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Edited by A. Dold and B. Eckmann

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To my Parents

Preface

It is a pleasure for me to acknowledge my debt to all those who have contributed to this book.

My interest in fixed point theory originates from lectures given by Albrecht Dold at the university of Heidelberg. I am especially thankful to him and to Dieter Puppe who taught me most of what I know about algebraic topology and homotopy theory. Concerning the book's actual contents, I could further profit from various papers of Tammo tom Dieck and his pupils treating related topics.

I am very grateful to my friends Mónica Clapp and Carlos Prieto for their repeated encouragements to publish this text, and I am obliged to the Springer-Verlag for accepting it into its Lecture Notes of Mathematics series. The project, however, could not have been realized without the generous assistance of IBM. I was impressed how the IBM research and development laboratory in Böblingen promoted the scientific interests of one of its employees which, by no means, were related to the company's vital business interests.

IBM subsidized the translation of the German original done by Andreas Schroth and Christiane Vierthaler. I owe both of them sincere thanks for their careful efforts. The tiresome work of typing the manuscript and formatting the text to its final layout was considerably facilitated by IBM's text processing tools. For printing, an IBM 3820 all-points-addressable printer was at my disposal. I am very much indebted to my managers Karl Niebel and Günther Sonntag who supported the publication and who gave me the time to complete it.

Finally, I wish to thank my wife for her patient understanding and for her moral help over the past year.

Tübingen, fall 1987 *Hanno Ulrich*

Contents

Introduction

In this book, we are developing an *algebraic fixed point theory* for *equivariant maps*, that is for maps with symmetry properties.

So, let E be a topological space with a continuous action of some topological group G, a *G-space* in short, and let f be a partially defined G-transformation on E: I.e. the domain of f has to be a G-subspace of E and f is to respect the action of G. Maps are always understood to be continuous.
More generally, we consider *continuous G-families* $f = \{f_b: V_b \to E_b,\ b \in B\}$ of such maps where G may act non-trivially on the parameter space B: An element $g \in G$ transports the fibre f_b of f to the fibre over $gb \in B$ and for $v \in V_b$, $f_{gb}(gv)$ equals $gf_b(v)$. If we let E denote the union of all E_b, $b \in B$, then G acts on E and $p: E \to B$, $p^{-1}(b) := E_b$, is a G-map. $V := \bigcup_{b \in B} V_b$ is a G-subspace of E over B and f becomes a *vertical G-map*

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

with $pf(v) = p(v)$, $f(gv) = gf(v)$, and $p(gv) = gp(v)$. We call f a *G-fixed point situation over B* if its fixed point set $\mathrm{Fix}(f) := \{v \in V, f(v) = v\}$ lies properly over B, if V is open in E, and if p is a G-ENR_B, in words a *G-euclidian neighbourhood retract over B*. So, there exist a *G-module M*, i.e. a linear representation of G in some euclidean space $M = \mathbb{R}^n$, and a vertical G-embedding of p into the projection $B \times M \to B$ such that therein, p is a vertical G-retract of some G-neighbourhood. We assume that the base B *is paracompact and compactly generated*. Then the fixed point set of f lies properly over B if and only if it is closed and fibrewise uniformly bounded as a subspace of $B \times M$.
In [Dold 2], any non-equivariant fixed point situation f over B gets assigned a *fixed point index* $I(f)$ in the *zeroth stable cohomotopy group* $\pi_s^0(B)$ of B constructed as follows: Each fibre of f has a Hopf index $I(f_b) \in \mathbb{Z}$ represented by a transformation $i(f_b)$ of the pointed n-sphere, or of the pair $(\mathbb{R}^n, \mathbb{R}^n - 0)$, of degree $I(f_b)$. The family of all these transformations $i(f_b)$ can be constructed so as to depend continuously on the parameter $b \in B$. It then constitutes a map

$$i(f): B \times (\mathbb{R}^n, \mathbb{R}^n - 0) \to (\mathbb{R}^n, \mathbb{R}^n - 0)$$

which represents the fixed point index $I(f) \in \pi_s^0(B)$. When f is a G-fixed point situation with G a compact (Lie) group, the analogous construction yields a G-map

$$i_G(f)\colon B \times (M, M-0) \to (M, M-0).$$

which will represent the *G-fixed point index $I_G(f)$ in the zeroth stable G-cohomotopy group* $\pi_G^0(B)$. The fixed point index of f in any other *multiplicative G-cohomology theory* h_G is the image of $I_G(f)$ under the degree map $u : \pi_G{}^*(B) \to h_G{}^*(B)$. The h_G-index of the identity on a proper G-ENR$_B$ p will be called the *h_G-characteristic of p*, denoted by $\chi_{h_G}(p)$. Two G-fixed point situations f_0 and f_1 over B are said to be *equivalent* if they can be connected through a G-fixed point situation $f = \{f_t\}$ over $B \times [0, 1]$. The equivalence classes form a ring with unit element, the *G-fixed point ring $Fix_G(B)$*. By virtue of its basic properties, the fixed point index in a multiplicative G-cohomology theory h_G defines a homomorphism of unitary rings

$$I_{h_G} : Fix_G(B) \to h_G^0(B).$$

We show that $I_G := I_{\pi_G}$ is an *isomorphism.* It will be indicated only how stable G-cohomotopy classes of non-trivial degree α can be realized by G-fixed point situations of degree α where α is an element of the real representation ring of G.
By interpreting maps between equivariant cohomotopy groups in terms of equivariant fixed point theory, we attain a geometrical view of the *equivariant group completion map* and thus of the mapping in the *Segal conjecture.* And by means of the *fixed point transfer in equivariant K-theory,* we can describe what is meant by *induced representations in the category of compact Lie groups.*

The *sum formula* in Chapter III, Section 5 is a crucial tool for calculating the equivariant fixed point index over a point: The h_G-index of a G-fixed point situation f decomposes into an integral linear combination of the h_G-characteristics of the orbit types of G around its fixed point set, that is

$$I_{h_G}(f) = \sum_{(H)} n_H(f) \cdot \chi_{h_G}(G/H).$$

χ_{h_G} vanishes on all orbit types (G/H) whose G-automorphism group $W(H) = N(H)/H$ is not finite whereas otherwise, $n_H(f)$ is the Hopf index

$$n_H(f) = \left(I(f^H) - I(f^{\underline{H}})\right) / \,|W(H)| \in \mathbb{Z}.$$

f^H and $f^{\underline{H}}$ denote the fixed point situations induced by f in the H-fixed point spaces $E^H := \{x \in E,\ hx = x \text{ for all } h \in H\}$ and $E^{\underline{H}} := \bigcup_{K>H} E^K$. For example, if $S \leq G$ is a torus, then the *Hopf index* $I(f)$ of f equals $I(f^S)$, and if G is a finite p-group, it is congruent to $I(f^G)$ modulo p. Therefore, the Hopf index of any odd transformation of a sphere is even. This is the *Borsuk-Ulam theorem.* Generalizations follow at once by means of the sum formula.

We will further discuss the relation between the index of a G-fixed point situation f and that of the map f/G induced on orbit spaces - provided of course, the latter's fixed point set is compact. To illustrate the ideas developed, we derive three results of A. Weil, H. Hopf, and D.H. Gottlieb which nowadays are folklore in the theory of compact Lie groups.

For the identity on a compact G-ENR E, the sum formula takes the form

$$\chi_{h_G}(E) = \sum_{(H)} \chi_c(E_{(H)}/G) \cdot \chi_{h_G}(G/H)$$

where $E_{(H)} \subset E$ is the G-subspace consisting of all points on orbits of type (G/H) and χ_c denotes the Euler-Poincaré characteristic in singular or Čech cohomology with compact support. Thus the *Euler-Poincaré characteristic of E* decomposes as

$$\chi(E) = \sum_{(H)} \chi_c(E_{(H)}).$$

Regarding again the sum formula for general f, the last result might suggest that the $(H)^{\text{th}}$ term would be counting the fixed points of f in $E_{(H)}$. This, however, is not correct as a fixed point of f living in E^K with $K > H$ may be counted by f^K with a multiplicity different from that counted by f^H.

Interpreted in stable G-cohomotopy theory, the sum formula purports that every G-fixed point situation over a point is equivalent to the identity on a suitable compact G-ENR. The *coefficient ring* $F(G) := Fix_G(\text{pt})$ of G-fixed point theory even coincides with the *Burnside ring* $A(G)$ of G and our isomorphism $I_G\colon F(G) \to \pi^0_G(\text{pt})$ provides a *geometrical* *view of the tom Dieck isomorphism* $A(G) \cong \pi_G^0(\mathrm{pt})$ [tom Dieck 2]. For, $A(G)$ is defined as the set of equivalence classes of compact G-ENRs under the equivalence relation $E \sim E'$ if $\chi(E^H) = \chi(E'^H)$ for all $H \leq G$; and $f \mapsto (I(f^H))$ induces an injective ring homomorphism $F(G) \to \prod_{H \leq G} \mathbb{Z}$. Indeed, using the sum formula, we see that already the corresponding homomorphism

$$I^*: F(G) \to \mathbb{Z}^{\Phi(G)}, \quad [f] \mapsto (I(f^H))$$

is injective where $\Phi(G)$ denotes the set of those orbit types of G whose G-automorphism group is finite. By inverting the matrix of I^* - as an abelian group, $F(G)$ is free on $\Phi(G)$ - we can *describe the image of* $F(G)$ *in* $\mathbb{Z}^{\Phi(G)}$ *by relations and congruences:* For every finite subset $\mathscr{H} \subset \Phi(G)$, there exists a relation specifying in $\mathbb{Z}^{\Phi(G)}$ a free subgroup $C_{\mathscr{H}}$ of rank $|\mathscr{H}|$ which contains the image of the subgroup of $F(G)$ free on $\mathscr{H}$, and further a set of congruences determining this image in $C_{\mathscr{H}}$.

The inverse of I^* is a three-dimensional matrix **M** over $\mathbb{Z}$. We present an explicit recipe to calculate its entries. For G finite, **M** reduces to a two-dimensional matrix. And if G is a finite cyclic group generated by some some g, **M** provides us with congruences between the Hopf indices of the iterates g^k of g.

In the union of all $C_{\mathscr{H}}$, we can describe the image of $F(G)$ by a closed set of congruences - even if G is not finite. It is derived from the following *congruence* holding for finite group actions:

$$\sum_{g \in G} I(f^g) \equiv 0 \mod |G|.$$

The superscript "g" stands for the subgroup of G generated by g. This relation, in turn, will be deduced by several ways: First, it is an immediate consequence of the sum formula which, for the identity on a compact G-ENR, reads

$$\sum_{g \in G} \chi(E^g) = \chi(E/G)\,|G|.$$

Second, we find an elementary approach by investigating local Hopf indices, and finally, we will encounter it again in the last section when we are analyzing the fixed point index in equivariant K-theory.

For *in equivariant K-theory*, the index of a G-fixed point situation f over a point is an element of the complex representation ring $R(G) = K_G(\mathrm{pt})$ of G. Employing the sum formula, we can calculate its character function as

$$I_{K_G}(f)\colon G \to \mathbb{Z}, \quad g \mapsto I(f^g).$$

In particular, the K_G-characteristic of a compact G-ENR is the virtual representation

$$\chi_{K_G}(E) = \sum (-1)^i H^i(E; \mathbb{C})$$

of G over $\mathbb{C}$. For a finite G-set E, this is the permutation representation $\mathbb{C}(E)$ of G.
We end our discourse with a description of the *K_G-transfer* in terms of *Atiyah's topological index homomorphism* t-ind and a final application of the *Atiyah-Singer index formula*.

These are the contents of **Chapter III.** For the sake of completeness, we have enclosed an **Appendix** on *proper maps*.

The results from the *theory of compact transformation groups* used in this book are listed in **Chapter I**. There we also prove a simplified version of the *equivariant transversality theorem* required lateron when we will discuss local Hopf indices.

G-ENR$_B$s are discussed in **Chapter II.** Throughout, G will be a compact group equipped with a Lie-structure if necessary. In the results stated, B is assumed to be paracompact.

We confirm the *properties of* G-ENR$_B$s well-known for the non-equivariant case. So, a closed vertical G-subspace q of a G-ENR$_B$ p is a G-ENR$_B$ if and only if the inclusion $q \to p$ is a *G-cofibration over B*. Proper G-ENR$_B$s are G-fibrations. Arbitrary G-ENR$_B$s, however, only enjoy the *G-beginning covering homotopy property* G-BCHP: In general, G-homotopies on the base can be G-lifted just a bit.
It turns out that a vertical G-space $p\colon E \to B$ over a G-ENR is a G-ENR$_B$ if and only if it has the G-BCHP and E is a G-ENR. This allows us to confirm a conjecture proposed in [Dold 3] for the non-equivariant case: Over a G-ENR B, a G-ENR$_B$ is characterized as a vertical G-space which has the G-BCHP and all of whose fibres are G-ENRs.

A smooth manifold is an ENR and a submersion of smooth manifolds is a vertical ENR since it comes with a family of local cross sections and hence is a C^0-submersion. We formulate this equivariantly in three versions of different strength and investigate how such *G-C^0-submersions* are related to spaces having the G-BCHP.
While the G-BCHP distinguishes vertical G-ENRs over a G-ENR, a vertical G-space whose total space is a G-ENR, proves to be a G-ENR$_B$ if and if only it is a G-C^0-submersion. We will see however that in a G-ENR$_B$ p with surjective mappings p^H for all $H \leq G$, the base is already a G-ENR if - and hence only if - the total space is one.

To prove the last two results, we employ a *vertical generalization of the Jaworowski criterion for G-ENRs with a compact Lie group*: A closed G-subspace $p\colon E \to B$ of a vertically trivial G-ENR$_B$ $B \times M \to B$ over a sufficiently nice base is a G-ENR$_B$ if and only if p^H is a vertical ENR over B^H for each orbit type (H) on $B \times M - E$.

Examples of G-ENR$_B$s are provided by *locally trivial fibre bundles with action of a compact Lie group G*. If such a fibre bundle is locally trivial in an equivariant sense, it proves to be an extra strong G-C^0-submersion and hence a G-fibration. This is the homotopy theorem in [tom Dieck 1].
If p is a G-locally trivial bundle of finite type over a suitable base, we can specify ENR-conditions for the non-equivariant fibre F which identify p as a G-ENR$_B$. In G-vector bundles, for instance, these conditions are satisfied. From this, we can derive an ENR-condition for F sufficient to make p a G-ENR$_B$ even if p is not G-locally trivial.
For illustration, we study the projection of a G-ENR E onto its orbit space: It will be a vertical G-ENR if and only if there is only one orbit type on E - locally at least.

We end the chapter with various examples. From an inductive criterion, we deduce that smooth G-manifolds of finite orbit structure are G-ENRs. Hence, a G-C^∞-submersion of such manifolds is a vertical G-ENR, for it is a G-C^0-submersion. Finally, with regard to our sum formula, we show that the saturations $p^{(H)} := G \cdot p^H$ and $p^{\underline{(H)}} := G \cdot p^{\underline{H}}$ of the H-fixed point spaces in a G-ENR$_B$ p are G-ENR$_B$s for themselves.

Further comments will be found in the extensive introductions preceding each chapter. We use the symbol □ to mark the end of a proof whereas the symbol ⌑ indicates that the proof remains to be completed in a subsequent section.

CHAPTER I
Preliminaries on Group Actions

The **first two sections** establish the notational conventions and list some classical results on compact transformation groups. For details, we refer the reader to the books [Bredon] and [Palais 2]. In addition, we provide the following two results required in the subsequent chapters.
Let G be a *compact Lie group*. The set of conjugacy classes of closed subgroups of G comes with a natural partial ordering and we first show how it may be refined to a well-ordering. Second, some useful *sum formulae of R.S. Palais*, relating the dimension of a G-space to that of its orbit space, will be generalized from separable, metrizable G-spaces to just metrizable and even to paracompact, perfectly normal ones.

The entire **Section 3** has been devoted to derive a simplified version of the *equivariant transversality theorem* following along the lines of [Hausschild 1].

In the final **Section 4**, we will sketch the local characterization of *G-cofibrations* and *G-fibrations* whereby G is merely a compact group.

1. Some Set Theory on Compact Lie Groups

1.1 Let G always denote a *compact topological group*. Continuous vector valued functions on G can thus be integrated. If necessary, G comes with the structure of a Lie group.
A *subgroup* $H \leq G$ is always to be a *closed* subspace of G. Hence, H as well as G is a compact (Lie) group. Let (H) denote the *conjugacy class* of H in G. $(H) \leq (K)$ means that H is subconjugate to K. G/H stands for the set of *left cosets* gH of H in G. By $W(H)$, we denote the *Weyl group* $N(H)/H$ where $N(H)$ is the *normalizer* of H in G. We write $H \trianglelefteq G$ in case H is a *normal* subgroup of G. $\langle g \rangle \leq G$ is the topological subgroup *generated* by $g \in G$. Its order is denoted by $|g|$.

If G is a compact Lie group, then, up to conjugation, the number of subgroups $H \leq G$ is countable ([Palais 2]). In this case, therefore, the partial ordering " $\leq$ " on the set of

conjugacy classes in G can readily be refined to a total ordering - simply by a numeration with real numbers:
One enumerates the conjugacy classes (H) with natural numbers and places (H_1) at the origin, say. If (H_2) and (H_1) are incomparable, we assign to (H_2) any real number different from zero. Otherwise, (H_2) gets a positive real number in case $(H_2) > (H_1)$ and a negative one else. We then place (H_3) at a suitable point and continue.
Of course, we will not get a well-ordering that way. But as indicated, we can show:

1.2 Theorem. *Let G be a compact Lie group. The partial ordering "$\leq$" on the set of conjugacy classes $\{(H)\,,\ H \leq G\}$ in G can be refined to a well-ordering "$\preccurlyeq$".*

PROOF. Let $\mathscr{H}$ be any non-empty set of conjugacy classes in G and select some $(H_1) \in \mathscr{H}$. If (H_1) is not minimal with respect to the partial ordering "$\leq$" on $\mathscr{H}$, then there exists a subgroup $H_2 < H_1$ with $(H_2) \in \mathscr{H}$. If (H_2) neither is minimal in $\mathscr{H}$, we find some (H_3) in $\mathscr{H}$ such that $H_3 < H_2 < H_1$. This chain will cease eventually since, as a proper submanifold, H_{i+1} has less components or a smaller dimension than H_i. Hence, there is a minimal element in $\mathscr{H}$.
In other words, the set of conjugacy classes in G satisfies the descending chain condition. The assertion now follows from the next lemma. □

1.3 Lemma. *Let (X, ω) be a non-empty, partially ordered set. The ordering ω can be refined to a well-ordering on X if and only if ω satisfies the descending chain condition.*

PROOF. The descending chain condition on ω is necessary because a totally ordered set is well-ordered if and only if it satisfies the descending chain condition.
Conversely, we consider the set W of all well-orderings w_A refining ω which are defined on subsets $A \subset X$ with the property that $a \in A$ implies $x \in A$ for all ω-predecessors of a.
By calling w_A smaller than w_B if w_A is an initial segment in w_B, we equip W with an inductive ordering: For, any chain $\{w_A\}$ in W defines, on the union of all its domains A, a well-ordering belonging to W which yields an upper bound for the chain. Clearly, W is non-empty - the empty set for instance belongs to W. Hence, Zorn's Lemma provides a maximal element w_M in W.
If $X - M$ were non-empty, then, because of the descending chain condition, we could find therein an element x minimal with respect to ω. Putting x behind all of M, we would define a well-ordering in W strictly geater than w_M: For, all ω-predecessors of x were in M because of the minimum choice of x while, by definition, none of the ω-successors of x could belong to M. □

2. On the Topology of Spaces with Group Action

2.1 Throughout the section, let G be a *compact group*.

By a G-space, we understand a topological space X equipped with a continuous action of G from the left, i.e. with a continuous multiplication $(-\cdot-) : G \times X \to X$. An equivariant mapping of X - a *G-map* in short - is a continuous map f from X to some other G-space respecting the group action: $f(gx) = gf(x)$. By a *G-transformation* of X, we mean an equivariant self mapping of X.
G/H for example is a G-space for any $H \leq G$. The set of its G-transformations is the Weyl group $W(H) = N(H)/H$.

2.2 For any G-subspace $A \subset X$ the closure $\overline{A}$, the interior $\overset{\circ}{A}$, and the complement $X - A$ are G-subspaces of X. If A is the zero-set of some function $\tau : X \to [0, 1]$, we may assume that τ is a G-function. For otherwise, we integrate τ over G.
The *orbit* of a point $x \in X$ is denoted by Gx and the *isotropy subgroup* of G at x by G_x. Since G is compact, the natural map $G/G_x \to Gx$, $gG_x \mapsto gx$ is a G-homeomorphism.
Every neighbourhood of an orbit in X contains a G-neighbourhood since G is compact. If X is a Hausdorff space, the closed G-neighbourhoods constitute neighbourhood bases around the orbits of G, for the action of a compact group is a closed map.

2.3 The *orbit space* of X will be denoted by X/G. The projection $X \to X/G$ is proper, hence in particular closed, and open. Therefore, X/G is a Hausdorff, a (completely) regular, or a (perfectly) normal space, just as X is. Furthermore, X/G will be (locally) compact, Lindelöf-compact, or compactly generated if and only if X is so. In particular, when X is separable and metrizable, then so is X/G. According to a theorem of E. Michael, X/G is paracompact exactly if X is. All these results are detailed in the book [Engelking].

2.4 By the *type of an orbit* we mean its equivariant topological type. The set of orbit types on X comes with a partial ordering, namely type$(Gx) \leq$ type(Gy) if there is a G-map from Gx to Gy. This in turn holds if and only if the isotropy group G_x is subconjugate to G_y, i.e. if $(G_x) \leq (G_y)$.
Without any harm, we may therefore identify the orbit type of Gx with the conjugacy class of G_x.

2.5 For $H \leq G$, let X^H denote the H-fixed point set of X, i.e. the set of all points in X whose isotropy group contains H. X^H carries an obvious action of $N(H)$ or $W(H)$. We write $X^{(H)}$ for the G-saturation GX^H. It consists of all points in X whose isotropy group H is subconjugate to.
By $X_{(H)}$, we denote the set of all points on orbits of type (G/H), and by $X_H \subset X_{(H)}$ the subspace of all points whose isotropy group is exactly H. The complement $X^{\underline{(H)}}$ of $X_{(H)}$ in $X^{(H)}$ consists of all those points whose isotropy group is of type strictly greater than (H).

2.6 A G-map $f: X \to Y$ induces maps $f^H: X^H \to Y^H, f^{(H)}: X^{(H)} \to Y^{(H)}$, and $f^{\underline{(H)}}: X^{\underline{(H)}} \to Y^{\underline{(H)}}$ since for every $x \in X$, we have $G_x \leq G_{f(x)}$.
Observe however that f maps the pair $(X_{(H)}, X_H)$ to $(Y_{(H)}, Y_H)$ only if $G_{f(x)}$ equals G_x for all points x on orbits of type (G/H). This is the case if and only if f is injective on each such orbit Gx, i.e. if Gx gets mapped homeomorphically onto $Gf(x)$. If this holds at all points in X, we say that f is *isovariant*.

2.7 By a *G-module M*, we mean a real vector space of finite dimension equipped with a linear G-action from the left. Thus, M is a linear representation of G over $\mathbb{R}$. We will emphasize specifically, when M is to be a complex G-module.
As G is a compact group, we can integrate over G. Therefore, any linear representation of G is equivalent to an orthogonal or a unitary one. More generally, we can equip any metrizable G-space X with a *G-invariant metric* which in turn induces a metric on X/G generating the identification topology.

2.8 The *Tietze-Gleason theorem* is the equivariant version of the Tietze extension theorem: If X is a normal G-space, then any G-map f from a closed G-subspace $A \subset X$ to a G-module M can be extended to a G-map $F: X \to M$.
In other words, a G-module is an *equivariant absolute extensor*, a G-AE in short, for the class of normal G-spaces.

If the G-map f given on A takes its values in some real interval I with trivial G-action, then, of course, we can arrange that its G-extension F remains in I as well.
Or, if A_0 and A_1 are disjoint closed G-subspaces of X, then there exists a G-function $\tau: X \to [0, 1]$ separating A_0 and A_1, i.e. such that $\tau(A_i) = i$ for $i = 0, 1$.

2.9 We show that paracompact G-spaces are *G-numerable*, that is numerable in an equivariant sense.

Proposition. *Let X be a G-space with G a compact group. If $\underline{U}$ is a numerable covering of X by G-subspaces, then $\underline{U}$ is numerable by G-functions.*

PROOF. Let $\{u,\ U \in \underline{U}\}$, be a partition of unity subordinate to $\underline{U}$. By integrating over G, we make the functions u G-functions $u^G\colon X \to [0, 1]$. The support of u^G is contained in U since U is a G-subspace of X. It remains to show that the familiy $\{u^G\}$ is a locally finite partition of unity.
$\{u\}$ is a locally finite family and therefore, keeping $x \in X$ fixed, we find, for every $g \in G$, a neighbourhood of gx in which only a finite number of the functions u does not vanish identically. I.e. for every $g \in G$, there are neighbourhoods $V_g \subset G$ of g and $W_g \subset X$ of x such that $u(V_g \cdot W_g) = 0$ holds for all u up to a finite number of exceptions, say $u \in \underline{U} - \underline{U}_g$. Now, take a finite subset of $\{V_g\}$ which covers G. Let W be the intersection of the corresponding subset of $\{W_g\}$ and $\underline{U}_x$ that of $\{\underline{U}_g\}$ Then W is a neighbourhood of x, $\underline{U} - \underline{U}_x$ is a finite set, and for each $U \in \underline{U}_x$, we have $u(GW) = 0$. On W, therefore, u^G vanishes for every $U \in \underline{U}_x$. Hence, the family $\{u^G\}$ is locally finite. And at each point $x \in X$, it sums up to

$$\sum_{\underline{U} - \underline{U}_x} u^G(x) = \int_G \sum_{\underline{U} - \underline{U}_x} u(gx) = \int_G \sum_{\underline{U}} u(gx)$$

which is 1. □

2.10 For the rest of the section, let G be a *compact Lie group.*

A G-space X now enjoys a nice local structure provided only, it is completely regular. For then, any point $x \in X$ lies on a G_x-slice. This is the *slice theorem of J.L. Koszul* ([Koszul]). Remember that for $H \leq G$, an *H-slice* in X is an H-subspace $S \subset X$ for which the multiplication $G \times_H S \to GS \subset X$ is an open G-embedding. We call S an *H-kernel* if we do not care about whether GS is open in X or not.

2.11 Among the various consequences of the slice theorem, we are particularly interested in the following results:

In a completely regular G-space X, with G a compact Lie group, every orbit is a G-neighbourhood retract. In particular, every point $x \in X$ has a neighbourhood in which all