Lecture Notes in Mathematics 1487

Eberhard Freitag

Singular Modular Forms and Theta Relations

Springer-Verlag

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Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

Author

Eberhard Freitag
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
W-6900 Heidelberg, FRG

Mathematics Subject Classification (1991): 11F, 10D

ISBN 3-540-54704-5 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-54704-5 Springer-Verlag New York Berlin Heidelberg

Printed in Germany

Typesetting: Camera ready by author
Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
46/3140-543210 - Printed on acid-free paper

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Introduction

Siegel modular forms are holomorphic functions on the generalized upper half plane $\mathbb{H}_n$, which consists of all symmetric complex $n \times n$-matrices with positive definite imaginary part. They admit Fourier expansions of the type

$$f(Z) = \sum_T a(T) \exp \pi i \sigma(TZ),$$

where T runs through a lattice of rational symmetric matrices.

A modular form is called **singular** if

$$a(T) \neq 0 \Longrightarrow T \quad \text{singular, i.e.} \quad \det T = 0.$$

Important examples of modular forms are **theta series**

$$\sum_G \exp \pi i \sigma(S[G]Z).$$

Here S is a rational symmetric positive definite $r \times r$-matrix and G runs through a lattice of rational $r \times n$-matrices. This theta series is singular if

$$r < n.$$

The theory of singular modular forms states in a very precise sense that each singular modular form is a linear combination of theta series.

In these notes we give an introduction to the theory of **Siegel modular forms**, especially singular ones. We aspire to highest generality, we consider arbitrary congruence subgroups of the Siegel modular group, the modular forms can be vector valued and the weights are allowed to be half integral. Before we describe the contents in more detail, we make some historical comments.

The notion of a singular modular form is due to H.Resnikoff [Re1], who considered scalar valued modular forms of the transformation type

$$f(M\langle Z\rangle) = v(M) \det(CZ + D)^{r/2} f(Z).$$

He proved that such a modular form can be singular only if r is integral and if $r < n$. The question arose whether in this case all modular forms are singular. In the paper [Fr1] an affirmative answer was given in the case $n = 2$. A little later the general case was independently solved by different methods in [Fr2] and [Re2].

The only known examples of singular modular forms were theta series (and linear combinations of them). So the question arose whether every singular modular form is a linear combination of theta series. I gave a very short and simple proof (s.[Fr4]) that scalar valued singular modular forms with respect to the full Siegel modular group are linear combinations of theta series

$$\sum_{G \text{ integral}} \exp \pi i \sigma(S[G]Z),$$

where S is an even unimodular positive matrix. (Even means that S is integral and that the diagonal is even.) A little later I generalized this result to vector valued modular forms with respect to the full Siegel modular group [Fr5]. Those forms are holomorphic functions

$$f : \mathbb{H}_n \longrightarrow \mathcal{Z}$$

with values in some finite dimensional vector space $\mathcal{Z}$, which transform as

$$f(M\langle Z\rangle) = \varrho(CZ + D)f(Z).$$

Here

$$\varrho : \mathrm{GL}(n, \mathbb{C}) \longrightarrow \mathrm{GL}(\mathcal{Z})$$

is some rational representation. The role of r is played by the biggest number k such that

$$\varrho(A) \det(A)^{-k}$$

is polynomial. In the vector valued case it is necessary to consider theta series with harmonic coefficients:

$$\sum_G P(S^{1/2}G) \exp \pi i \sigma(S[G]Z).$$

Here P is a harmonic polynomial with the transformation property

$$P(GA) = \varrho(A') \det(A)^{-r/2} P(G).$$

(For simplicity we assume $r \equiv 0 \bmod 2$ at the moment.) It is a very remarkable fact that vector valued singular modular forms automatically "produce" such harmonic coefficients.

The method which I used depended heavily on the restriction to the full modular group. As has been pointed out in [Al], [Zw] and [En], the method generalizes to congruence groups which contain all unimodular substitutions,

$$Z \longmapsto Z[U], \quad U \in \mathrm{GL}(n, \mathbb{Z}).$$

But the general case of an arbitrary congruence subgroup or at least of a cofinal system of congruence subgroups was not obvious.

In an important paper [Ho] R.Howe proved a theorem about singular representations of $\mathrm{Sp}(n, \mathbf{A})$ ($\mathbf{A}$ denotes the ring of adeles). In classical language his result can be formulated as follows:

Each singular Siegel modular form is linear combination of theta series.

(HOWE actually considered only square integrable modular forms with respect to the PETERSSON inner product. A little later WEISSAUER proved [Wel] that singular forms are always square integrable.).

In some sense the result of HOWE is not satisfactory. Let Γ be a fixed congruence subgroup of the Siegel modular group. As usual we denote by $[\Gamma, \varrho]$ the space of all modular forms with the transformation law

$$f(M\langle Z\rangle) = \varrho(CZ + D)f(Z) \quad \text{for all } M \in \Gamma.$$

This space is of finite dimension. Very often the dimension has geometric or arithmetic meaning and one would like to compute or estimate it.

For this reason one would like to have a **finite system of theta series** *which generates* $[\Gamma, \varrho]$ *and one would like to describe all linear relations between the generators.*

The representation theoretic result of HOWE seems not to give an answer to this refined question. For example, he always has to consider besides the theta series

$$\sum_{G \text{ integral}} P(S^{1/2}G) \exp \pi i \sigma(S[G]Z)$$

all satellites

$$\sum_{G \text{ integral}} P(S^{1/2}G) \exp \pi i \sigma(S[G]Z + 2G'V)$$

with arbitrary rational characteristics V. But they generate a vector space of infinite dimension and it is not clear which linear combinations of them belong to a given group Γ. This question is difficult because there are non-trivial relations between them. For example, the **classical Riemann theta relations** are of this type.

The first step into a more concrete representation theorem has been done by R.ENDRES [En]. He treated the case of scalar valued modular forms of weight $1/2$ ($r = 1$). Some of his ideas have proved to be essential for the general case.

In these notes we prove a refined representation theorem for singular modular forms, which gives a finite system of generators and describes all linear relations between them.

Actually the proof is complete only for $n \geq 2r$ (instead of $n > r$) and some other cases.

Our method is elementary and not representation theoretic. It depends heavily on the analysis of the Fourier-Jacobi expansion of a modular form. Now we describe the contents in more detail.

Chapter I contains an introduction to the theory of Siegel modular forms. We consider vector valued forms and also admit half integral weights. Therefore we have to deal with multiplier systems. The choice of a multiplier system is not too important, because in the case $n > 1$, two multiplier systems always agree on a suitable congruence subgroup. We investigate the standard multiplier system –the so called **theta multiplier system**– in some detail and express it as a Gauss sum. This Gauss sum will be computed in important special cases. All the results about the theta multiplier system are already in the literature but are scattered. It seemed me to be worth while to include this theory with complete proofs.

As already mentioned, vector valued modular forms involve finite dimensional rational representations of $\mathrm{GL}(n)$. We include the theory of highest weights of such representations without proofs. Readers who are interested only in scalar valued modular forms can pass over this part.

Chapter II is devoted to the **theta transformation formalism**. Examples of modular forms are theta series. In the vector valued case (and only in this case) one needs theta series with harmonic coefficients. For our purposes it is necessary to develop the transformation formalism for arbitrary polynomial coefficients. One reason is that in the Fourier Jacobi expansion of a vector valued modular form, theta series arise with polynomial coefficients which are not known to be harmonic in advance. For the proof of the transformation formalism we use EICHLER's imbedding trick. This is a very convenient method to reduce the transformation formalism to the full modular group, where simple generators are available. The transformation formalism simplifies considerably if the coefficients are harmonic forms. For the purposes of these notes not much more than the definition of a harmonic form is needed. Nevertheless we have included some of the beautiful results of KASHIWARA-VERGNE [KV], who classified all harmonic forms.

Chapter III contains the proof that non-vanishing modular forms are singular if and only if $r < n$. The main tool is the **Fourier Jacobi expansion** of a modular form. The transformation properties of those coefficients lead to the notion of a **Jacobi form**. We prove a variant of the SHIMURA-isomorphism [Sh], which states that Jacobi forms correspond to finite systems of usual modular forms. The proof of this correspondence is tedious in the vector valued case and depends heavily on the general theta transformation formalism. We prove the correspondence between Jacobi forms and usual modular forms only for varying levels, i.e., we do not get information about precise levels.

Chapter IV describes a central part of the theory. First of all we describe a certain space of Fourier series $\mathbf{M}$, which contains the space of modular forms of a fixed level q. This inclusion is nothing else but a reformulation of the classification of singular weights. The space $\mathbf{M}$ has the advantage that in its definition no multiplier system or sophisticated congruence groups have to be considered. At first glance the space $\mathbf{M}$ looks tremendously big. Actually our general representation theorem is valid for arbitrary elements of $\mathbf{M}$. We will investigate the Fourier Jacobi expansion of elements of $\mathbf{M}$. The representation theorem will be reduced to an elementary statement (called the fundamental lemma in these notes) which has nothing to do with modular forms. Unfortunately this lemma seems to be very hard.

Chapter V is devoted to the fundamental lemma. We give a complete proof in the case $n \geq 2r$ and in some other cases.

In the last chapter we formulate the results and point out the connection with the theory of theta relations. We work out a formula for the dimension of $\mathbf{M}$ (and as a consequence of certain spaces of singular modular forms), which allows one in principle to compute the dimension explicitly by a calculator. We include some numerical results.

But we confess that the connection between our results and the classical theta relations is not understood satisfactorily.

Most of the material of these notes has been published in the preprint series of the "Forschungsschwerpunkt Geometrie, Heidelberg" [Fr7-10].

In particular I would like to thank Dr. Dipendra Prasad who brought my attention to various mistakes in the original manuscript.

Heidelberg, 1990

I Siegel modular forms

1 The symplectic group

We introduce the symplectic group and recall some of its basic properties. A detailed treatment can be found in [Fr4].

Let R be a commutative ring with unit element $1 = 1_R$. We denote by

$$E = E^{(n)} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

the $n \times n$-unit matrix with coefficients in R and by

$$I = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$$

the standard alternating matrix. The symplectic group of degree n with coefficients in R consists of all $2n \times 2n$-matrices

$$M \in R^{(2n,2n)},$$

such that

$$I[M] = I.$$

Here we use the usual notation

$$A[B] = B'AB \quad (B' = \text{transpose of } B)$$

for matrices $A \in R^{(n,n)}$, $B \in R^{(n,m)}$. We denote the symplectic group by

$$\mathrm{Sp}(n, R).$$

It is often useful to decompose a symplectic matrix into four $n \times n$-blocs:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

1.1 Remark. *1) A matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is symplectic if and only if the relations*

$$A'D - C'B = E, \quad A'C = C'A, \quad B'D = D'B$$

hold. Especially

$$\mathrm{Sp}(1,R) = \mathrm{SL}(2,R).$$

2) One has $I^{-1} = -I$. *Therefore the transpose* M' *of a symplectic matrix* M *is symplectic, i.e.*

$$AD' - BC' = E, \quad AB' = BA', \quad CD' = DC'.$$

3) The inverse of a symplectic matrix is

$$M^{-1} = I^{-1}M'I = \begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix}.$$

4) Some examples of symplectic matrices are

a) $\begin{pmatrix} E & S \\ 0 & E \end{pmatrix}, \quad S = S';$

b) $\begin{pmatrix} U' & 0 \\ 0 & U^{-1} \end{pmatrix}, \quad U \in \mathrm{GL}(n,R);$

c) $I = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}.$

1.2 Proposition. *Let* R *be either* $\mathbb{Z}$ *or a field. The group* $\mathrm{Sp}(n,R)$ *is generated by the special matrices*

$$\begin{pmatrix} E & S \\ 0 & E \end{pmatrix}, S = S'; \quad \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}.$$

The symplectic group with coefficients in $\mathbb{Z}$ is sometimes called the Siegel modular group. We denote it by

$$\Gamma_n = \mathrm{Sp}(n,\mathbb{Z}).$$

Congruence subgroups

The kernel of the natural restriction homomorphism mod q

$$\Gamma_n[q] := \ker\bigl(\mathrm{Sp}(n,\mathbb{Z}) \longrightarrow \mathrm{Sp}(n,\mathbb{Z}/q\mathbb{Z})\bigr)$$

is called the **principal congruence subgroup** of level q.

1.3 Definition. *A subgroup*

$$\Gamma \subset \mathrm{Sp}(n,\mathbb{R})$$

is called a congruence subgroup if it contains some principal congruence subgroup

$$\Gamma_n[q] \subset \Gamma$$

as a subgroup of finite index.

1.4 Theorem. *Assume $n > 1$. Let*

$$\Gamma \subset \mathrm{Sp}(n,\mathbb{Z})$$

be a normal subgroup which is not contained in the central subgroup $\{\pm E^{(2n)}\}$. Then Γ is a congruence subgroup.

Corollary. *Each subgroup $\Gamma \subset \mathrm{Sp}(n,\mathbb{Z})$ of finite index is a congruence subgroup.*

We don't have to make use of this beautiful result of MENNICKE [Me]. But it will sometimes be helpful to have it in mind.

There are several "standard" congruence subgroups which we will use in these notes.

1) *The* **theta group**

$$\Gamma_{n,\vartheta} = \{M \in \Gamma_n, \quad AB' \text{ and } CD' \text{ have even diagonal entries}\}.$$

We will see later that $\Gamma_{n,\vartheta}$ actually is a group.

2) *The generalized* **Hecke group**

$$\Gamma_{n,0}[q] := \{M \in \Gamma_n, \quad C \equiv 0 \bmod q\}.$$

3) *The "theta variant" of 2)* [En]

$$\Gamma_{n,0,\vartheta}[q] = \{M \in \Gamma_n; \quad C \equiv 0 \bmod q, \text{ the diagonal entries of } (CD')/q \text{ are even}\}.$$

4) IGUSA's group [Ig1]

$$\Gamma_n[q,2q] = \{M \in \Gamma_n[q], \quad \text{the diagonal entries of } AB'/q \text{ and } CD'/q \text{ are even}\}.$$

One has

$$\Gamma_n[2q] \subset \Gamma_n[q,2q] \subset \Gamma_n[q].$$

Obviously

$$\Gamma_n = \Gamma_n[1]; \quad \Gamma_{n,\vartheta} = \Gamma_n[1,2].$$

1.5 Remark. *The Igusa group*

$$\Gamma[q,2q] \subset \Gamma_{n,\vartheta}$$

is a normal subgroup of $\Gamma_{n,\vartheta}$.
For even q, the group $\Gamma_n[q,2q]$ is normal in the full modular group Γ_n.

The proof of this remark can be found in [Ig1].

2 The Siegel upper half space

We introduce the action of the real symplectic group on the Siegel upper half space.

In the following, we denote by

$$\mathcal{Z}_n = \{Z = Z' \in \mathbb{C}^{(n,n)}\}$$

the vector space of all symmetric complex $n \times n$-matrices.

2.1 Definition. *The Siegel upper half space of degree n consists of all symmetric complex $n \times n$-matrices whose imaginary part is positive (definite).*

$$\mathbb{H}_n = \{Z = X + iY \in \mathcal{Z}_n, \quad Y > 0\}.$$

2.2 Remark. *The Siegel upper half space is an open convex subdomain of $\mathcal{Z}_n$.*

2.3 Remark. *Let*

$$f : \mathbb{H}_n \longrightarrow \mathbb{C}$$

be a holomorphic function without zeros. Then there exists a holomorphic square root of f:

$$h : \mathbb{H}_n \longrightarrow \mathbb{C},$$

i.e. h is holomorphic and

$$h(Z)^2 = f(Z).$$

Proof. Consider for a fixed Z, the function

$$\alpha : [0,1] \longrightarrow \mathbb{C},$$
$$\alpha(t) = f\big(iE + t(Z - iE)\big).$$

The function

$$H(Z) := \int_0^1 \dot{\alpha}(t)/\alpha(t)\, dt$$

is holomorphic and has the property

$$e^{H(Z)} = f(Z).$$

The function

$$h(Z) = e^{H(Z)/2}$$

has the desired property.

2.4 Lemma. *Let*

$$Z \in \mathbb{H}_n, \quad M \in \mathrm{Sp}(n, \mathbb{R}).$$

Then the matrix

$$CZ + D \qquad \left(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)$$

is invertible and

$$M\langle Z \rangle := (AZ + B)(CZ + D)^{-1}$$

is again contained in $\mathbb{H}_n$. *This defines an action of* $\mathrm{Sp}(n, \mathbb{R})$ *on* $\mathbb{H}_n$, *i.e.*

a) $E\langle Z \rangle = Z$,

b) $M\langle N\langle Z \rangle\rangle = (MN)\langle Z \rangle \qquad$ *for* $M, N \in \mathrm{Sp}(n, \mathbb{R})$.

The map

$$\mathbb{H}_n \longrightarrow \mathbb{H}_n,$$
$$Z \longmapsto M\langle Z \rangle,$$

is of course biholomorphic. It can be shown that each biholomorphic map of $\mathbb{H}_n$ onto itself is symplectic (i.e. of this form).

2.5 Remark. *Two symplectic matrices* $M, N \in \mathrm{Sp}(n, \mathbb{R})$ *have the same action on* $\mathbb{H}_n$ *if and only if*

$$M = \pm N.$$

Examples of symplectic substitutions are

1) $M = I = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$; $\qquad I\langle Z \rangle = -Z^{-1}$,

2) $M = \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}, S = S'$; $\qquad M\langle Z \rangle = Z + S$,

3) $M = \begin{pmatrix} U' & 0 \\ 0 & U^{-1} \end{pmatrix}, U \in \mathrm{GL}(n, \mathbb{R})$; $\qquad M\langle Z \rangle = U'ZU$.

For proofs we refer to [Fr4].