FUNCTIONAL ANALYSIS

an introduction

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PREFACE

The exposition in the following pages is based on lectures I gave to second year mathematics graduate students at Wesleyan University during the academic years 1970 - 71 and 1971 - 72. The intent of the lectures was to provide the student with an introduction to functional analysis that not only presented the basic notions, theorems, and techniques of the subject, but also gave a modest sampling of the applications of functional analysis. This remains the goal of the present book. The choice of the topics and applications is clearly idiosyncratic, and I make no claim to a balanced, let alone definitive, treatment. I will consider the book a success if it but convinces the reader of the beauty, power, and utility of functional analysis.

An intelligent reading of the book presupposes at most the usual mathematical equipment possessed by second year mathematics graduate students. The main items required are some knowledge of point set topology, linear algebra, and elementary complex analysis, together with a good background in measure and integration theory, say, for example, as in Royden's book "Real Analysis" [Ry]. Results with which it is assumed the reader is familiar are frequently cited without further elaboration. However, in almost all such instances an appropriate reference is given.

The last section in each chapter consists of problems that hopefully are do-able with the material developed up to that point. An asterisk before a problem generally indicates that the problem may be of a more substantial nature or, in some cases, that it contains a result of particular importance. Most of the results in the body of the text that are formally stated without proof appear again in the problem sections and such problems are cross referenced accordingly.

The conclusion of proofs is indicated by the symbol □ at the right hand margin.

I would like to thank all of the graduate students at Wesleyan who passed through my course while this book was evolving for their comments and suggestions. In particular, I would like to thank David DeGeorge, Hans Engenes, Polly Moore Hemstead, and Michael Paul for their often perspicacious observations and questions that more than once kept me from foolish error. Those errors that remain, foolish or otherwise, are of course my own responsibility.

I am especially grateful to Polly Moore Hemstead, who not only passed through the course but also provided me with valuable editorial assistance and collected and organized the problem sets at the end of each chapter. Her efforts have greatly enhanced the final form of the book.

I would also like to thank Helen Diehl, who typed all of the original manuscript and a good deal of the final one.

Finally, thanks are due to the editors and staff of Marcel Dekker for their cheerful and expert cooperation during the production of the book.

Middletown, Connecticut
January, 1973

CONTENTS

CHAPTER 1

SEMINORMED AND NORMED LINEAR SPACES

1.0. Introduction. In this chapter we develop some of the basic concepts and results concerning seminormed and normed linear spaces. We begir with the fundamental definitions and some elementary theorems about the construction of new normed linear spaces from given normed linear spaces. This is followed by a collection of examples, by no means exhaustive, of seminormed and normed linear spaces. The verifications are left to the reader and are, in some cases, nontrivial. References are cited at the end of the section.

Section 1.3 contains a discussion of finite-dimensional normed linear spaces in which it is shown that all such spaces are topologically ismorphic and that a normed linear space is finite-dimensional if and only if closed bounded sets are compact. In the next two sections we discuss the concept of the gauge of a convex balanced absorbing set, the relationship between gauges and seminorms, and the introduction of a topology into a seminormed linear space.

1.1. Basic Definitions. We begin with several fundamental definitions. Throughout, Φ will denote either the real-number field or the complex-number field, $\mathbb{R}$ and $\mathbb{C}$ will stand for the real and complex fields, respectively, and $\mathbb{Z}$ will always denote the integers.

Definition 1.1.1. A linear space V over a field Φ is a commutative group under a binary operation +, together with an operation of scalar multiplication from $\Phi \times V$ to V, denoted by juxtaposition, such that

(i) $a(x + y) = ax + ay$,

(ii) $(a + b)x = ax + bx$,

(iii) $a(bx) = (ab)x$,

(iv) $1x = x$ $\qquad (a,b \in \Phi;\ x,y \in V)$.

Of course, 1 denotes the multiplicative identity in Φ, and the operations of addition and multiplication in Φ have been indicated in the usual manner; 0 will be used to denote the additive identity in both V and Φ. The context will make clear which is meant.

Definition 1.1.2. Let V be a linear space over Φ. Given $E \subset \Phi$; $a \in \Phi$; $A,B \subset V$; and $x_o \in V$, we set

$$A + B = \{x + y \mid x \in A,\ y \in B\},$$
$$x_o + B = \{x_o + y \mid y \in B\},$$
$$EA = \{ax \mid a \in E,\ x \in A\},$$
$$aA = \{ax \mid x \in A\}.$$

Definition 1.1.3. Let V be a linear space over Φ and let $W \subset V$. Then W is a linear subspace of V if $W + W \subset W$ and $\Phi W \subset W$. Furthermore, W is convex if $ax + (1 - a)y \in W$ whenever $x,y \in W$ and $a \in \Phi$, $0 \leq a \leq 1$; W is symmetric if $-1W = -W = W$; and W is balanced if $aW \subset W$, $a \in \Phi$, $|a| \leq 1$.

Clearly every balanced set is symmetric, but the converse need not be true. Moreover, a nonempty balanced set contains the origin.

It should be remarked that the terminology with regard to balanced sets is not universally the same. Many authors call a balanced set "circled" (for example, see [K, p. 176; KeNa, p. 14]), whereas others use the term "balanced" [Bb, p. 8; E_1, p. 50; T, p. 123; W_1, p. 22; Y, p. 24].

Definition 1.1.4. Let V be a linear space over Φ and suppose that $p : V \to \mathbb{R}$. Then p is a seminorm on V if

(i) $p(x + y) \leq p(x) + p(y)$

(ii) $p(ax) = |a|p(x)$ $\qquad (x,y \in V;\ a \in \Phi)$.

Property (i) of p is known as the <u>triangle inequality</u>, for obvious reasons. It is also referred to as the <u>subadditivity</u> of p.

We shall shortly see some examples of seminorms. First, however, we wish to prove the following proposition:

<u>Proposition 1.1.1</u>. Let V be a linear space over Φ and let p be a seminorm on V. Then

(i) $p(0) = 0$,

(ii) $p(x) \geq 0$,

(iii) $p(x - y) \geq |p(x) - p(y)|$ $\qquad (x,y \in V)$.

<u>Proof</u>. $p(0) = p(0x) = |0|p(x) = 0$, $x \in V$, and part (i) follows. Clearly part (ii) is a consequence of part (iii). But, for any $x,y \in V$, the subadditivity of p reveals that

$$p(x) = p(x - y + y) \leq p(x - y) + p(y),$$

and hence $p(x) - p(y) \leq p(x - y)$. However, from the homogeneity property of p we deduce that

$$p(x - y) = p[-(y - x)] = p(y - x) \geq p(y) - p(x),$$

and so $p(x - y) \geq |p(x) - p(y)|$. □

We could, of course, have included the results of Proposition 1.1.1 as part of the definition of a seminorm, and this is done by some writers.

<u>Definition 1.1.5</u>. Let V be a linear space over Φ and let p be a seminorm on V. If $p(x) = 0$ implies $x = 0$, then p is said to be a <u>norm</u> on V.

Definition 1.1.6. Let V be a linear space over Φ. Then V is said to be a seminormed linear space if there exists a family $P = \{p_\beta \mid \beta \in \Lambda\}$ of seminorms on V such that $p_\beta(x) = 0$, $\beta \in \Lambda$, implies $x = 0$. If V is a seminormed linear space and $P = \{p\}$, that is, P is a singleton set, then V is said to be a normed linear space.

Evidently the condition on the family of seminorms $P = \{p_\beta \mid \beta \in \Lambda\}$ required for V to be a seminormed linear space is a replacement for the positive definiteness displayed by a norm. It will become clear in the sequel how this property of $\{p_\beta\}$ is utilized.

In the case that V is a normed linear space, we shall generally call $p(x)$ the norm of x and write $p(x) = \|x\|$. It is then easily seen that the equation $\rho(x,y) = \|x - y\|$, $x,y \in V$, defines a metric ρ on V and that a net $\{x_\alpha\} \subset V$ converges in this metric topology to $x \in V$ if and only if $\lim_\alpha \|x_\alpha - x\| = 0$. The details of these assertions are left to the reader. We shall generally refer to this metric topology as the norm topology. The next definition now clearly makes sense.

Definition 1.1.7. A normed linear space V over Φ is said to be a Banach space if it is a complete metric space with the metric $\rho(x,y) = \|x - y\|$, $x,y \in V$.

Since there may be many families of seminorms under which a given linear space is a seminormed linear space, we shall write these spaces as pairs (V,P), where P is the relevant family of seminorms. When $P = \{p\}$, that is, V is a normed linear space, we shall generally write $(V,\|\cdot\|)$.

Before we turn to some examples, we wish to state two theorems concerning normed linear spaces. The proofs are straightforward and are left to the reader.

<u>Theorem 1.1.1.</u> Let $(V,\|\cdot\|)$ be a normed linear space over Φ.

(i) If $W \subset V$ is a linear subspace, then the closure of W in V is a linear subspace of V.

(ii) If $W \subset V$ is a linear subspace, then $(W,\|\cdot\|)$ is a normed linear space. If $(V,\|\cdot\|)$ is a Banach space and W is a closed linear subspace, then $(W,\|\cdot\|)$ is a Banach space.

(iii) If $W \subset V$ is a closed linear subspace, then the quotient space V/W is a normed linear space with the norm

$$|||x + W||| = \inf_{y \in W} \|x + y\| \qquad (x \in V).$$

If $(V,\|\cdot\|)$ is a Banach space and W is a closed linear subspace, the $(V/W,|||\cdot|||)$ is a Banach space.

(iv) There exists a Banach space $(V_1,\|\cdot\|_1)$ over Φ and a mapping $\varphi : V \to V_1$ such that

(a) φ is an isomorphism.
(b) $\varphi(V)$ is dense in $(V_1,\|\cdot\|_1)$.
(c) $\|\varphi(x)\|_1 = \|x\|$ $\qquad (x \in V)$.

Moreover, if $(V_2,\|\cdot\|_2)$ is another Banach space that satisfies properties (a), (b), and (c), then there exists a mapping $\psi : V_1 \to V_2$ that is a surjective isomorphism such that $\|\psi(x)\|_2 = \|x\|_1$, $x \in V_1$.

The space $(V_1,\|\cdot\|_1)$ described in Theorem 1.1.1(iv) is, of course, called the <u>completion</u> of V. Clearly Theorem 1.1.1(i) also remains valid for any seminormed linear space (V,P).

<u>Theorem 1.1.2.</u> Let $(V_1,\|\cdot\|_1),(V_2\|\cdot\|_2),\ldots,(V_n,\|\cdot\|_n)$ be normed linear spaces over Φ and let V denote the topological product $V_1 \times V_2' \times \cdots \times V_n$ of the topological spaces $V_1,V_2,\ldots,V_n$, with the respective norm topologies and with linear space addition and scalar multiplication defined componentwise. Then $(V,\|\cdot\|)$ is a normed linear space over Φ such that the norm topology is equivalent to the product topology if $\|\cdot\|$ is defined as any of the following:

(i) $\|x\| = \sup_{k=1,2,\ldots,n} \|x_k\|_k$,

(ii) $\|x\| = \left[\sum_{k=1}^{n} (\|x_k\|_k)^p\right]^{1/p}$ $(1 \leq p < \infty)$,

where $x = (x_1, x_2, \ldots, x_n)$. If $(V_1, \|\cdot\|_1), (V_2, \|\cdot\|_2), \ldots, (V_n, \|\cdot\|_n)$ are Banach spaces, then so is $(V, \|\cdot\|)$.

It should be noted that the topological product of infinitely many normed linear spaces cannot be provided with a norm for which the norm topology is equivalent with the product topology (see, for example, [K, p. 150]).

1.2. Examples of Seminormed Linear Spaces. In this section we give a number of examples of seminormed and normed linear spaces. No proofs are provided for the various assertions. Some proofs will appear in later chapters, whereas others are left for the reader to establish.

Example 1.2.1. Let X be a Hausdorff topological space. By $C(X)$, $C_o(X)$, and $C_c(X)$ we denote, respectively, the linear spaces over $\mathbb{C}$ of all continuous complex-valued functions on X that are bounded, vanish at infinity, or have compact support. The linear space operations are the usual ones of pointwise addition and scalar multiplication. If f is an element of any of these spaces, then we set

$$\|f\| = \|f\|_\infty = \sup_{t \in X} |f(t)|.$$

Then $(C(X), \|\cdot\|_\infty)$ and $(C_o(X), \|\cdot\|_\infty)$ are Banach spaces, whereas $(C_c(X), \|\cdot\|_\infty)$ is a normed linear space. Clearly if X is compact, then $C(X) = C_o(X) = C_c(X)$, and $(C_c(X), \|\cdot\|_\infty)$ is a Banach space only in this case.

If X is noncompact, then we denote by $C'(X)$ all continuous complex-valued functions on X. Then clearly $\|\cdot\|_\infty$ no longer defines a norm on $C'(X)$. However, if for each compact set $K \subset X$ we set

$$p_K(f) = \sup_{t \in K} |f(t)| \qquad (f \in C'(X)),$$

then $P = \{p_K \mid K \subset X, K \text{ compact}\}$ is a family of seminorms on $C'(X)$ for which $(C'(X),P)$ is a seminormed linear space. Moreover, it can be shown that $C'(X)$ is never a normed linear space when X is noncompact.

We shall denote by $C^R(X)$, $C_o^R(X)$, and $C_c^R(X)$ the real parts of the functions in $C(X)$, $C_o(X)$, and $C_c(X)$, respectively. Obviously these are linear spaces over $\mathbb{R}$ with the same properties as the analogous spaces of complex functions. They are equivalently the spaces of continuous real-valued functions on X that are bounded, vanish at infinity, or have compact support.

Example 1.2.2. Let X be any set. We denote by $B(X)$ the linear space under pointwise operations of all bounded complex-valued functions defined on X. The linear space $B(X)$ is a Banach space with the norm

$$\|f\|_\infty = \sup_{t \in X} |f(t)| \qquad (f \in B(X)).$$

If X is a locally compact Hausdorff topological space, then it is evident that $C(X)$ is a linear subspace of $B(X)$.

Example 1.2.3. Let $a < b$ and let n be a nonnegative integer. We denote by $C^n([a,b])$ the linear space of n-times continuously differentiable real-valued functions on $[a,b]$. If we define

$$\|f\|_n = \sum_{k=0}^{n} \|f^{(k)}\|_\infty \qquad (f \in C^n([a,b])),$$

where $f^{(k)}$ denotes the kth derivative of f, then $(C^n([a,b]),\|\cdot\|_n)$ is a Banach space over $\mathbb{R}$.

Furthermore, set $C^\infty([a,b]) = \bigcap_{n=0}^\infty C^n([a,b])$. Then $C^\infty([a,b])$