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ABSTRACT.

A pseudo-tree is a partially ordered set such that the set of all predecessors of any element is linearly ordered. Clearly, each linearly ordered set is a pseudo-tree and pseudo-trees are, in general, much more complicated objects than chains. The aim of this paper is to develop a theory of natural order topologies on pseudo-trees which extends the theories of linearly ordered topological spaces and GO-spaces. Moreover, applications are given for some classes of continua which admit a natural ordering.

To the memory of my Mother

CONTENTS

1. INTRODUCTION

The main purpose of this paper is to build up a theory of natural topologies on pseudo-trees, i.e., partially ordered sets which fulfil the following acyclicity condition: for each point the set of all its predecessors is linearly ordered. The developed theory appears to be a natural extension of the theories of linearly ordered topological spaces and GO-spaces (see Remark 6.12). It contains a part of the research due to T.B.Muenzenberger, R.E.Smithson and L.E.Ward (see Remark 5.16). The obtained results are used to solve problems of J. van Mill, B.J.Pearson and J.J.Charatonik.

Chapter 2 contains all the preliminary definitions and facts. The theory of topologies on pseudo-trees is developed in Chapters 3-7. It turns out that, dealing with pseudo-trees, there are two immediate ways to generalize the construction of the natural topology of a linearly ordered set. They lead to the interval topology T_J and order topology $T_{\leq}$. Unfortunately, T_J and $T_{\leq}$ do not behave well (even in the simplest cases they need not be Hausdorff, etc.). However, all the pathologies are omitted when one concernes the topology of a type $T'_{\leq}$ which makes pseudo-trees being monotonically normal, and compact pseudo-trees being regular supercompact. We show that the restriction to pseudo-trees in the class of all partially ordered sets is essential (Example 3.3). We do not state the corollaries for trees (which are the most important class of pseudo-trees), this can be easily done by the reader, however, see also Remark 6.17.

Chapters 8-10 are of a different nature than the predeceding part of the paper. They deal with three classes of uniquely arcwise connected

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continua (dendrons, dendroids and hyperspaces of all subcontinua of hereditarily indecomposable continua). Since each uniquely arcwise connected space admits a natural acyclic order structure, we are able to apply topological properties of pseudo-trees to get some information on continua considered in the last three chapters.

Now, recall a general question of M.A.Maurice (see [53, p. 289]): *which theorems known for linearly ordered topological spaces can be proved for topological spaces whose topology is less directly tied to linear orderings*?

2. PRELIMINARIES

We include a subject index (which contains also all abbreviations) at the end of the paper. If a confusion is possible we will equip our abbreviations with additional subscripts (for example we will sometimes write $r_X(x)$, $r_{\le}(x)$, $r_{(X,\le)}(x)$ instead of $r(x)$; etc.).

We will often write iff to mean "if and only if".

A. Set theory

An ordinal number is identified with the set of all its predecessors. A cardinal number is the first ordinal of a given cardinality. Nonnegative integers are treated as ordinals. Recall that 0 is a limit ordinal. By ω and ω_1 we will denote the first infinite and the first uncountable ordinals, respectively. If α is a cardinal number, then α^+ denotes the first cardinal greater than α. $|A|$ denotes the cardinality of a set A.

By **R** we denote the set of real numbers. If we write $a \le b$, for some $a,b \in \mathbf{R}$, then we always mean the standard ordering of reals. If $a,b \in \mathbf{R}$, then $[a,b]$ denotes the interval with end-points a,b; $[a,b[\; = [a,b] - \{b\}$, etc.

Let A be a family of subsets of a set X. We will say that A is binary [58, p. 7] provided for each subfamily $B \subset A$ such that $\bigcap B = \emptyset$ there are $U,V \in B$ such that $U \cap V = \emptyset$. Moreover, we will say that A is cross-free [61, p. 60] if for any $U,V \in A$ either $U \subset V$, or $V \subset U$, or $U \cap V = \emptyset$, or $U \cup V = X$. By $\vee.A$ (resp. $\wedge.A$) we will denote the family of all unions (resp. intersections) of finitely many members of A. Observe that $\vee.\wedge.A = \wedge.\vee.A$ is the least family of subsets of X which contains A and is closed with respect to taking finite unions and intersections.

B. Ordered sets

We will denote partially ordered sets by $(X,\le)$. "Isomorphism" will always mean "isomorphism of partially ordered sets". We will write $\neg\, x < y$ to mean that x is not less than y, and $x \perp y$ to mean that $\neg\, x \le y$ and $\neg\, x \ge y$.

Let $(X,\le)$ be a partially ordered set, $A,B,Y \subset X$, $B \ne \emptyset$. $(Y,\le)$ will always denote the set Y together with its partial ordering induced from $(X,\le)$. Moreover, we will write:

$A \le B$ if $a \le b$ for any $a \in A$, $b \in B$; and

$A < B$ if $a < b$ for any $a \in A$, $b \in B$.

We also will use the following conventions: $\emptyset < B$; $\emptyset < x$; and $\neg\, x < \emptyset$, for each $x \in X$. Let: $M(A) = \{x \in X : A \le x\}$, $m(A) = \{x \in X : A < x\}$, $L(B) = \{x \in X : x \le B\}$ and $l(B) = \{x \in X : x < B\}$. We will always write $M(x)$ instead of $M(\{x\})$, etc.

By $\inf(Y)$ (resp. $\sup(Y)$) we will denote the greatest (resp. the least) element c of $(X,\le)$ such that $c \le Y$ (resp. $c \ge Y$), provided such an element c exists. In particular, $\sup(\emptyset)$ denotes the least element of $(X,\le)$.

We say that Y is a <u>chain</u> of $(X,\le)$ provided $(Y,\le)$ is a linearly ordered set. Maximal chains (with respect to the inclusion) are called <u>branches</u>. We say that Y is an <u>anti-chain</u> of $(X,\le)$ if $x \perp y$ for any $x,y \in Y$, $x \ne y$.

$(X,\le)$ is said to be a <u>semi-lattice</u> provided $\inf(\{x,y\})$ exists in $(X,\le)$ for any $x,y \in X$. If $\inf(C)$ exists for each nonempty $C \subset X$, then we say that $(X,\le)$ is a <u>complete semi-lattice</u>.

B^1. Linearly ordered sets

We will often write "chain" instead of "linearly ordered set".

Let $(X,\le)$ be a chain and $Y \subset X$. We say that:

$(X,\le)$ <u>contains a jump</u> if there are $x,y \in X$ such that $L(x) = l(y)$;

$(X,\le)$ <u>contains a gap</u> if there exist nonempty $A,B \subset X$ such that $A < B$, $A \cup B = X$, A has no greatest element and B has no least element;

a set Y is <u>dense</u> in $(X,\le)$ provided, for any $x,y \in X$, if $x < y$, then there exists a point $t \in Y$ such that $x \le t \le y$.

B^2. Pseudo-trees

We say that a partially ordered set $(X,\le)$ is a pseudo-tree [46, Definition 6, p. 83] if the following acyclicity condition holds:

if $x,y,z \in X$, $x < z$ and $y < z$, then

either $x = y$, or $x < y$, or $x > y$.

Saying equivalently, $(X,\le)$ is a pseudo-tree iff $L(x)$ is a chain in $(X,\le)$, for each $x \in X$. Clearly, each linearly ordered set is a pseudo-tree.

Let $(X,\le)$ be a pseudo-tree and $Y \subset X$. We say that:

Y is a semi-branch of $(X,\le)$ if Y is a chain of $(X,\le)$ and $L(y) \subset Y$, for each $y \in Y$ (hence $\emptyset$ is a semi-branch);

Y is convex in $(X,\le)$ provided $M(L(x) \cap L(y)) \cap (L(x) \cup L(y)) \subset Y$, for any $x,y \in Y$ (see also Remark 5.14, below).

Let D be a semi-branch of $(X,\le)$ and set

A_D = {A : A is a maximal family of branches of $(X,\le)$ such that $C \cap C' = D$ for any $C,C' \in A$, $C \ne C'$}.

Note that $|A| = |A'|$ for any $A,A' \in A_D$. Hence we can define a cardinal number $r'(D)$ as $|A|$, for any $A \in A_D$. We define also $r(D)$ as follows: $r(D) = r'(D)$ provided either $D = \emptyset$ or $D = \{d\}$ for some $d \in X$, and $r(D) = r'(D) + 1$ otherwise (see also remarks in the begining of Chapter 8). If $x \in X$ we will often write $r(x)$ instead of $r(L(x))$.

Let $(X,\le)$ and $(Y,\le')$ be pseudo-trees, $(J,\le'')$ a chain, and $f\colon X \to Y$ and $j\colon X \to J$ be functions. We will say that:

$f\colon (X,\le) \to (Y,\le')$ is semi-convex if $f(L(x)) = L(f(x))$ for each $x \in X$;

$f\colon (X,\le) \to (Y,\le')$ is convex provided f is semi-convex and $f^{-1}(B)$ is convex in $(X,\le)$ for each convex subset B of $(Y,\le')$;

j embeds $(X,\le)$ into $(J,\le'')$ if $f(x) < f(y)$ for any $x,y \in X$ such that $x < y$ (see also [24, p. 15]);

j $(J,\le'')$-folds $(X,\le)$ provided j is a semi-convex map which embeds $(X,\le)$ into $(J,\le'')$;

j strongly $(J,\le'')$-folds $(X,\le)$ if j $(J,\le'')$-folds $(X,\le)$ and $j(C) = J$ for each branch C of $(X,\le)$.

Let $(X,\le)$ be a pseudo-tree, α an ordinal number and

$Q = (X_\beta : \beta < \alpha)$ a (transfinite) sequence of subsets of X. We will say that Q is a description of $(X,\le)$ if the following conditions hold:

(i) if either $\beta = 0$ or $\beta = \gamma + 1$, then $X_\beta = \bigcup P_\beta$, where P_β is a maximal family of pairwise disjoint branches of a sub-pseudo-tree $(X - \bigcup \{X_\delta : \delta < \beta\}, \le)$;

(ii) if $\beta \ne 0$ and β is a limit ordinal, then x belongs to X_β iff $x \notin \bigcup \{X_\gamma : \gamma < \beta\}$ and $l(x) \subset \bigcup \{X_\gamma : \gamma < \beta\}$; and

(iii) $X = \bigcup \{X_\beta : \beta < \alpha\}$.

Clearly, each pseudo-tree admits many descriptions.

B^3. Trees

A pseudo-tree $(X,\le)$ is said to be a tree if a chain $(L(x),\le)$ is a well-ordered set, for each $x \in X$. The definitions of a length of a tree, an Aronszajn tree and a Souslin tree can be found for example in [46] and [24]. Recall only that a tree $(X,\le)$ is said to be **Q**-embeddable (resp. **R**-embeddable) provided that there exists a function which embeds $(X,\le)$ into the set of all rational numbers (resp. real numbers). Recall also that **Q**-embeddable trees do exist (see for example [46], it is easy to check that the Aronszajn tree constructed there in the proof of Theorem 2 on p. 330-332 is **Q**-embeddable).

C. Topology

Topological spaces will be often denoted by (X,T), where X is a set and T a family of all open subsets of X. However, we will often use the following convention: if (X,T) is a topological space and $Y \subset X$, then (Y,T) denotes the topological space Y whose topology is induced from (X,T). This will never lead to a confusion.

We will say that a topological space X is:

supercompact if X admits a subbasis S for closed sets such that S is a binary family (clearly, each supercompact space is compact - but not conversely - see for example [58]);

regular supercompact [58, p. 43-44] provided X admits a closed subbasis S such that S is a binary family and $\vee.\wedge.S$ consists of closed domains (i.e., $cl(int(U)) = U$ for each $U \in S$);

monotonically normal [33, p. 481-482] if X is a T_1-space and there exists an operator H which assignes to each pair (p,C), where $C \subset X$ is closed and $p \in X - C$, an open set $H(p,C) \subset X$ such that: (1) $p \in H(p,C) \subset X - C$; (2) if $D \subset X$ is closed and $p \notin C \supset D$, then $H(p,C) \subset H(p,D)$; and (3) if $p,q \in X$ and $p \neq q$, then $H(p,\{q\}) \cap H(q,\{p\}) = \emptyset$;

rim-finite (resp. rim-compact) if each point of X has an arbitrarily small open neighbourhood with the finite (resp. compact) boundary;

semi-locally connected provided each point of X admits an arbitrarily small open neighbourhood the complement of which has finitely many components;

zero-dimensional if X admits a basis consisting of closed-open sets;

a continuum if X is a compact connected Hausdorff space;

a Suslinian curve provided X is a continuum such that each family of pairwise disjoint nondegenerate subcontinua of X is countable.

A closure, an interior and a boundary of a set A will be denoted by cl(A), int(A) and bd(A), respectively. We will also use standard symbols Ls, Li and Lim to denote topological limits of a sequence of sets.

If X is a metric space, d a distance function on X, $A \subset X$ and ε is a positive real number, then diam(A) denotes the diameter of A, and $B(A,\varepsilon) = \{x \in X : d(A,x) < \varepsilon\}$ denotes the open ball of the radius ε and with the center A.

By C(X) we will denote the hyperspace of all subcontinua of a continuum X. C(X) will always be equipped with the Vietoris topology - see e.g. [72].

C^1. Topologies on partially ordered sets

Let X be a set, T a topology on X and $\leq$ a partial ordering on X. We will say that $\leq$ is a continuous ordering of (X,T) if the set $\{(x,y) \in X \times X : x \leq y\}$ is closed in the space $(X,T) \times (X,T)$.

Let $(X,\leq)$ be a chain. The family $S = \{L(x) : x \in X\} \cup \{M(x) : x \in X\}$ is a subbasis for closed sets of the order topology T on X. The space $(X,\leq,T)$ is called a linearly ordered topological space (LOTS). The basic information on linearly ordered topological spaces can be found in [27] and [53]. Recall only that (X,T) is a connected space iff $(X,\leq)$ has no jumps and gaps, and (X,T) is compact iff $(X,\leq)$ has no gaps and contains the least and greatest elements.

Let T' be a topology on the chain $(X,\leq)$. We will say that $(X,\leq,T')$ is a generalized ordered space (GO-space; see e.g. [52], [53]) provided $T \subset T'$ and there exists a basis S' for the space (X,T') such that S' consists of sets of the form either $m(x) \cap l(y)$, or $m(x) \cap L(y)$, or $M(x) \cap l(y)$, or $M(x) \cap L(y)$, $x,y \in X$.

C^2. Arcs, arcwise connected spaces

We say that a topological space (X,T) is an arc provided it is a continuum with exactly two non-cut points. Then there exists a linear ordering $\leq$ on X such that $(X,\leq,T)$ is a LOTS; moreover, if $(X,\leq)$ is a chain, T' denotes the order topology of $(X,\leq)$ and the space (X,T') is compact connected, then (X,T') is an arc [35,Theorem 2-27, p. 54].

We say that a topological space X is uniquely arcwise connected if, for any $x,y \in X$, $x \neq y$, there exists exactly one arc in X the end-points of which are x,y. This arc will be denoted by $[x,y]$. If, for any distinct points $x,y \in X$, the arc $[x,y]$ is separable, then X is called an I-connected space.

A uniquely arcwise connected space X is said to be a nested space provided, for each family A of arcs of X such that A is linearly ordered by inclusion, the set $\bigcup A$ is contained in some arc.

C^3. Dendritic spaces, dendrons

We will say that a topological space X is a dendritic space provided X is connected and for any $x,y \in X$, $x \neq y$, there exists $z \in X$ such that x and y lie in distinct components of $X - \{z\}$. If X is a dendritic space, $x \in X$ and U is a component of $X - \{x\}$, then U is an open set and $bd(U) = \{x\}$, [83, Theorem 4, p. 296]. Hence each dendritic space is Hausdorff. Recall that there exist dendritic spaces which are either not locally connected or not arcwise connected (see Example 5.15 (ii)).

If X is a dendritic space and x,y are distinct points of X, then an <u>interval</u> [x,y] of X with end-points x,y is defined to be the set $\{x,y\} \cup \{z \in X : x,y$ lies in distinct components of $X - \{z\}\}$. The introduced notation will never lead to a confusion because if X is a uniquely arcwise connected dendritic space, $x,y \in X$, $x \neq y$, then the arc of X with end-points x,y is the same as the interval [x,y], [71, Theorem 3, p. 109].

Compact dendritic spaces are called <u>dendrons</u>. Recall that each <u>dendrite</u> (i.e., a metrizable, acyclic and locally connected continuum) is a dendron, and each metrizable dendron is a dendrite. Moreover, there exists a universal dendrite in the plane $\mathbf{R}^2$, [45, Example (v), p. 300].

3. ORDER TOPOLOGIES ON PSEUDO-TREES

Let $(X,\le)$ be a partially ordered set and set $S_1 = \{M(x) : x \in X\}$, $S_2 = \{X - m(x) : x \in X\}$ and $S_\le = S_1 \cup S_2$. Let $T_\le$ denote the topology on X for which $S_\le$ is a closed subbasis.

3.1. PROPOSITION. If $(X,\le)$ is a partially ordered set then $T_\le$ is a T_1-topology on X.

3.2. PROPOSITION. If C is a branch of a pseudo-tree $(X,\le)$, then the topology of C induced from the space $(X,T_\le)$ is precisely the usual order topology of C.

Proof. Indeed, if $x \notin C$, then $M(x) \cap C = \emptyset$ and $C \subset X - m(x)$.

3.3. EXAMPLE. The assumption that $(X,\le)$ is a pseudo-tree is essential in Proposition 3.2. In fact, let $X = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$ and let $(x,y) \le (u,v)$ provided either $x = 0$ and $y = -1$, or $u = 0$ and $v = 1$, or $xu > 0$ and $y \le v$. Observe that $(X,T_\le)$ is a Hausdorff space. Let $C = \{(x,y) \in X : x \le 0\}$. Then C is a branch of $(X,\le)$ and C equipped with the topology induced from $(X,T_\le)$ is homeomorphic to $[0,1[\cup \{2\}$.

3.4. THEOREM. Let $(X,\le)$ be a pseudo-tree and x,y be distinct points of X. Then there are no disjoint open neighbourhoods of x and y in $(X,T_\le)$ <u>iff</u> at least one of the following conditions hold:

(a) $l(x) = l(y) \ne \emptyset$ and $\sup(l(x))$ does not exist;

(b) $l(x) = L(y)$ (resp. $l(y) = L(x)$) and there a branch C of $(X,\le)$ such that $y \in C$ (resp. $x \in C$) and $y = \inf(C \cap m(y))$ (resp. $x = \inf(C \cap m(x))$);

(c) x,y are minimal elements of $(X,\le)$ and there exists a branch C without a least element;

(d) there exist distinct points $x_1, x_2, \ldots \in X$ such that $l(x) = l(y) = l(x_1) = \ldots$;

(e) there exist distinct points $x_1, x_2, \ldots \in X$ such that either $L(y) = l(x) = l(x_1) = \ldots$ or $L(x) = l(y) = l(x_1) = \ldots$.

Proof. Suppose that U, V are disjoint open subsets of (X,T) such that $x \in U$, $y \in V$. We may assume that
$U = m(a) \cap \bigcap \{X - M(a_i) : 1 \le i \le m\}$ and
$V = m(b) \cap \bigcap \{X - M(b_i) : 1 \le i \le n\}$, for some
$a, a_1, \ldots, a_m, b, b_1, \ldots, b_n \in X$ such that $a < x$, $a < a_i$, either $x < a_i$ or $x \perp a_i$, $a_i \perp a_j$ for $i \ne j$, $b < y$, $b < b_i$, either $y < b_i$ or $y \perp b_i$, $b_i \perp b_j$ for $i \ne j$.

If (a) holds, then $a, b \in l(x) = l(y)$ and there is $c \in X$ such that $a < c < x$ and $b < c < y$. Hence $c \in U \cap V$ – a contradiction. Suppose that (b) holds. Let $w = \min\{z : z \in C \cap m(y)$ and either $z = a_i$ or $z = b_i$, for some $i\}$. Since $\inf(C \cap m(y)) = y$, the set $W = \{z : y < z < w\}$ is nonempty. Observe that $W \subset U \cap V$. If (c) holds, then $\emptyset \ne l(t) \subset U \cap V$, where $t = \min\{z : z \in C$ and either $z = a_i$ or $z = b_i$, for some $i\}$. Suppose that either (d) or (e) holds. Note that $\{x_1, x_2, \ldots\} \subset m(a) \cap m(b)$. Moreover, $\{x_1, x_2, \ldots\} \subset X - M(z)$ if $z \notin \{x_1, x_2, \ldots\}$, and $\{x_1, x_2, \ldots\} - \{x_n\} \subset X - M(z)$ if $z = x_n$. Hence infinitely many points of $\{x_1, x_2, \ldots\}$ are contained in $U \cap V$, a contradiction.

Now, assume that x, y have no disjoint neighbourhoods in $(X, T_{\le})$.

<u>Case 1:</u> $x > y$. Hence there is no $z \in X$ such that $x > z > y$, i.e., $l(x) = L(y)$. Suppose that (b) and (e) do not hold. Let $A = \{z : z = \inf(C \cap m(y))$ for some branch C of $(X, \le)$ such that $y \in C\}$. Hence $y \notin A$ and $m(y) = \bigcup \{M(z) : z \in A\}$. Moreover, A is finite – because $A = \{z \in X : l(x) = l(z)\}$. Set $U = m(y)$ and $V = \bigcap \{X - M(z) : z \in A\}$. Then U, V are disjoint neighbourhoods of x, y, respectively; a contradiction.

<u>Case 2:</u> $x \perp y$. Hence there is no $z \in X$ such that either $l(x) \cap l(y) < z < x$ or $l(x) \cap l(y) < z < y$, i.e., $l(x) = l(y)$. Suppose that (a)-(d) do not hold. Since (a) does not hold, either $l(x) = \emptyset$ or $l(x) = L(z)$ for some $z \in X$. Since (d) does not hold, the set $B = \{t : l(x) = l(t)\}$ is finite. Moreover, if $l(x) = \emptyset$ then $X = \bigcup \{M(t) : t \in B\}$, and if $l(x) = L(z)$ then $m(z) = \bigcup \{M(t) : t \in B\}$