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PREFACE OF THE SECOND EDITION

This book is the second edition of a text written in 1986, 1987 and reproduced in 1988 in the preprint series Notas de Matemática of the Universidade Estadual de Campinas. This second edition has benefitted from a few improvements but it is not substantially different. The set of references has however been enriched by many more recent papers.

INTRODUCTION

In the year 1954 L.Schwartz published a celebrated result on the "impossibility of the multiplication of distributions", proving that there does not exist a differential algebra A containing the space $\mathcal{D}'$ of distributions (on the real line) and having the classical properties relative to differentiation and to the algebraic operations of addition and multiplication. At that time it had been recognized that physicists had been using "illegal multiplications of distributions", even in classical physical theories such as Continuum Mechanics. In many cases this leads to satisfactory results, some of them consisting of numerical codes used in industry. This situation suggests to mathematicians that they should reconsider the problem so as to try to find a solution in form of a suitable underlying mathematical theory.

Seven years ago, a differential algebra $\mathcal{G}$ containing $\mathcal{D}'$, and having all natural properties (of course one of them is in a weakened form relative to Schwartz' impossibility result), has been constructed. Recently it was recognized that this theory was perfectly well-adapted to the solution of problems of physics and engineering involving such multiplications.

The aim of this book is to provide a simple introduction to this nonlinear theory of generalized functions introduced by J.F.Colombeau. Now this theory extends from pure mathematics to physics, passing through the theory of partial differential equations, both from the theoretical and the numerical viewpoints.

pure mathematics: the theory presents a faithful generalization of the classical theory of C^∞ functions, encompassing all of its main properties. Further it provides a synthesis of most existing multiplications of distributions.

physics: in some cases in which the equations of physics involve "multiplications of distributions" and give "ambiguous results" this theory allows and suggests more precise formulations of the equations, thus leading to new (and unambiguous) formulas, in agreement with the experimental facts.

theoretical solutions of partial differential equations: in this theory one can obtain general existence-uniqueness results for large classes of equations which have no solutions in the setting of distributions. These new solutions are coherent with the classical solutions when they exist. In the case of equations of physics it has often been checked that these new solutions are in fact classical functions, for instance piecewise C^∞ functions, corresponding to the solutions expected by physicists and engineers.

numerical solutions: this theory provides new numerical methods which permit one to compute solutions of systems used in industry (numerical simulations of collisions for instance). It gives the possibility of mastering the treatment of systems in nonconservative form, which is made more important by the fact that this theory allows one to transform conservative systems into equivalent systems in nonconservative form (in certain circumstances one obtains in this way very efficient numerical schemes even for the classical system of fluid dynamics).

This text presents basic concepts and results that until now were only published in article form. It is intended for mathematicians, but, since we do not dissociate the theory from its applications, it may also be useful for physicists and engineers. The needed prerequisites for its reading are essentially reduced to the classical notions of differential calculus and the theory of integration over n-dimensional euclidean space E_n. Since it is recent and very original, this theory is not widely understood; we give an account of the basic points using its simplest presentation. The applications are yet more recent and most of them still in preprint form. We give a sketch of some of them: semilinear hyperbolic systems, nonlinear parabolic equations, systems in non-conservative form, new formulas and new numerical schemes. Therefore we believe this survey will be useful to a wide audience and should facilitate the reading of articles and books on this subject.

Chapter 1 is an introduction to these generalized functions on an arbitrary open subset Ω of the euclidean space E_n. In § 1.1 we sketch how J.F.Colombeau has obtained this concept from a study of C^∞ or holomorphic functions over the locally convex spaces $\mathcal{D}(\Omega)$ and $\mathcal{E}'(\Omega)$. Fortunately the reader who is not familiar with the concept of locally convex spaces may drop this section without any trouble for the understanding of the remainder of the book. For readers acquainted with infinite dimensional spaces, we believe this section might be useful to explain how this original concept can be obtained, in a natural way, from a purely mathematical viewpoint. We also expose successive natural simplifications of the first construction and then an elementary definition is studied in §§ 1.2 to 1.6 in a way which is significantly different from the other texts on this subject. We obtain all the desired properties (coordinate invariance, free restrictions to subspaces and composition products, independence of the set of generalized numbers of the dimension of E_n). In §1.7 we introduce natural topologies on $\mathcal{G}(\Omega)$, as well as concepts of strong and weak convergence in this space. In §1.8 we define a subspace of $\mathcal{G}(\Omega)$, $\mathcal{G}_s(\Omega)$, which although simpler than $\mathcal{G}(\Omega)$, is a sufficient setting for most of the applications in Chapter 3. There is no natural inclusion of the space $\mathcal{D}'(\Omega)$ of the distributions on Ω in $\mathcal{G}_S(\Omega)$: several elements of $\mathcal{G}_S(\Omega)$ may represent equally well any given distribution. Nevertheless this is perhaps clearer and more adequate for some physical applications. We work in $\mathcal{G}_S(\Omega)$ in this book since it gives shorter proofs; anyway we could have worked in $\mathcal{G}(\Omega)$ just by modifying the proofs in a routine way. One might prefer (§1.10) a slightly more sophisticated definition of $\mathcal{G}(\Omega)$ in which generalized solutions of algebraic equations are automatically classical solutions; the price to pay consists in straightforward additional technicalities in some proofs of results in Chapters 2 and 3. In the Appendix of Chapter 1 we expose how Colombeau's theory unifies the previous definitions of multiplications of distributions proposed by various authors in special cases.

A novelty relative to distribution theory is that the definition of these generalized functions extends easily to any not necessarily open subset X of $\mathbb{R}^n$, thus generalizing the classical C^∞ functions on X in Whitney's sense. Chapter 2 is devoted to this natural extension, which is used in some applications and which has not yet been published in book form. We recall the classical concept of the Whitney C^∞ functions on X and state without proof the Whitney extension theorem. A similar result holds for Colombeau's generalized functions. The proof we sketch is obtained from an analysis of a proof of the classical Whitney extension theorem (§2.4). We prove in detail two particular cases: when X is reduced to a single point (Borel's theorem) and when X is a closed half space (by following a very simple proof due to R.T.Seeley).

Several applications are exposed in Chapter 3. They concern nonlinear partial differential equations; some of them consist in new formulas and new numerical schemes of interest to physicists and engineers. In elasticity and elastoplasticity engineers state Hooke's law in nonconservative form, see Appendix 1. Numerical tests make it evident that the systems of partial differential equations thus obtained have shock-wave solutions, represented mathematically by classical discontinuous functions. Since these systems are in nonconservative form, the concept of discontinuous solutions does not make sense mathematically: this gives rise to "meaningless" or "ambiguous" multiplications of distributions. Colombeau's generalized functions provide a setting in which these problems can be treated successfully. This has been done recently by Colombeau, Le Roux and their co-authors, and a sketch of their theory is given in §§3.1 to 3.4 and Appendices 2,3 and 4. In §§3.1 and 3.4 we show in models of elasticity and elastoplasticity how we can compute jump formulas for systems in nonconservative form. In §3.2 we show for these systems how we may solve Cauchy problems with discontinuous Cauchy data, in agreement with the observations of numerical analysis, see Appendix 2. In §3.3 we obtain a new (nonconservative) formulation of hydrodynamics which is equivalent to the classical formulation and which gives rise to new numerical schemes, described in Appendix 3. These new formulas and numerical schemes are in agreement with the experimental observations. They are used for numerical simulations of collisions; these phenomena last only a few microseconds, which makes the experimentation very difficult (besides the obvious fact that such experimentation is extremely expensive) hence the interest of these numerical simulations. Of course in this book we only sketch a few schemes in one space dimension. We refer the reader to specialized papers for more results and for schemes in two and three space dimensions. In §3.5 we present a general existence-uniqueness result for semilinear hyperbolic systems with Cauchy data distributions. In §3.6 we present an existence-uniqueness result for a nonlinear parabolic equation with Cauchy data distributions. It is known that in general such equations do not have solutions in the classical sense; the solutions we obtain agree with the classical solutions when they exist. We only give the simpler results and we refer the reader to more specialized papers.

In short we hope that this book gives an easy and reasonably self-contained panorama of this new theory (thereby making its applications accessible to a wider audience).

We became convinced of the interest of Colombeau's nonlinear theory of generalized functions because it satisfies at the same time the following requirements of mathematical beauty and of efficiency:

a) $\mathcal{G}(\Omega)$ is a differential algebra containing naturally a copy of the vector space $\mathcal{D}'(\Omega)$ (see Rosinger's recent book [4] for the natural character of this inclusion) on which it induces the classical concept of distributional derivatives, and $C^\infty(\Omega)$ is a subalgebra of $\mathcal{G}(\Omega)$. This is the best possible situation for a differential algebra containing $\mathcal{D}'(\Omega)$. The algebra $\mathcal{C}(\Omega)$ of all continuous functions cannot be a subalgebra in any case (Schwartz's impossibility result) but in Colombeau's theory this difficulty is mastered and overcome.

b) at the same time this theory provides a very convenient setting for finding and studying solutions of nonlinear partial differential equations, both from a purely mathematical viewpoint (existence, uniqueness, coherence with classical solutions when they exist) and from the viewpoint of numerical analysis, engineering and physics (explicit computations of solutions, new numerical schemes). Often one can prove that the "abstract solutions" have a very classical aspect, for instance, discontinuous functions representing shock waves; even in physical cases these solutions often do not exist within distribution theory since they are not solutions in the sense of distributions.

We have developed this book as an attempt to present these facts very clearly. The overlap with Rosinger's recent book [4] treating Colombeau's theory, of applications to partial differential equations, and of connections with Rosinger's algebras is quite reduced, and so these books complete each other. The overlap with Colombeau's original books [2,3] is also quite reduced since our book treats developments and applications which did not exist at the time those books were written. Further our presentations of Colombeau's generalized functions is original and has not been published up to now.

I am very much indebted to J.F.Colombeau for having intro-

duced me to his research already in 1982 when he was developing it, for several discussions and for having sent me manuscripts of his papers. I am also very much indebted to J.Aragona, J.T. Donohue, J.E.Galé, M.Langlais, A.Y.Le Roux, A.Noussair, M.Oberguggenberger and B.Perrot for their help and for corrections in the text.

CHAPTER 1

GENERALIZED FUNCTIONS ON AN OPEN SUBSET OF E_n

1.1 - THE ORIGINAL DEFINITION.

J.F. Colombeau, trying to find a general multiplication of distributions, had successive ideas until he arrived at a simple definition which requires only a very elementary knowledge of differential calculus.

We begin this chapter with these ideas so that the reader might follow Colombeau's reasoning. This paragraph may be dropped by those who are not familiar with the theories of locally convex spaces and distributions.

If Ω denotes an open subset of $\mathbb{R}^n$ and $\mathcal{D}(\Omega)$ the space of C^∞ complex valued functions on Ω with compact support, Colombeau's first idea was to use C^∞ or holomorphic functions on $\mathcal{D}(\Omega)$. He thought that, if T_1 and T_2 were distributions on Ω, their product might be the map

$$\varphi \in \mathcal{D}(\Omega) \longmapsto \langle T_1, \varphi\rangle . \langle T_2, \varphi\rangle \in \mathbb{C} ,$$

where $\langle T, \varphi\rangle \in \mathbb{C}$ denotes the value of the distribution T on the test function φ ; but this definition would not even generalize the usual multiplication of C^∞ functions since for f_1, $f_2 \in C^\infty(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ we have, in general,

$$(1) \qquad \int f_1(x)\ \varphi(x)\ dx\ .\ \int f_2(x)\ \varphi(x)\ dx \neq \int f_1(x)\ f_2(x)\ \varphi(x)\ dx .$$

Without abandoning the idea of using C^∞ or holomorphic functions on $\mathcal{D}(\Omega)$ and in order to identify the two members of (1), he considered the idea of taking a quotient.

We recall that $\mathcal{D}(\Omega)$ is contained and dense in $\mathcal{E}'(\Omega)$ (the space of all distributions on Ω with compact support $\mathcal{E}'(\Omega)$ is the topological dual of $C^\infty(\Omega)$), which is a strong dual of a Fréchet-Schwartz space. $C^\infty(\mathcal{E}'(\Omega))$ and $C^\infty(\mathcal{D}(\Omega))$ denote respectively the spaces of all complex valued C^∞ functions on $\mathcal{E}'(\Omega)$ and on $\mathcal{D}(\Omega)$, see Colombeau [1]. Then the restriction map

$$C^\infty(\mathcal{E}'(\Omega)) \longrightarrow C^\infty(\mathcal{D}(\Omega))$$

$$\emptyset \longmapsto \emptyset|_{\mathcal{D}(\Omega)}$$

is injective, see Colombeau [1, 0.6.9 and 1.1.6]. So we may consider that

$$C^\infty(\mathcal{E}'(\Omega)) \subset C^\infty(\mathcal{D}(\Omega)) .$$

If $f_1, f_2 \in C^\infty(\Omega)$, the following map

$$T \in \mathcal{E}'(\Omega) \longmapsto \langle T, f_1\rangle \langle T, f_2\rangle \in \mathbb{C}$$

coincides in the set $\{\delta_x : x \in \Omega\}$ (δ_x is the Dirac measure at x) with the classical product $f_1.f_2$. This fact led Colombeau to consider in $C^\infty(\mathcal{E}'(\Omega))$ the equivalence relation r :

$$R_1 \ r \ R_2 \Leftrightarrow R_1(\delta_x) = R_2(\delta_x) \quad \text{for all} \quad x \in \Omega .$$

If A denotes the map

$$C^\infty(\mathcal{E}'(\Omega)) \longrightarrow C^\infty(\Omega)$$
$$R \longmapsto (x \mapsto R(\delta_x))$$

then

$$R \in \text{Ker } A \Leftrightarrow R \ r \ 0 .$$

Therefore the algebras $C^\infty(\mathcal{E}'(\Omega))/\text{Ker } A$ and $C^\infty(\Omega) = L(\mathcal{E}'(\Omega), \mathbb{C})$ are isomorphic (if E is a locally convex space, $L(E,\mathbb{C})$ denotes the space of all continuous linear maps from E into $\mathbb{C}$; one proves classically that $L(\mathcal{E}'(\Omega), \mathbb{C}) = C^\infty(\Omega)$: reflexivity of $C^\infty(\Omega)$).

A new concept of derivatives in $C^\infty(\mathcal{D}(\Omega))$ was defined which generalized the derivation in the sense of distributions and which corresponded, via the map A, to the usual derivatives in $C^\infty(\Omega)$.

For these reasons, Colombeau was seeking an ideal of $C^\infty(\mathcal{D}(\Omega))$ whose intersection with $C^\infty(\mathcal{E}'(\Omega))$ would be Ker A.

The following result gave a convenient characterization of Ker A: given $R \in \text{Ker } A$, $q \in \mathbb{N}$, $\varphi \in \mathcal{A}_q := \{\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $\int \psi(x)\, dx = 1$, $\int x^i \psi(x)\, dx = 0$ if $i \in \mathbb{N}^n$, $1 \leqslant |i| \leqslant q\}$ and K a compact subset of Ω ($K \subset\subset \Omega$, for short) there are $c>0$ and $\eta>0$ such that

$$(2) \qquad |R(\varphi_{\varepsilon,x})| \leqslant c\varepsilon^{q+1}$$

for all $x \in K$ and $0<\varepsilon<\eta$. Here $\varphi_{\varepsilon,x}$ denotes the element of $\mathcal{A}_q$ defined by

$$\varphi_{\varepsilon,x}(\lambda) = \varepsilon^{-n} \varphi(\varepsilon^{-1}(\lambda - x))$$

for all $\lambda \in \mathbb{R}^n$. That is, $R \in C^\infty(\mathcal{E}'(\Omega))$ satisfies inequality (2) for given φ and K if, and only if, $R \in \text{Ker } A$.

It was this characterization of Ker A that would produce the definition of an ideal of $C^\infty(\mathcal{D}(\Omega))$. However, when multiplying an element of Ker A by an element of $C^\infty(\mathcal{D}(\Omega))$, the product might grow very fast with ε^{-1}, when $\varepsilon \to 0$. Therefore he decided to consider only elements of $C^\infty(\mathcal{D}(\Omega))$ with a moderate growth in ε^{-1}, that is, elements R with the property :

for every $K \subset\subset \Omega$ and $D = \dfrac{\partial^{|k|}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}$ $(0 \leqslant |k| < \infty)$ partial derivation operator, there is $N \in \mathbb{N}$ such that, for each $\varphi \in \mathcal{A}_N$ there are $c>0$ and $\eta>0$ satisfying

$$|(DR)(\varphi_{\varepsilon,x})| \leqslant c\varepsilon^{-N} .$$

for all $x \in K$ and $0<\varepsilon<\eta$.

The set of these elements of $C^\infty(\mathcal{D}(\Omega))$ is denoted by $\mathcal{E}_M(\mathcal{D}(\Omega))$. It is obvious that $DR \in \mathcal{E}_M(\mathcal{D}(\Omega))$ if $R \in \mathcal{E}_M(\mathcal{D}(\Omega))$ and that the product in $C^\infty(\mathcal{D}(\Omega))$ of two elements of $\mathcal{E}_M(\mathcal{D}(\Omega))$ is still in $\mathcal{E}_M(\mathcal{D}(\Omega))$.

As some important examples there are the elements of $C^\infty(\mathcal{E}'(\Omega))$ and the distributions on Ω.

Now, an ideal to be considered is the set $\mathcal{N}$ of all $R \in \mathcal{E}_M(\mathcal{D}(\Omega))$ with the following property :

(N) | for every $K \subset\subset \Omega$ and D derivation operator there is $N \in \mathbb{N}$ such that, for each $\varphi \in \mathcal{A}_q$, with $q \geqslant N$, there are $c>0$ and $\eta>0$ satisfying

$$|(DR)(\varphi_{\varepsilon,x})| \leqslant c\varepsilon^{q-N}$$

for all $x \in K$ and $0<\varepsilon<\eta$.

This ideal $\mathcal{N}$ satisfies the initial requirement :

$$\mathcal{N} \cap C^\infty(\mathcal{E}'(\Omega)) = \operatorname{Ker} A .$$

Finally the quotient

$$\mathcal{G}(\Omega) := \mathcal{E}_M(\mathcal{D}(\Omega))/\mathcal{N}$$

is an algebra containing $C^\infty(\Omega)$ as a subalgebra, since $C^\infty(\Omega)=C^\infty(\mathcal{E}'(\Omega))/\operatorname{Ker} A$, and containing $\mathcal{D}'(\Omega)$, since $\mathcal{N}\cap\mathcal{D}'(\Omega)=\{0\}$.

Many results were proved by using this definition but, working with representatives of elements of $\mathcal{G}(\Omega)$, Colombeau realized that they did not need to be defined on the whole $\mathcal{D}(\Omega)$, since only their values on functions of the kind $\varphi_{\varepsilon,x}$, with $\varphi \in \mathcal{A}_q$ (q large enough), $\varepsilon>0$ small enough and $x \in \Omega$, were required by the definition. Then he replaced $C^\infty(\mathcal{D}(\Omega))$ by an algebraic inductive limit of spaces $C^\infty(U)$, where U is an open set of $\mathcal{D}(\Omega)$ with the following property :

(P_1) | for every $K \subset\subset \Omega$ there is $N \in \mathbb{N}$ such that for all $\varphi \in \mathcal{A}_N$ there is $\eta>0$ with $\varphi_{\varepsilon,x} \in U$ for all $x \in K$ and $0<\varepsilon<\eta$.

This space was denoted by $\mathcal{E}(\Omega_{\mathcal{D}(\Omega)})$ and its moderate elements were defined in the same manner as those of $C^\infty(\mathcal{D}(\Omega))$, their set being denoted by $\mathcal{E}_M(\Omega_{\mathcal{D}(\Omega)})$.

Again, working on applications, another difficulty used to appear : an element of $\mathcal{E}_M(\Omega_{\mathcal{D}(\Omega)})$ was obtained first on the set $\{\varphi_{\varepsilon,x}\}$, then it had to be extended to an open subset U of $\mathcal{D}(\Omega)$ and sometimes this extension led to complications in proofs.

Therefore the sets $C^\infty(U)$ were replaced by sets $\mathscr{E}^*(U)$, where U might be a subset of $\mathscr{D}(\Omega)$, not necessarily open, satisfying property (P_1), where

$\mathscr{E}^*(U)$ = {R : U → $\mathbb{C}$ such that, if $\varphi \in \mathscr{A}_N$ and $\varepsilon>0$ are fixed such that the set $\{\varphi_{\varepsilon,x} : x \in \omega\}$ is contained in U, where ω is an open subset of Ω, then the function $x \in \omega \longmapsto R(\varphi_{\varepsilon,x})$ is C^∞}.

In this definition differentiability in φ and ε is dropped. We shall come back to this point later. The new algebraic inductive limit was denoted by $\mathscr{E}^*(\Omega_{\mathscr{D}(\Omega)})$.

This is not the simplest definition yet. It was noticed that, when a representative of a generalized function on $\mathbb{R}^n$ was constructed, the chosen set U was often $\bigcup_{x\in\mathbb{R}^n} \tau_x\mathscr{A}_0$, where τ_x is the translation operator ($(\tau_x\varphi)(y)=\varphi(y-x)$). Then, why not to take this set U as the common domain of all representatives of generalized functions on $\mathbb{R}^n$? It is proved in Colombeau [2], §7.7, that the consideration of functions with domain $\mathbb{R}^n\times\mathscr{A}_0$ is indeed equivalent to the more complicated use of the above inductive limit.

Besides the simplification sketched above, there is also a modification on the sets $\mathscr{A}_q$. In order to define composition products and restrictions of generalized functions to subspaces we shall define different sets $\overline{\mathscr{A}}_q$, which further, like the concepts sketched above, give a concept of generalized functions which does not depend on a change of basis, a change of origin and of the norm.

Also one has to consider instead of the bound $c\varepsilon^{q-N}$ in (N) more general bounds $c\varepsilon^{\gamma(q)-N}$, where γ is an increasing function from $\mathbb{N}$ into $\mathbb{R}^+$ tending to $+\infty$ at infinity. This more technical bound is needed for certain properties, see 1.2.16 below.

The elementary definition as given in Colombeau [3] and Rosinger [4] is exactly the one given in § 1.2 below in which our sets $\overline{\mathscr{A}}_q$ would be replaced by the sets $\mathscr{A}_q$. In the definition given in Colombeau [3] and Rosinger [4], the restriction of a generalized function on $\mathbb{R}^n$ to a subspace of dimension less than n is not defined as a generalized function on the subspace, since the sets $\mathscr{A}_q$ depend too much on the space dimension. This is quite natural