Lecture Notes in Mathematics

1595

Robert D. M. Accola

Topics in the Theory of Riemann Surfaces

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§ 3. Final manuscripts should preferably be in English. They should contain at least 100 pages of scientific text and should include

- a table of contents;
- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

Further remarks and relevant addresses at the back of this book.

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Preface

These are lecture notes for a course given during the Fall of 1988 at Brown University. The students were assumed to have had a previous course on the theory of Riemann surfaces or algebraic curves. Chapter One of these notes gives a review of most of the basic material needed later, but few proofs are given, and the reader is assumed to have some previous acquaintance with the material.

Besides giving the author's point of view and notational conventions, the introduction gives a few proofs (or rather demonstrations) intended to bridge the gap between the more analytic approach to the subject as exemplified by the books of Ahlfors-Sario [4], Farkas-Kra [11] or Forster [12] and the more algebraic approach as exemplified by Walker [26]. In particular the proof of the genus formulas for a plane curve based on the Riemann-Hurwitz formula and other materials in Section 1.3 were inspired by a seminar given in the early '60's at Brown by S. Lefschetz.

Despite the recent appearance of several excellent books on the theory of compact Riemann surfaces (which is, of course, basically the same as the theory of algebraic curves over the complex numbers), most of the material in these notes does not appear in these books. The two main subjects treated here are exceptional points on Riemann surfaces (Weierstrass points, higher-order Weierstrass points) and automorphisms of Riemann surfaces. A foundational treatment of the theory of automorphisms from the viewpoint of Galois coverings of Riemann surfaces is given in Chapters Four and Five, following and expanding to some extent the treatment of Ahlfors-Sario [4] and Seifert-Threlfall [25]. The treatment is technically different from that of A. M. Macbeath [19] and his students, although fundamentally it is the same. In the treatment here, no mention of Fuchsian groups occurs. Also a treatment of the extremely useful inequality of Castelnuovo-Severi is included, a treatment for which the author knows no reference. (It is difficult, however, to believe that anything on this venerable subject can be really new.)

The treatment of all subjects is basically elementary.

The author heartily thanks Natalie Ruth Johnson for an excellent job in preparing this manuscript.

Table of Contents

Chapter 1. Review of some basic concepts in the theory of Riemann surfaces.

1.1 Coverings. A surface X is a connected Hausdorff space which satisfies the second axion of countability and has a basis for the open sets of sets homeomorphic to open sets in $\mathbb{C}$, the complex numbers. A Riemann surface is a surface with an open cover $\{U_\alpha \mid \alpha \in A\}$, for some index set A, and homeomorphisms $\{\varphi_\alpha\}$, $\varphi_\alpha : U_\alpha \to \mathbb{C}$, where $\varphi_\beta \circ \varphi_\alpha^{-1}$ is biholomorphic wherever it is defined. A pair $(U_\alpha, \varphi_\alpha)$ is called a chart and the set $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ is called an atlas. φ_α will be called a local parameter. All Riemann surfaces are orientable.

A continuous function $f : X \to Y$ between Rieman surfaces will be said to be holomorphic if it is holomorphic (analytic) when expressed in local parameters. If f is holomorphic, injective, and surjective then f is said to be biholomorphic and X and Y are said to be conformally equivalent. A biholomorphic map of a Riemann surface onto itself will be called an automorphism, and the group (under composition) of automorphisms of a Riemann surface X will be denoted $\mathrm{Aut}(X)$. An automorphism of period 2 will be called an involution.

A holomorphic (meromorphic) function on a Riemann surface X is a holomorphic mapping of X into $\mathbb{C}$ ($\mathbb{P}^1$, the projective line, or Riemann sphere).

If $f : X \to Y$ is a non-constant holomorphic mapping of Riemann surfaces and $x \in X$, then in suitable local coordinates at x and $f(x)$, f looks like $z \to z^n$, n a positive integer. We say f is equivalent to $z \to z^n$ at x. (In particular, f is an open mapping.) n is said to be the multiplicity of f at x, and $n - 1$ is said to be the branching or ramification of f at x, denoted $\mathrm{ram}_x(f)$. The ramified points x, where $\mathrm{ram}_x(f) > 0$, are a discrete set in X.

Theorem. Let $f : X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then there exists a positive integer n so that for any $y \in Y$

$$n = \mathrm{card}\left\{f^{-1}(y)\right\} + \sum_{f(x)=y} \mathrm{ram}_x(f) .$$

n will be called the number of sheets in the covering $X \to Y$. If $Y = \mathbb{P}^1$ then f is a non-constant meromorphic function on X and

n is called the order of f, denoted $o(f)$.

A non-constant holomorphic map of compact Riemann surfaces will often be called a covering.

The Riemann-Hurwitz Formula for coverings.

Let $f : X \to Y$ be a non-constant n-sheeted holomorphic map between compact Riemann surfaces. Let p and q be the genus of X and Y respectively. Let $\mathrm{ram}(f) = \sum_{x \in X} \mathrm{ram}_x(f)$. Then

$$2p - 2 = n(2q - 2) + \mathrm{ram}(f).$$

Definition. If X_p is a compact Riemann surface, the subscript p will always denote the genus of X_p.

Corollary. Let $f : X_p \to X_q$ be a non-constant n-sheeted holomorphic mapping of compact Riemann surfaces where $n \geq 2$. Then $p \geq q$ with equality possible only if $p = 0$ or 1.

1.2 Function Fields. If X is a Riemann surface let $M(X)$ denote the field of meromorphic functions on X. If $f : X \to Y$ is an n-sheeted holomorphic map then $f^* : M(Y) \to M(X)$ maps $M(Y)$ onto a field which is of index n in $M(X)$. Conversely, if X is a compact Riemann surface and K is a subfield of $M(X)$ of index n, then there exists an n-sheeted covering $f : X \to Y$ of compact Riemann surfaces and $f^*\big(M(Y)\big) = K$.

A rational function field is a field isomorphic to $M(\mathbb{P}^1)$, that is, the rational functions. A meromorphic function field on a Riemann surface of genus 1 wil be called an elliptic function field. A Riemann surface X_p, $p \geq 2$, will be called hyperelliptic if $M(X_p)$ admits a rational subfield of index 2. This is equivalent to X_p admitting a 2-sheeted covering $X_p \to \mathbb{P}^1$. The interchange of the two sheets of this covering is an automorphism of X_p of order 2 called the hyperelliptic involution. A hyperelliptic Riemann surface, X_p, is the Riemann surface for a polynomial $P(z,w) \in \mathbb{C}[z,w]$ where

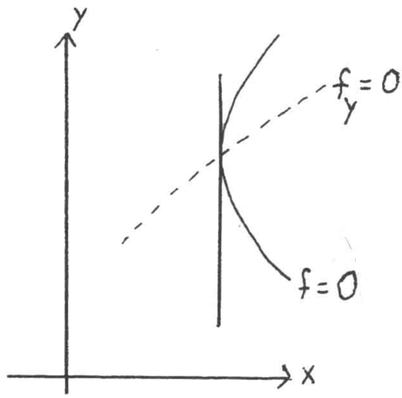
$$P(z,w) = w^2 - \prod_{j=1}^{2p+2} (z - a_j) \text{ where } a_i \neq a_j.$$

A Riemann surface of genus greater than one will be called elliptic-hyperelliptic if it admits a 2-sheeted covering of a Riemann surface of genus one ; that is, its function field admits an elliptic function field of index 2 .

In general, if X is a compact Riemann surface then $M(X)$ is the Riemann surface for some (many) irreducible polynomial $P(z,w) \in \mathbb{C}[z,w]$.

1.3 Plane Curves [26]. Let $f(x,y,z)$ be an irreducible homogeneous polynomial of degree n . Let $C_n = \{(x,y,z) \in \mathbb{P}^2(\mathbb{C}) \mid f(x,y,x) = 0\}$, an irreducible plane curve. Suppose C_n is non-singular, that is, $f_x(x,y,z) = 0 = f_y(x,y,z) = f_z(x,y,z)$ implies $(x,y,z) = (0,0,0)$.

We show that as a Riemann surface, C_n has genus $\frac{(n-1)(n-2)}{2}$. C_n has only a finite number of inflection points (where the tangent line has more than two intersections with C_n). Choose coordinate axes in $\mathbb{P}^2$ so that none of the inflectional tangent lines pass through $(0,1,0)$ (i.e. no inflectional tangent line is parallel to the y-axis). Then each tangent line to C_n passing through $(0,1,0)$ has only two intersections where it is tangent. Such points occur where $f = 0$ and $f_y = 0$. At such points the projection of C_n onto the x-axis is locally two-to-one; that is, this projection (a holomorphic map onto the x-axis $(= \mathbb{P}^1)$) has ramification one at each of these points. The number of such points is $n(n-1)$.



Using the Riemann-Hurwitz formula we have

$$2p - 2 = -2n + n(n-1) \; ; \; 2p - 2 = n(n-3)$$

$$\text{or } p = \frac{n^2 - 3n + 2}{2} .$$

We can use the same method if C_n has singularities. Suppose C_n has an ordinary singularity of multiplicity k at $(0,0,1)$. Dehomogenizing $f(x,y,z)$ to $f(x,y,1)$ we have

$$f = \prod_{i=1}^{k} (\alpha_i x + \beta_i y) + \text{higher order terms}$$

where the k tangents lines $\alpha_i x + \beta_i y = 0$ are all distinct. Thus

$f_y = \prod_{i=1}^{k-1} (\gamma_i x + \delta_i y)$ + higher order and so $f = 0$ and $f_y = 0$ have $k(k-1)$ intersections at $(0,0,1)$. Now assume C_n has s ordinary singularities of multiplicities $k_1,\ldots,k_s$. Then the total ramification for the projection of C_n onto the x-axis is $n(n-1) - \sum k_j(k_j - 1)$ or

$$p = \frac{(n-1)(n-2)}{2} - \sum \frac{k_j \cdot (k_j - 1)}{2}.$$

Let us look more closely at singularities of multiplicity two at, say, $(0,0,1)$.

$f = ax^2 + bxy + cy^2$ + higher order.
$f_y = bx + 2cy$ + higher order.

Local parametric equations for $f_y = 0$ are

$$x = 2ct + c_2 t^2 + \cdots$$
$$y = -bt + b_2 t^2 + \cdots .$$

Thus $f(x,y) = a(4c^2t^2 + \cdots) + b(-2bct^2 + \cdots) + c(b^2t^2 + \cdots) + o(t^3)$
$= c(4ac - b^2)t^2 + o(t^3)$.

If $4ac - b^2 \neq 0$, $f = 0$ and f_y have two intersections (node).

If $4ac - b^2 = 0$, $f = 0$ and $f_y = 0$ have at least three intersections (cusp).

If $f = (ax + by)^2 + (ax + by)Q(x,y) + 4^{th}$ degree terms and higher order, then $f = 0$ and $f_y = 0$ have 4 intersections (tacnode).

Nodes add nothing to the ramification of the propection map. Cusps add one to this ramification and tacnodes add nothing.

If δ = # of nodes, κ = # of simple cusps, η = # of simple tacnodes for C_n and there are no further singularities then

$$2p - 2 = -2n + n(n-1) - 2\delta - 3\kappa - 4\eta + \kappa$$

$$p = \frac{(n-1)(n-2)}{2} - \delta - \kappa - 2\eta .$$

For a general plane curve of degree n we have

$$p = \frac{(n-1)(n-2)}{2} - \delta$$

where δ will be called the δ-invariant (sometimes called the number of nodes suitable counted).

Ordinary k-fold singularities add $\frac{k(k-1)}{2}$ to the δ-invariant.

Simple cusps add 1 to the δ-invariant.

Simple tacnodes add 2 to the δ-invariant.

1.4 Divisors on Riemann surfaces. A divisor D on a Riemann surface X is a singular zero-chain, written $D = \sum n_i z_i$ for $n_i \in Z$ and $z_i \in X$. The degree of D, $\deg D$, is $\sum n_i$, an integer. If $n_i \geq 0$ for all i then D is said to be positive (or integral) and we write $D \geq 0$. The zero divisor is positive.

If $D = \sum n_i z_i$ and $D' = \sum n'_i z_i$ are two divisors then by the greatest common divisor of D and D', written (D,D'), we mean the divisor $\sum \min(n_i, n'_i) z_i$. Thus to write $(E,z) = 0$ is to say that the coefficient of z in E is zero. If $(E,F) = E$ we will write $E \subseteq F$.

If f is a meromorphic function on X, the divisor of f is written (f) where (f) = (zeros of f) - (poles of f)

$$= (f)_0 - (f)_p .$$

Note that $\deg(f) = 0$.

Two divisors D, E are said to be (linearly) equivalent if there exist an $f \in M(X)$ and $(f) = D - E$ (an equivalence relation).

Notation: $D \equiv E$.

Definition. If D is a divisor, $L(D) = \left\{ f \in M(X) \,\middle|\, (f) + D \geq 0 \right\}$.

Theorem. If X is compact then $L(D)$ is a finite dimensional vector space over $\mathbb{C}$.

Definition. If D a divisor on X, $\ell(D) = \dim_{\mathbb{C}} L(D)$, the affine dimension of $L(D)$. $r(D) = \ell(D) - 1$, the projective dimension of $L(D)$.

A meromorphic (or abelian) differential ω is a one-form that locally can be written

$$\omega = f(z)\,dz$$

where f is meromorphic. The value of a meromorphic differential at a point of X is not well defined; however, zeros and poles are well defined so we can consider the divisor of ω,

$$(\omega) = (\text{zeros of } \omega) - (\text{poles of } \omega).$$

If ω_1 and ω_2 are two meromorphic differentials then ω_1/ω_2 is a well defined meromorphic function on X. Since $\deg(\omega_1/\omega_2) = 0$ we see that $\deg(\omega_1) = \deg(\omega_2)$.

Theorem. If ω is a meromorphic differential on X_p then

$$\deg(\omega) = 2p - 2.$$

Definition. If D is a divisor

$$\Omega(D) = \{\text{meromorphic differentials } \omega \mid (\omega) - D \geq 0\}$$

$i(D)$ = dimension of $\Omega(D)$ as a vector space over $\mathbb{C}$.

Lemma. If K is the divisor of a meromorphic differential then $i(D) = \ell(K - D)$.

Now we list some of the basic theorems concerning compact Riemann surfaces.

Riemann-Roch theorem. If D is a divisor on a compact Riemann surface of genus p, then

$$r(D) = \deg D - p + i(D).$$

Brill-Noether formulation of the Riemann-Roch theorem. If D and D' are two divisors so that $D + D' \equiv (\omega)$ where ω is a meromorphic differential, then $\deg D - 2r(D) = \deg D' - 2r(D')$.

Notice that the divisors in the Brill-Noether theorem need not be positive.

Definition. A meromorphic differential without poles is called holomorphic (or analytic, or regular, or an abelian differential of the first kind).

Definition. If $D \geq 0$ and $i(D) > 0$ then D is said to be special and $i(D)$ is often called the index (of speciality) of D.

Clifford's theorem. If D is special then $\deg D - 2r(D) \geq 0$.

Definition. If D is a divisor, the Clifford index of D, denoted $c(D)$, is:

$$c(D) = \deg D - 2r(D) = p - r(D) - i(D).$$

Strong form of Clifford's theorem. If X_p admits a special divisor D where $\deg D \neq 0$, $\deg D \neq 2p-2$ and $c(D) = 0$, then X_p is hyperelliptic.

Definition. $\Omega(X_p) = \{\text{holomorphic differentials on } X_p\}$.

Corollary. $\dim_{\mathbb{C}} \Omega(X_p) = p$. If $\omega \in \Omega(X_p)$ then $i((\omega)) = 1$.

1.5 Linear series. Let D be a divisor on X_p. Then the complete linear series determinant by D, denoted $|D|$, is

$$|D| = \{D' \geq 0 \mid D' \equiv D\}.$$

$|D|$ can be empty. If $D \geq 0$ then $|D|$ is a set of divisors parametrized by the projectification of $L(D)$.

$$|D| = \{D' \mid D' - D = (f),\ f \in L(D)\}.$$

Let S be a linear subspace of $L(D)$ of dimension $r+1$, $r \leq r(D)$. The set of divisors parametried by S

$$\{D' \mid D' - D = (f),\ f \in S\}$$

is called a linear series and denoted g^r_n where $n = \deg$ and $r+1 = \dim S$.

If $r < r(D)$, g^r_n is said to be incomplete.

Example: K = divisors of holomorphic differentials = g^{p-1}_{2p-2}. K is the unique linear series of dimension $p-1$ and degree $2p-2$.

A linear series g^r_n can have base (or fixed) points, that is, a divisor common to all divisors in g^r_n. If F of degree f is such a divisor then $g^r_n - F = g^r_{n-f}$ stands for the linear series without base points obtained by subtracting F from each divisor in g^r_n.

Let g^r_n be a linear series without base points. Then we can map X into $\mathbb{P}^r$ as follows. Suppose

$$g^r_n = \{D' \mid D' - D = (f),\ f \in S\},\ S \subset L(D).$$

Let $f_0, \cdots, f_r$ be a basis for S. If $x \in X$ let

$$\Theta(x) = (f_0(x), \cdots, f_r(x)) \in \mathbb{P}^r(\mathbb{C}).$$

If we pick a local parameter z at x_0, $z(x_0) = 0$ then

$$f_i = z^{n_i} g_i(z) \text{ where } g_i(0) \neq 0 .$$

Locally z is mapped by Θ into $(\cdots, z^{n_i} g_i(z), \cdots) \in \mathbb{P}^r$.

Let $n = \min\{n_i\}$. Then $z \overset{\Theta}{\mapsto} (\cdots, z^{n_i - n} g_i(z), \cdots)$ and so

$x \overset{\Theta}{\mapsto} (\cdots, z^{n_i - n} g_i(z) \big|_{z=0}, \cdots)$ and at least one component is non-zero. That is, Θ is well defined on all of X.

Let $C = \Theta(X) \subset \mathbb{P}^r$ be the image of X under Θ. Then the hyperplane sections of C pull back via Θ to the divisors of g^r_n. For

if $\sum_{i=0}^{r} a_i y_i = 0$ is a hyperplane H in $\mathbb{P}^r$ then

$\Theta(x) \in H \cap C \Leftrightarrow \sum a_i f_i(x) = 0 \Leftrightarrow x$ is a zero of $\sum a_i f_i$. If $f_r \equiv 1$ then D is the pull back of the hyperphane (at ∞) $y_r = 0$.

If the map Θ is one-to-one in general, then g^r_n is said to be <u>simple</u>. In this case X is (conformally equivalent to) the Riemann surface (or normalization) of C. (C can, of course, have singularities.) Each hyperplane in $\mathbb{P}^r$ cuts C in n points (counting multiplicities) so we write C_n for C, a curve of degree n in $\mathbb{P}^r$.

If the map Θ is not one-to-one in general then g^r_n is said to be <u>composite</u>. Then X is a t-sheeted covering ($t \geq 2$) of (the normalization of) C and C has degree n/t, since g^r_n has no base points. (Thus a linear series of dimension $r \geq 2$ of prime degree greater than r without base points is simple.) In this case, if Y_q is the normalization of C then Y_q admits a $g^r_{n/t}$. If $\pi : X_p \to Y_q$ is the t-sheeted covering then $g^r_{n/t}$ on Y_q lifts to g^r_n on X_p. The fibers of this map π are called an <u>involution of genus q</u>