeing, to the usual scientific standards, of the individual contributions. If the resulting reports can be forwarded to us (series editors or Springer) this is very helpful. If no reports are forwarded or if other questions remain unclear in respect of homogeneity etc, the series editors may wish to consult external referees for an overall evaluation of the volume. A final decision to publish can be made only on the basis of the complete manuscript, however a preliminary decision can be based on a pre-final or incomplete manuscript. The strict minimum amount of material that will be considered should include a detailed outline describing the planned contents of each chapter.

   Volume editors and authors should be aware that incomplete or insufficiently close to final manuscripts almost always result in longer evaluation times. They should also be aware that parallel submission of their manuscript to another publisher while under consideration for LNM will in general lead to immediate rejection.

Continued on inside back-cover

Lecture Notes in Mathematics 1812

Authors

Luigi Ambrosio
Scuola Normale Superiore
Piazza dei Cavalieri 7
56126 Pisa, Italy

e-mail: luigi@ambrosio.sns.it

Klaus Deckelnick
Institute for Analysis and Numerical Analysis
Otto-von-Guericke University of Magdeburg
Universitätsplatz 2
39106 Magdeburg, Germany

e-mail:Klaus.Deckelnick@mathematik.uni-magdeburg.de

Gerhard Dziuk
Institute for Applied Mathematics
University of Freiburg
Hermann-Herderstr. 10
79104 Freiburg, Germany

e-mail: gerd@mathematik.uni-freiburg.de

Masayasu Mimura
Department of Mathematical
and Life Sciences
Institute of Nonlinear Sciences
and Applied Mathematics
Graduate School of Sciences
Hiroshima University
Higashi-Hiroshima 739-8526, Japan

e-mail: mimura@math.sci.hiroshima-u.ac.jp

Vsevolod A. Solonnikov
Saint-Petersburg Department of the
Steklov Mathematical Institute
Fontanka 27
191011 St. Petersburg, Russia

e-mail: solonnik@pdmi.ras.ru

Halil Mete Soner
Department of Mathematics
Koc University
Rumeli Fener Yolu Sariyer
80910 Istanbul, Turkey

e-mail: msoner@ku.edu.tr

Editors

Pierluigi Colli
Department of Mathematics 'F. Casorati'
University of Pavia
Via Ferrata 1
27100 Pavia, Italy

e-mail: pier@imati.cnr.it

José-Francisco Rodrigues
CMAF, University of Lisboa
Av. Prof. Gama Pinto 2
1649-003 Lisboa, Portugal

e-mail: rodrigue@ptmat.fc.ul.pt

Cataloging-in-Publication Data applied for
Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at http://dnb.ddb.de
Mathematics Subject Classification (2000): 35-02, 35Bxx, 35QXX, 35K57, 53C44, 76D05, 80A22

ISSN 0075-8434
ISBN 3-540-14033-6 Springer-Verlag Berlin Heidelberg New York

Springer-Verlag Berlin Heidelberg New York a member of BertelsmannSpringer
Science + Business Media GmbH

http://www.springer.de

Printed in Germany

Typesetting: Camera-ready TEX output by the authors

SPIN: 10922378 41/3142/du - 543210 - Printed on acid-free paper

Preface

The Euro-Summer School on Mathematical Aspects of Evolving Interfaces gathered senior experts and young researchers at the University of Madeira, Funchal, Portugal, in the week July 3-9, 2000. This meeting arose as a joint school of CIM (Centro Internacional de Matemática, Portugal) and CIME (Centro Internazionale Matematico Estivo, Italy).

The school was intended to present an advanced introduction and state of the art of recent analytic, modeling and numerical techniques to the mathematical representation and description of moving interfaces. Five complementary courses were delivered and this volume collects the notes of the lectures.

Interfaces are geometrical objects modeling free or moving boundaries and arise in a wide range of phase change problems in physical and biological sciences, in particular in material technology and in dynamics of patterns. Especially in the end of last century, the rigorous study of evolving interfaces in a number of applied fields becomes increasingly important, so that the possibility of describing their dynamics through suitable mathematical models became one of the most challenging and interdisciplinary problems in applied mathematics.

It was recognized that essential problems related to evolving interfaces can be modelled by means of partial differential equations and systems in domains whose boundary depends on time. In many complicated cases these boundaries are themselves unknown, and correspond, e.g., to a particular level set, or to the discontinuity set, of some physical quantity. In particular, free boundary problems are boundary value problems for differential equations set in a domain where part of the boundary is "free" and further conditions allow to exclude undeterminacy.

Although the first modern work in a free boundary problem was written by Lamé and Clapeyron in 1831, who considered a simple model for the solidification of a liquid sphere, in the last decades of the XXth century this interdisciplinary field developed tremendously with many new computational demands and new problems from industry and applied sciences, as well as with increasing contributions from Mathematics. Indeed, problems of this sort are concerned with several phenomena of high applied interest. Examples include Stefan type problems, where, tipically, the free boundary is the moving interface between liquid and solid, as well as, more general models of phase transitions. Another important example arises in filtration through porous media; here free boundaries occur as fronts between saturated and unsaturated regions. Interesting examples also come from reaction-diffusion models, fluid dynamics, contact mechanics, superconductivity and so on. Several of these problems are also of direct industrial interest, and offer an interesting opportunity of collaboration among theoretical analysts, mathematical physicists and applied scientists.

The Madeira school reported on mathematical advances in some theoretical, modeling and numerical issues concerned with dynamics of interfaces and free boundaries. Specifically, the five courses dealt with an assessment of recent results on the optimal transportation problem (L. Ambrosio), the numerical approximation of moving fronts evolving by mean curvature (G. Dziuk), the dynamics of patterns and interfaces in some reaction-diffusion systems with chemical-biological applications (M. Mimura), evolutionary free boundary problems of parabolic type or for Navier-Stokes equations (V.A. Solonnikov), and a variational approach to evolution problems for the Ginzburg-Landau functional (H.M. Soner).

We expect that these lecture notes will be useful not only to experienced readers, to find a detailed description of results and a presentation of techniques, but also to the beginners that aim to learn some of the mathematical aspects behind the different fields.

The editors

Pierluigi Colli, Pavia
José Francisco Rodrigues, Lisboa

CIME's activity is supported by:
Ministero dell'Istruzione, dell'Università e della Ricerca; Ministero degli Affari Esteri – Direzione Generale per la Promozione e la Cooperazione – Ufficio V; Istituto Nazionale di Alta Matematica "Francesco Severi"; E.U. under the Training and Mobility of Researchers Programme and UNESCO-ROSTE, Venice Office.

Fondazione C.I.M.E.

Centro Internazionale Matematico Estivo
International Mathematical Summer Center
http://www.math.unifi.it/~cime
cime@math.unifi.it

2003 COURSES LIST

Stochastic Methods in Finance

July 6–13, Cusanus Akademie, Bressanone (Bolzano)
Joint course with European Mathematical Society

Course Directors:

Prof. Marco Frittelli (Univ. di Firenze), marco.frittelli@dmd.unifi.it
Prof. Wolfgang Runggaldier (Univ. di Padova), runggal@math.unipd.it

Hyperbolic Systems of Balance Laws

July 14–21, Cetraro (Cosenza)

Course Director:

Prof. Pierangelo Marcati (Univ. de L'Aquila), marcati@univaq.it

Symplectic 4-Manifolds and Algebraic Surfaces

September 2–10, Cetraro (Cosenza)

Course Directors:

Prof. Fabrizio Catanese (Bayreuth University)
Prof. Gang Tian (M.I.T. Boston)

Mathematical Foundation of Turbulent Viscous Flows

September 1–6, Martina Franca (Taranto)

Course Directors:

Prof. M. Cannone (Univ. de Marne-la-Vallée)
Prof.T. Miyakawa (Kobe University)

Druck und Bindung: Strauss Offsetdruck GmbH

4. Manuscripts should in general be submitted in English. Final manuscripts should contain at least 100 pages of mathematical text and should include
   - a general table of contents;
   - an informative introduction, with adequate motivation and perhaps some historical remarks: it should be accessible to a reader not intimately familiar with the topic treated;
   - a global subject index: as a rule this is genuinely helpful for the reader.

5. Lecture Notes are printed by photo-offset from the master-copy delivered in camera-ready form by the authors. Springer-Verlag provides technical instructions for the preparation of manuscripts. We strongly recommend that all contributions in a volume be written in the same LaTeX version, preferably LaTeX2e. Macro-packages in LaTeX2e are available from Springer's web-pages at

   http://www.springer.de/math/authors/index.html .

   Careful preparation of manuscripts will help keep production time short and ensure satisfactory appearance of the finished book. After acceptance of the manuscript authors/volume editors will be asked to prepare the final LaTeX source files (and also the corresponding dvi- or pdf-file) together with the final printout made from these files. The LaTeX source files are essential for producing a unified full-text online version of the book

   (http://www.springerlink.com/link/service/series/0304/tocs.htm).

   The actual production of a Lecture Notes volume takes approximately 12 weeks.

6. Volume editors receive a total of 50 free copies of their volume to be shared with the authors, but no royalties. They and the authors are entitled to a discount of 33.3 % on the price of Springer books purchased for their personal use, if ordering directly from Springer-Verlag.

Commitment to publish is made by letter of intent rather than by signing a formal contract. Springer-Verlag secures the copyright for each volume. Authors are free to reuse material contained in their LNM volumes in later publications: A brief written (or e-mail) request for formal permission is sufficient.

Addresses:

Professor J.-M. Morel, CMLA,
Ecole Normale Supérieure de Cachan,
61 Avenue du Président Wilson, 94235 Cachan Cedex, France
E-mail: Jean-Michel.Morel@cmla.ens-cachan.fr

Professor F. Takens, Mathematisch Instituut,
Rijksuniversiteit Groningen, Postbus 800,
9700 AV Groningen, The Netherlands
E-mail: F.Takens@math.rug.nl

Professor B. Teissier, Université Paris 7
Institut Mathématique de Jussieu, UMR 7586 du CNRS
Equipe "Géométrie et Dynamique", 175 rue du Chevaleret
75013 Paris, France
E-mail: teissier@math.jussieu.fr

Springer-Verlag, Mathematics Editorial, Tiergartenstr. 17,
69121 Heidelberg, Germany,
Tel.: +49 (6221) 487-8410
Fax: +49 (6221) 487-8355
E-mail: lnm@Springer.de

Table of Contents

Lecture Notes on Optimal Transport Problems
Luigi Ambrosio 1
1 Some elementary examples 3
2 Optimal transport plans: existence and regularity 5
3 The one dimensional case 14
4 The ODE version of the optimal transport problem 16
5 The PDE version of the optimal transport problem and the p-laplacian approximation 29
6 Existence of optimal transport maps 32
7 Regularity and uniqueness of the transport density 38
8 The Bouchitté–Buttazzo mass optimization problem 41
9 Appendix: some measure theoretic results 43
References 50

Numerical Approximation of Mean Curvature Flow of Graphs and Level Sets
Klaus Deckelnick, Gerhard Dziuk 53
1 Introduction 53
Notations 55
2 Mean Curvature Flow of Graphs 55
2.1 The Differential Equation 55
2.2 Some analysis for the problem 56
2.3 Discretization 58
Discretization in Space 58
Fully discrete schemes and stability 59
Convergence of the semi-discrete scheme 61
2.4 Convergence of the fully discrete semi implicit scheme 63
3 Mean Curvature Flow of Level Sets 75
3.1 Viscosity Solutions 75
3.2 Regularization 76
3.3 Convergence of the numerical scheme for the level set problem 78
The semi-discrete scheme 78
The fully discrete scheme 78

3.4 Numerical Tests for the Level Set Algorithm 81
The regular case. 81
Mildly singular case. 82
The highly singular case. 83
The dumb-bell problem. 83
References 86

Reaction-Diffusion Systems Arising in Biological and Chemical Systems: Application of Singular Limit Procedures
Masayasu Mimura 89
1 What is diffusion? 89
1.1 Discrete diffusion models 89
1.2 Continuous models 90
2 Paradox of diffusion 91
3 Diffusive patterns and waves - bistable RD equations 94
3.1 Scalar bistable RD equation 95
3.2 Two component systems of bistable RD equations 98
4 Resource-consumer RD systems 101
4.1 Bacterial colony model 101
4.2 Grey-Scott model 107
5 Competition-diffusion systems and singular limit procedures 109
5.1 Spatial segregating limits of two competing species model 111
5.2 Dynamics of triple junctions arising in three competing species model 113
6 Chemotactic patterns - aggregation and colonies 115
6.1 Aggregation of Dictyostelium discoideum 116
6.2 New pattern arising in a chemotaxis-diffusion-growth system 117
References 119

Lectures on Evolution Free Boundary Problems: Classical Solutions
Vsevolod A. Solonnikov 123
1 Introduction 124
2 The model problem 128
3 Transformation of the problem (1) 134
4 Proof of Theorem 3.1 138
5 Introduction 142
6 Linear and model problems 146
7 Lagrangean coordinates and local existence theorems 154
8 Proof of Theorem 5.3 162
9 Scheme of the proof of Theorem 5.4 166
References 173

Variational and Dynamic Problems for the Ginzburg-Landau Functional
Halil Mete Soner ... 177
1 Introduction ... 177
2 Energy Lower Bounds ... 183
2.1 Degree and Jacobian ... 183
2.2 Covering argument ... 184
2.3 Main lower bound ... 193
3 Jacobian and the GL Energy ... 201
3.1 Jacobian estimate ... 201
3.2 Compactness in two dimensions ... 207
4 Gamma Limit ... 213
4.1 Functions of BnV ... 215
4.2 Gamma limit of I^ϵ ... 217
5 Compactness in Higher Dimensions ... 222
6 Dynamic Problems: Evolution of Vortex Filaments ... 225
6.1 Energy identities ... 226
6.2 Mean curvature flow and the distance function ... 227
6.3 Convergence ... 227
References ... 231

List of Participants ... 235

Lecture Notes on Optimal Transport Problems

Luigi Ambrosio*

Scuola Normale Superiore
Piazza dei Cavalieri 7, I-56126, Pisa, Italy
luigi@ambrosio.sns.it

Contents

1. Some elementary examples
2. Optimal transport plans: existence and regularity
3. The one dimensional case
4. The ODE version of the optimal transport problem
5. The PDE version of the optimal transport problem and the p-laplacian approximation
6. Existence of optimal transport maps
7. Regularity and uniqueness of the transport density
8. The Bouchitté–Buttazzo mass optimization problem
9. Appendix: some measure theoretic results

Introduction

These notes are devoted to the Monge–Kantorovich optimal transport problem. This problem, in the original formulation of Monge, can be stated as follows: given two distributions with equal masses of a given material $g_0(x)$, $g_1(x)$ (corresponding for instance to an embankment and an excavation), find a transport map ψ which carries the first distribution into the second and minimizes the transport cost

$$C(\psi) := \int_X |x - \psi(x)| g_0(x)\, dx.$$

The condition that the first distribution of mass is carried into the second can be written as

$$\int_{\psi^{-1}(B)} g_0(x)\, dx = \int_B g_1(y)\, dy \qquad \forall B \subset X \text{ Borel} \tag{1}$$

* This work has been partially supported by the GNAFA project "Problemi di Monge–Kantorovich e strutture deboli"

or, by the change of variables formula, as

$$g_1(\psi(x))\,|\det\nabla\psi(x)| = g_0(x) \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in B$$

if ψ is one to one and sufficiently regular.

More generally one can replace the functions g_0, g_1 by positive measures f_0, f_1 with equal mass, so that (1) reads $f_1 = \psi_\# f_0$, and replace the euclidean distance by a generic cost function $c(x, y)$, studying the problem

$$\min_{\psi_\# f_0 = f_1} \int_X c(x, \psi(x))\, df_0(x). \tag{2}$$

The infimum of the transport problem leads also to a c-dependent distance between measures with equal mass, known as Kantorovich–Wasserstein distance.

The optimal transport problem and the Kantorovich–Wasserstein distance have a very broad range of applications: Fluid Mechanics [7, 8]; Partial Differential Equations [32, 29]; Optimization [13, 14] to quote just a few examples. Moreover, the 1-Wasserstein distance (corresponding to $c(x, y) = |x - y|$ in (2)) is related to the so-called flat distance in Geometric Measure Theory, which plays an important role in its development (see [6], [24], [30], [27], [44]). However, rather than showing specific applications (for which we mainly refer to the Evans survey [21] or to the introduction of [7]), the main aim of the notes is to present the different formulations of the optimal transport problem and to compare them, focussing mainly on the linear case $c(x, y) = |x-y|$. The main sources for the preparation of the notes have been the papers by Bouchitté–Buttazzo [13, 14], Caffarelli–Feldman–McCann [15], Evans–Gangbo [22], Gangbo–McCann [26], Sudakov [42] and Evans [21].

The notes are organized as follows. In Section 1 we discuss some basic examples and in Section 2 we discuss Kantorovich's generalized solutions, i.e. the transport plans, pointing out the connection between them and the transport maps. Section 3 is entirely devoted to the one dimensional case: in this situation the order structure plays an important role and considerably simplifies the theory. Sections 4 and 5 are devoted to the ODE and PDE based formulations of the optimal transport problem (respectively due to Brenier and Evans–Gangbo); we discuss in particular the role of the so-called transport density and the equivalence of its different representations. Namely, we prove that any transport density μ can be represented as $\int_0^1 \pi_{t\#}(|y - x|\gamma)\, dt$, where γ is an optimal planning, as $\int_0^1 |E_t|\, dt$, where E_t is the "velocity field" in the ODE formulation, or as the solution of the PDE $\mathrm{div}(\nabla_\mu u\mu) = f_1 - f_0$, with no regularity assumption on f_1, f_0. Moreover, in the same generality we prove convergence of the p-laplacian approximation.

In Section 6 we discuss the existence of the optimal transport map, following essentially the original Sudakov approach and filling a gap in his original proof (see also [15, 43]). Section 7 deals with recent results, related to those obtained in [25], on the regularity and the uniqueness of the transport density.

Section 8 is devoted to the connection between the optimal transport problem and the so-called mass optimization problem. Finally, Section 9 contains a self contained list of the measure theoretic results needed in the development of the theory.

Main notation

X	a compact convex subset of an Euclidean space $\mathbb{R}^n$
$\mathcal{B}(X)$	Borel σ-algebra of X
$\mathcal{L}^n$	Lebesgue measure in $\mathbb{R}^n$
$\mathcal{H}^k$	Hausdorff k-dimensional measure in $\mathbb{R}^n$
$\mathrm{Lip}(X)$	real valued Lipschitz functions defined on X
$\mathrm{Lip}_1(X)$	functions in $\mathrm{Lip}(X)$ with Lipschitz constant not greater than 1
Σ_u	the set of points where u is not differentiable
π_t	projections $(x, y) \mapsto x + t(y - x)$, $t \in [0, 1]$
$\mathcal{S}_o(X)$	open segments $]x, y[$ with x, $y \in X$
$\mathcal{S}_c(X)$	closed segments $[\![x, y]\!]$ with x, $y \in X$, $x \neq y$
$\mathcal{M}(X)$	signed Radon measures with finite total variation in X
$\mathcal{M}_+(X)$	positive and finite Radon measures in X
$\mathcal{M}_1(X)$	probability measures in X
$\|\mu\|$	total variation of $\mu \in [\mathcal{M}(X)]^n$
μ^+, μ^-	positive and negative part of $\mu \in \mathcal{M}(X)$
$f_{\#}\mu$	push forward of μ by f

1 Some elementary examples

In this section we discuss some elementary examples that illustrate the kind of phenomena (non existence, non uniqueness) which can occur. The first one shows that optimal transport maps need not exist if the first measure f_0 has atoms.

Example 1.1 (Non existence). Let $f_0 = \delta_0$ and $f_1 = (\delta_1 + \delta_{-1})/2$. In this case the optimal transport problem has no solution simply because there is no map ψ such that $\psi_{\#} f_0 = f_1$.

The following two examples deal with the case when the cost function c in $X \times X$ is $|x - y|$, i.e. the euclidean distance between x and y. In this case we will use as a test for optimality the fact that the infimum of the transport problem is always greater than

$$\sup \left\{ \int_X u \, d(f_1 - f_0) : \; u \in \mathrm{Lip}_1(X) \right\}. \tag{3}$$

Indeed,

$$\int_X u\,d(f_1 - f_0) = \int_X u(\psi(x)) - u(x)\,df_0(x) \leq \int_X |\psi(x) - x|\,df_0(x)$$

for any admissible transport ψ. Actually we will prove this lower bound is sharp if f_0 has no atom (see (6) and (13)).

Our second example shows that in general the solution of the optimal transport problem is not unique. In the one-dimensional case we will obtain (see Theorem 3.1) uniqueness (and existence) in the class of nondecreasing maps.

Example 1.2 (Book shifting). Let $n \geq 1$ be an integer and $f_0 = \chi_{[0,n]}\mathcal{L}^1$ and $f_1 = \chi_{[1,n+1]}\mathcal{L}^1$. Then the map $\psi(t) = t + 1$ is optimal. Indeed, the cost relative to ψ is n and, choosing the 1-Lipschitz function $u(t) = t$ in (3), we obtain that the supremum is at least n, whence the optimality of ψ follows. But since the minimal cost is n, if $n > 1$ another optimal map ψ is given by

$$\psi(t) = \begin{cases} t + n & \text{on } [0,1] \\ t & \text{on } [1,n]. \end{cases}$$

In the previous example the two transport maps coincide when $n = 1$; however in this case there is one more (and actually infinitely many) optimal transport map.

Example 1.3. Let $f_0 = \chi_{[0,1]}\mathcal{L}^1$ and $f_1 = \chi_{[1,2]}\mathcal{L}^1$ (i.e. $n = 1$ in the previous example). We have already seen that $\psi(t) = t + 1$ is optimal. But in this case also the map $\psi(t) = 2 - t$ is optimal as well.

In all the previous examples the optimal transport maps ψ satisfy the condition $\psi(t) \geq t$. However it is easy to find examples where this does not happen.

Example 1.4. Let $f_0 = \chi_{[-1,1]}\mathcal{L}^1$ and $f_1 = \delta_{-1} + \delta_1$. In this case the optimal transport map ψ is unique (modulo $\mathcal{L}^1$-negligible sets); it is identically equal to -1 on $[-1,0)$ and identically equal to 1 on $(0,1]$. The verification is left to the reader as an exercise.

We conclude this section with some two dimensional examples.

Example 1.5. Assume that $2f_0$ is the sum of the unit Dirac masses at $(1,1)$ and $(0,0)$, and that $2f_1$ is the sum of the unit Dirac masses at $(1,0)$ and $(0,1)$. Then the "horizontal" transport and the "vertical" transport are both optimal. Indeed, the cost of these transports is 1 and choosing $u(x_1,x_2) = x_1$ in (3) we obtain that the infimum of the transport problem is at least 1.

Example 1.6. Assume that f_1 is the sum of two Dirac masses at A, $B \in \mathbb{R}^2$ and assume that f_0 is supported on the middle axis between them. Then

$$\int_X |x - \psi(x)|\,df_0(x) = \int_X |x - A|\,df_0(x)$$

whenever $\psi(x) \in \{A, B\}$, hence *any* admissible transport is optimal.