Lecture Notes in Mathematics
1623

H. P. Yap

Total Colourings of Graphs

Springer

H. P. Yap

Total Colourings of Graphs

Springer

Author

Hian-Poh Yap
National University of Singapore
Department of Mathematics
Lower Kent Ridge Road
Singapore 0511, Republic of Singapore

Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Yap, Hian Poh:
Total colourings of graphs / Hian Poh Yap. - Berlin ; Heidelberg ; New York ; Barcelona ; Budapest ; Hong Kong ; London ; Milan ; Paris ; Tokyo : Springer, 1996
(Lecture notes in mathematics ; 1623)
ISBN 3-540-60717-X
NE: GT

Mathematics Subject Classification (1991): 05C15, 05C35, 05C75

ISBN 3-540-60717-X Springer-Verlag Berlin Heidelberg New York

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, re-use of illustrations, recitation, broadcasting, reproduction on microfilms or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1996
Printed in Germany

Typesetting: Camera-ready TEX output by the author
SPIN: 10479714 46/3142-543210 - Printed on acid-free paper

Preface

I started writing this book in 1990 and completed the first draft in October 1991. It then took me another one and a half years (June 1992 to December 1993) to revise the first draft. My objective in writing this book is to give an up-to-date account of total colourings of graphs which can be used as a graph theory course/seminar materials for advanced undergraduate and graduate students and as a reference for researchers. To achieve the objectives, easy-to-read, detailed proofs of almost all of the theorems presented in this book, and numerous examples and exercises are provided here. Many open problems are also mentioned. I hope that through this rapid introduction I shall be able to bring the readers to the frontier of this currently very active field in graph theory.

After the first draft of this manuscript was completed, I used it as lecturing material in my graph theory course offered to the advanced undergraduate students of the National University of Singapore (NUS). I thank my students for their patience in attending my lectures and for giving me their valuable feedback.

I would like to thank the NUS for granting me a 10-month (1 July 1991 to 30 April 1992) sabbatical leave so that I could concentrate on writing and revising the manuscript. I would also like to thank the NUS for granting me conference leaves (June 1993 and June 1995) and to the Japan Society for the Promotion of Science for sponsoring my visit to three universities in Japan (12 April to May 1, 1993) so that I could have direct discussions with many graph theorists working on total colourings, and at the same time popularize this subject by giving many survey talks on various topics covered in this book. I had altogether given at least forty survey talks on total colourings of graphs to the following 18 institutions in Taiwan, People's Republic of China, USA, Japan and Singapore during the past three and half years: Academia Sinica (Taipei) and National Chiao Tung University, Hsingchu, Taiwan (1 - 30 November, 1991); Beijing Institute of Technology, Tsinghua University, Institute of Applied Mathematics, Academia Sinica (Beijing), Lanzhou Railway Institute, Lanzhou University, Shaanxi Normal University (Xian), Zhengzhou University and Institute of Systems Science, Academia Sinica (Beijing) (8 December 1991 to 6 January 1992); West Virginia University, USA (14 January to 8 March, 1992); Spring School and International Conference on Combinatorics held at Lushan Mountain and Hungshan Mountain, People's Republic of China (10 April to 30 April, 1992); Science University

of Tokyo, Ibaraki University and Keio University, Japan (12 April to May 1, 1993); Inner Mongolia University and Taiyuan Institute of Machinery, People's Republic of China (June, 1993); Spring School and International Conference on Combinatorics held at Hefei, People's Republic of China (May 22 to June 3, 1995). It is also a great pleasure for me to acknowledge the helpful comments and suggestions received from many friends who hosted my visits : Dr. Bor-Liang Chen, Professors A. J. W. Hilton, Zhang Zhongfu, Yoshimi Egawa, Mikio Kano, Hikoe Enomoto, Ku Tung-Hsin and Li Jiong-Sheng. I would like to express my deepest gratitude to Dr. Hugh R. Hind for carefully reading the first draft and making many valuable comments and suggestions. I am also thankful to Dr. Abdón Sánchez-Arroyo, Mr. Zhang Yi and Professor A. D. Keedwell in proofreading the second draft, to Mr. Liu Qizhang for using computers to draw the figures, to Professors J. C. Bermond, O. V. Borodin, A. V. Kostochka and C. J. H. McDiarmid for sending preprints and reprints of their papers to me, and to Miss D. Shanthi for typing this book in PCTEX.

Finally, a few words on the reference system and the exercises of this book. When a research paper by XX is referenced in the text as XX [93], it denotes that the paper by XX in the List of References was published in 1993. When a paper is referenced as YY [-a], it is unpublished and the ordering a, b, c, ... reflects the ordering of the unpublished papers of YY in the List of References. When an exercise is marked with a minus sign or a plus sign, it means that the exercise is easy or hard/time-consuming respectively; and if it is marked with a star, it means that it is an open problem or a conjecture.

H. P. Yap
November 8, 1995

试读结束，需要全本PDF请购买 www.ertongbook.com

CONTENTS

CHAPTER 1

BASIC TERMINOLOGY AND INTRODUCTION

§1. Basic terminology

In this section we define some basic terms that will be used in this book. Other terms will be defined when they are needed.

Unless stated otherwise, all graphs dealt in this book are finite, undirected, simple and loopless. Let $G = (V, E)$ be a graph, where $V = V(G)$ is its vertex set and $E = E(G)$ is its edge set. For a graph G, we denote $VE(G) = V(G) \cup E(G)$. The order of G is the cardinality $|V|$ of V and is denoted by $|G|$ or $v(G)$. The size of G is the cardinality $|E|$ of E and is denoted by $e(G)$. Two vertices u and v of G are said to be adjacent if $uv \in E$. If $e = uv \in E$, then we say that u and v are the end-vertices of e and that the edge e is incident with u and v. Two edges e and e' of G are said to be adjacent if they have one common end-vertex. If $uv \in E$, then we say that v is a neighbour of u. The set of all neighbours of u is called the neighbourhood of u and is denoted by $N_G(u)$ or simply by $N(u)$ if there is no danger of confusion. The degree (valency) of a vertex u is $|N(u)|$ and is denoted by $d_G(u)$ or simply by $d(u)$. The maximum (resp. minimum) of the vertex degrees of G is called the maximum (resp. minimum) degree of G and is denoted by $\Delta(G)$ (resp. $\delta(G)$).

A graph H is said to be a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $H \subseteq G$ if H is a subgraph of G. A subgraph H of G such that whenever $u, v \in V(H)$ are adajcent in G then they are also adjacent in H is called an induced subgraph of G. We write $H \leq G$ if H is an induced subgraph of G. An induced subgraph of G having vertex set V' is denoted by $G[V']$. The subgraph of G induced by $V(G) \backslash \{v_1, ..., v_k\}$, where $\{v_1, ..., v_k\} \subseteq V(G)$, is written as $G - \{v_1, ..., v_k\}$ or $G - v_1 - ... - v_k$ when k is small. The subgraph of G having edge set $E' \subseteq E(G)$ and vertex set the set of end-vertices of all edges in E' is denoted by $G[E']$. The subgraph of G having vertex set $V(G)$ and edge set $E(G) \setminus E'$, where $E' \subseteq E(G)$, is denoted by $G - E'$, and in particular, if E' consists of only a small number of edges $e_1, ..., e_k$, then this subgraph is denoted by $G - e_1 - ... - e_k$. The complement $\bar{G}$ of G is the graph having vertex set $V(G)$ and edge set $\{uv | u, v \in V(G), uv \notin E(G)\}$. If $E' \subseteq E(\bar{G})$, then $G + E'$ is the graph having vertex set $V(G)$ and edge set $E(G) \cup E'$.

If all the vertices of a graph G have the same degree d, then we say that G is regular of degree d, or G is a d-regular graph. The degree of a regular graph G is written as $\deg(G)$. A regular graph of degree 3 is called a cubic graph. If G is regular graph of order n such that $\deg(G) = 0$ (resp. $n - 1$), then G is called a null graph (resp. complete graph) and is denoted by O_n (resp. K_n).

A set of vertices S of a graph G is said to be independent if any two vertices u and v in S are not adjacent in G. The maximum cardinality of an independent set of vertices of G is called the vertex-independence number and is denoted by $\alpha(G)$. Analogously, a set of edges E' of a graph G is said to be independent if any two edges e and e' in E' are not adjacent in G. The maximum cardinality of an independent set of edges of G is called the edge-independence number of G is denoted by $\alpha'(G)$. An independent set of edges of G is also called a matching in G. A matching in G that includes (saturates) every vertex of G is called a perfect matching or a 1-factor of G. A matching of G that saturates every vertex, except one, of G is called a near-perfect matching of G. Thus G has a perfect matching only if $|G|$ is even and G has a near perfect matching only if $|G|$ is odd.

If the vertex set of a graph G can be partitioned into r independent sets $V_1, ..., V_r$ then G is called an r-partite graph (when $r = 2$, G is called a bipartite graph having bipartition (V_1, V_2)). Moreover, if every vertex of V_i is joined to every vertex of V_j, $j \neq i$, then G is called a complete r-partite graph. We denote a complete r-partite graph by $O_{p_1} + O_{p_2} + ... + O_{p_r}$ where $p_i = |V_i|$, $i = 1, 2, ..., r$. If $G = O_{p_1} + O_{p_2} + ... + O_{p_r}$ and $p_1 = p_2 = ... = p_r$, then we call G a balanced complete r-partite graph and we denote such a graph by $K(r, n)$, where $n = p_1 = p_2 = ... = p_r$. A spanning subgraph of a balanced complete r-partite graph is called a balanced r-partite graph. A bipartite graph having bipartition (V_1, V_2) such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. The graph $K_{1,r}$ is called a star and is denoted by S_r.

A cycle of length n is denoted by C_n and a (shortest) path of length n is denoted by P_n. If G has a cycle C that includes every vertex of G, then C is called a Hamilton cycle of G and G is said to be hamiltonian. If G has a path P that includes every vertex of G, then P is called a Hamilton path of G. Two graphs G and H are said to be disjoint if they have no vertex in common. The join $G + H$ of two disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$. The union $G \cup H$ of two disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Suppose G and H are two disjoint graphs. If there exists an injection $\phi : V(G) \rightarrow V(H)$ such that $\phi(x)\phi(h) \in E(H)$ if $xy \in E(G)$, then we say that ϕ is an embedding of G in H. If such an embedding exists, then we say that G is embeddable in H.

A multigraph permits more than one edge joining two of its vertices. In a multigraph G the number of edges joining two vertices x and y of G is called the multiplicity of xy and is denoted by $\mu(x, y)$. The multiplicity $\mu(G)$ of a multigraph G is max $\mu(x, y)$ taken over all pairs of adjacent vertices x and y of G.

A (proper) vertex-colouring (resp. edge-colouring) of a graph G is a mapping φ from $V(G)$ (resp. $E(G)$) to a set $\mathcal{C}$ such that no adjacent vertices (resp. edges) of G have the same image. If $\varphi : V(G) \rightarrow \mathcal{C}$ (resp. $\varphi : E(G) \rightarrow \mathcal{C}$) is a vertex-colouring (resp. edge-colouring) of G and $|\mathcal{C}| = k$, a positive integer, then we say that G is k-colourable (resp. k-edge-colourable), and φ is called a k-colouring or k-vertex-colouring (resp. k-edge-colouring) of G. The minimum cardinality of $\mathcal{C}$ for which there exists a vertex-colouring $\varphi : V(G) \rightarrow \mathcal{C}$ (resp. an edge-colouring $\varphi : E(G) \rightarrow \mathcal{C}$) is called the chromatic number (resp. chromatic index) of G, and is denoted by $\chi(G)$ (resp. $\chi'(G)$). If φ is a k-vertex-colouring (resp. k-edge-colouring) of G, then φ yields a partition of $V(G)$ (resp. $E(G)$) into independent sets $V_1, ..., V_k$ (resp. $E_1, ..., E_k$). These independent sets $V_1, ..., V_k$ of vertices (resp. $E_1, ..., E_k$ of edges) of G are called the colour classes of φ.

Suppose $Z_n = \{0, 1, ..., n-1\}$ is the group under addition modulo n. Let $S \subseteq Z_n$ be such that $0 \notin S$ and if $s \in S$, then $-s \in S$. The circulant graph $G(Z_n, S)$ is the simple graph having vertex set Z_n and edge set $\{\{g, h\} \mid h - g \in S\}$. The set S is called the symbol of the circulant graph $G(Z_n, S)$.

A planar graph G is outerplanar if it can be drawn on a plane in such a way that G has no crossings and that all its vertices lie on the boundary of the same face.

§2. Total-colouring of a graph - introduction

A total-colouring π of a graph G is a mapping from $VE(G)$ to a set $\mathcal{C}$ satisfying :

(i) no two adjacent vertices or edges of G have the same image; and

(ii) the image of each vertex of G is distinct from the images of its incident edges.

If $\pi : VE(G) \to \mathcal{C}$ is a total-colouring of G and $|\mathcal{C}| = k$, a positive integer, then we say that G is k-total-colourable. The minimum cardinality of $\mathcal{C}$ for which there exists a total-colouring $\pi : VE(G) \to \mathcal{C}$ is called the total chromatic number of G, and is denoted by $\chi_T(G)$. Thus if π is a total-colouring of G, then $\pi|_{V(G)}$, the restriction of π on $V(G)$, is a vertex-colouring of G. Similarly, $\pi|_{E(G)}$ is an edge-colouring of G. From this, it follows that a total-colouring π of G yields a partition of $VE(G)$ into independent sets $V_1 \cup E_1$, $V_2 \cup E_2, \ldots$, where $V_1, V_2, \ldots$ are independent sets of vertices of G and $E_1, E_2, \ldots$ are independent sets of edges of G, and that no vertex in V_i is incident with any edge in E_i. These independent sets $V_1 \cup E_1$, $V_2 \cup E_2, \ldots$ are called the colour classes of π. Conversely, any partition of $VE(G)$ into independent sets $V_1 \cup E_1, V_2 \cup E_2, \ldots, V_k \cup E_k$ gives rise to a k-total-colouring of G.

Undoubtedly, vertex-colourings and edge-colourings are among the main streams in Graph Theory. These two topics both have long histories. They are very important, difficult, and they have many real-life applications to storage problem, timetabling problem, electrical networks, production scheduling, and designs for experiments etc. Since a total-colouring of G is a vertex-colouring and at the same time an edge-colouring of G, the degree of difficulty of this subject is obvious and its importance is anticipated. Moreover, it is not surprising that very soon some nontrivial and important applications of total-colourings of graphs will be found.

The notion of a total-colouring of a graph was introduced and studied by Behzad and independently Vizing around the year 1965. Clearly, for any graph G, $\Delta(G)+1 \leq \chi_T(G)$, where $\Delta(G)$ is the maximum degree of G. The following conjecture was posed independently by Behzad and Vizing in 1965.

Total Colouring Conjecture (TCC) : For any graph G,

$$\chi_T(G) \leq \Delta(G) + 2.$$

(In fact, Vizing posed a more general conjecture which says that for any multigraph

试读结束，需要全本PDF请购买 www.ertongbook.com

G, $\chi_T(G) \leq \Delta(G) + \mu(G) + 1$.)

The TCC was proved true for a few classes of graphs in the 1970s. Only very recently, some new techniques have been introduced and used to prove that the TCC holds for some more classes of graphs, especially graphs having high maximum degree. In this book, we shall give an up-to-date account on results obtained in this area.

In chapter 2 some basic results on total-colourings of graphs are given. These basic results will be used very often throughout this book. Amongst these basic results are : (i) a powerful lemma which says that if a graph G contains an independent set of vertices S such that $|S| \geq |G| - \Delta(G) - 1$, then $\chi_T(G) \leq \Delta(G) + 2$ (This lemma will be used in Chapter 3 to show that complete r-partite graphs satisfy the TCC and it will also be used in Chapter 6 to show that graphs of high maximum degree satisfy the TCC); (ii) a useful theorem saying that if G is a graph of order $2n$ and $\chi_T(G) = t + 1$, then $e(\bar{G}) + \alpha'(\bar{G}) \geq n(2n - t)$. (This theorem will be used to show that some complete r-partite graphs G of even order has $\chi_T(G) = \Delta(G) + 2$ in Chapter 3 and many other results in Chapter 6.)

In Chapter 3 the exact value of $\chi_T(G)$ for $G = K_n$ and $G = K_{m,n}$ are determined. The main objectives of this chapter are: (i) to prove that the complete r-partite graphs satisfy the TCC; (ii) to prove that every complete r-partite graph of odd order has total chromatic number $\Delta(G) + 1$; (iii) to give a complete classification of balanced complete r-partite graphs according to their total chromatic numbers.

In Chapter 4 different proof techniques are used to show that the TCC holds for graphs G having $\Delta(G) = 3$ and $\Delta(G) = 4$.

In Chapter 5 it is proved that the TCC holds for graphs G having $\Delta(G) \geq |G| - 5$ and for graphs G having $\Delta(G) \geq \frac{3}{4}|G|$.

In Chapter 6 the exact value of $\chi_T(G)$ for graphs G having $\Delta(G) \geq |G| - 2$ are determined. The exact value of $\chi_T(G)$, where $G = K_{n,n} - E(J)$, $J \subseteq K_{n,n}$ and $\Delta(G) = n$ is given without proof. Some partial results on the total chromatic number of graphs having $\Delta(G) = |G| - 3$ are presented. Finally a complete classification (according to their total chromatic numbers) of regular graphs G whose complement $\bar{G}$ is bipartite is also stated without proof.

Chapter 7 is devoted to the study of total chromatic number of planar graphs. In this chapter we prove that the TCC holds for planar graphs G having $\Delta(G) \geq 8$

and we also prove that for planar graphs G having $\Delta(G) \geq 14$, $\chi_T(G) = \Delta(G) + 1$.

In Chapter 8 the following upper bounds for $\chi_T(G)$ are presented: (i) $\chi_T(G) \leq \chi'(G) + \lfloor \frac{1}{3}\chi(G) \rfloor + 2$; (ii) $\chi_T(G) \leq \chi'(G) + 2\lceil \sqrt{\chi(G)} \rceil$; (iii) $\chi_T(G) \leq \chi'(G) + k$ where k is the smallest positive integer such that $k! \geq v(G)$; and (iv) $\chi_T(G) \leq \Delta(G) + 2\lceil \frac{v(G)}{\Delta(G)} \rceil + 1$. Again, the techniques used to prove these results are totally different.

In Chapter 9 we mention some other results on/or related to total-colourings of graphs which have not been discussed in the previous sections. Probably they too will have some impact on future research on total-colourings.

Exercise 1

1. Prove that for any integer $n \geq 3$,

$$\chi_T(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \ (mod\ 3) \\ 4 & \text{otherwise.} \end{cases}$$

2. Show that $\chi_T(G) = \Delta(G) + 1$ where G is a tree of order at least 3.

3. Let G be a graph having $\chi_T(G) = t$. Suppose for any t-total-colouring π of G and for any colour class $V_i \cup E_i$ ($V_i \subseteq V(G)$ and $E_i \subseteq E(G)$), we have $|E_i| \geq 2$. Prove that for any edge e of G, $\chi_T(G - e) = t$.

4.* Let G be a graph having $\chi_T(G) = t$. Suppose G has a t-total-colouring π such that π has a colour class $V_m \cup E_m$ for which $|E_m| \geq 3$ is minimum among all possible colour classes of any t-total-colouring of G. Clearly if $e' \in E_m$, then by Exercise 1(2), $G' = G - e'$ has $\chi_T(G') = t$ and G' has a t-total-colouring φ such that φ has a colour class $V'_m \cup E'_m$ for which $|E'| \leq |E_m| - 1$. Is it true that $|E_m| - 1$ is the minimum cardinality of E'_i for any colour class $V'_i \cup E'_i$ of any t-total-colouring of G'?

5. Prove that for any graph $G \neq K_2$ and any edge e of G,

$$\chi_T(G - e) \geq \chi_T(G) - 1.$$

(Behzad [71b])

CHAPTER 2

SOME BASIC RESULTS

Similar to the study of vertex-colourings and edge-colourings of graphs, in the study of total-colourings of a graph G, we shall always assume that G is connected. In this chapter we present some basic results which will be used very often in this book.

The following lemma is often used either implicitly or explicitly. This lemma requires no proof.

Lemma 2.1 For any subgraph H of a graph G, $\chi_T(H) \leq \chi_T(G)$.

The following theorem is due to König [36; Chapter 11].

Theorem 2.2 Every graph (resp. multigraph) G having maximum degree k can be embedded into a k-regular graph (resp. multigraph).

Proof. We take two copies of G and join two corresponding vertices v and v' by an edge if $d(v) < k$. (If G is a multigraph we join the two corresponding vertices by $k - d(v)$ edges and we straightaway obtain a k-regular multigraph H in which G is embedded.) Now the minimum degree of this new graph G_1 is $\delta(G) + 1$ and $\Delta(G_1) = \Delta(G)$. We continue the same process if G_1 is not regular and eventually (after at most k steps) we obtain a k-regular graph H in which G is embedded. //

From Lemma 2.1 and Theorem 2.2 we can deduce the following theorem, which is useful in proving that the TCC holds for graphs having low maximum degree.

Theorem 2.3 (Behzad [71b]) If the TCC holds for all Δ-regular graphs, then it holds for any graph G having maximum degree Δ.

Proofs of the following theorem can be found in many books on graph theory, for instance, in Yap [86].

Theorem 2.4 (Vizing [64]) If G is a multigraph having maximum degree Δ and maximum multiplicity μ, then

$$\chi'(G) \leq \Delta + \mu.$$

In particular, if G is a simple graph having maximum degree Δ, then $\chi'(G) = \Delta$ or $\chi'(G) = \Delta + 1$.

A graph G is said to be Class 1 if $\chi'(G) = \Delta(G)$ and Class 2 if $\chi'(G) = \Delta(G)+1$. If the TCC holds for a certain class of graphs G, then we say that G is Type 1 if $\chi_T(G) = \Delta(G) + 1$ and is Type 2 if $\chi_T(G) = \Delta(G) + 2$. This definition is analogous to the above definition of Class 1 and Class 2 graphs in edge-colourings of a graph.

If v is a vertex of degree $\Delta(G)$ in G, then v is called a major vertex of G, otherwise a minor vertex of G. Suppose G has maximum degree Δ. The core of G is the subgraph of G induced by the major vertices of G and is denoted by G_Δ.

The following lemma follows immediately from some results of Vizing (Theorem 3.3 and Corollary 3.6 in Yap [86]).

Lemma 2.5 Suppose G is a graph having maximum degree Δ. If G_Δ is a forest, then G is Class 1.

The new technique used in the proof of the following lemma was introduced independently and almost at the same time (around 1986) by A. G. Chetwynd and A. J. W. Hilton, as well as by H. P. Yap, Wang Jian-Fang and Zhang Zhongfu.

Lemma 2.6 (Yap, Wang and Zhang [89]) Let G be a graph of order n and let $\Delta = \Delta(G)$. If G contains an independent set S of vertices, where $|S| \geq n - \Delta - 1$, then

$$\chi_T(G) \leq \Delta + 2.$$

Proof. Let M be a maximal matching in $G - S$ and let G^* be a graph obtained by adjoining a new vertex $v^* \notin V(G)$ to $G - M$ and adding an edge joining v^* to each vertex in $G - M - S$. Now $\Delta + 1 \geq \Delta(G^*) \geq \Delta$. If $\Delta(G^*) = \Delta$, then by Theorem 2.4, $\chi'(G^*) \leq \Delta + 1$. On the other hand, if $\Delta(G^*) = \Delta + 1$, then the core of G^* is a forest and thus by Lemma 2.5, $\chi'(G^*) = \Delta + 1$. Let φ be an edge-colouring of G^*

using colours $1, 2, ..., \Delta + 1$. We now modify φ to a total-colouring π of G by setting:

$$\begin{aligned}
\pi(v) &= \varphi(v^*v) && \text{for each } v \in V(G - S),\\
\pi(v) &= \Delta + 2 && \text{for each } v \in S,\\
\pi(e) &= \varphi(e) && \text{for each } e \in E(G - M), \text{ and}\\
\pi(e) &= \Delta + 2 && \text{for each } e \in M. \qquad //
\end{aligned}$$

The following is a generalization of a result of Hilton [89/90]. We shall see in the subsequent chapters that this generalized result unifies several previous results and proof techniques of J. C. Bermond, B. L. Chen and H. L. Fu, A. J. W. Hilton, as well as K. H. Chew and H. P. Yap.

Theorem 2.7 (Hilton [89/90]; Yap [95]) Suppose G is a graph of order $2n$ and $\chi_T(G) = t + 1$. Then

$$e(\bar{G}) + \alpha'(\bar{G}) \geq n(2n - t).$$

Proof. Let $m = \alpha'(\bar{G})$. Suppose φ is a $(t+1)$-total-colouring of G. Let

$$\varphi(V(G)) = \{\varphi(v) | v \in V(G)\} = \{c_1, c_2, ..., c_k\} = C.$$

It is clear that each colour class V_i of vertices (of φ) forms a clique in $\bar{G}$. If $|V_i| = 2s$ or $2s + 1$, we add s independent edges of $\bar{G}$ in $G[V_i]$. Let E' be the set of edges added in $G[V_1] \cup ... \cup G[V_k]$. Then $p = |E'| \leq m$.

Next, let H be the graph obtained from $G + E'$ by adjoining a new vertex v^* and adding an edge joining v^* to each vertex in $V(G) \setminus V(E')$. Then $\varphi|_{E(G)}$, the restriction of φ to $E(G)$, can be extended to a $(t+1)$-edge-colouring of H by setting $\varphi(e') = \varphi(w)$ if $e' \in E'$ and w is an end-vertex of e', and $\varphi(v^*u) = \varphi(u)$ for any $u \in V(G) \setminus V(E')$. Observe that

$$\begin{aligned}
e(H) &= e(G) + (2n - 2p) + p = (n(2n-1) - e(\bar{G})) + 2n - p\\
&= n(2n+1) - (e(\bar{G}) + p).
\end{aligned}$$

Since each edge-colour class E_i of φ contains at most $\lfloor \frac{v(H)}{2} \rfloor = n$ edges and $\chi'(H) \leq (t+1)$, we have

$$n(2n+1) - (e(\bar{G}) + p) \leq (t+1)n.$$

Consequently,

$$e(\bar{G}) + \alpha'(\bar{G}) \geq e(\bar{G}) + p \geq n(2n - t). \qquad //$$

From the last line of the proof of Theorem 2.7, we know that if $e(\bar{G}) + \alpha'(\bar{G}) = n(2n - t)$, then $p = |E'| = \alpha'(\bar{G}) = m$. Hence if $\bar{G}$ does not induce K_4, then $|V_i| = 2$ or 3 for exactly m colour classes V_i. Thus we have the following corollary.

Corollary 2.8 Let G be a graph of order $2n$. If G satisfies $e(\bar{G}) + \alpha'(\bar{G}) = n(2n - \Delta(G))$, $\bar{G}$ does not induce K_4, and G is Type 1, then for any $(\Delta(G) + 1)$-total-colouring of G, there are $m = \alpha'(\bar{G})$ pairs of pairwise nonadjacent vertices $\{x_i, y_i\}$, $i = 1, 2, ..., m$ receiving m distinct colours. In particular, if G is regular, $\bar{G}$ does not induce K_4, and is Type 1, then $\alpha'(\bar{G}) = n$ and thus $\bar{G}$ contains a 1-factor $x_i y_i$, $i = 1, 2, ..., n$ such that $\{x_i, y_i\}$, $i = 1, 2, ..., n$ receive n distinct colours in any $(\Delta(G) + 1)$-total-colouring of G.

Remarks. Suppose G is a graph of order $2n$ and G is Type 1. Then $e(\bar{G}) + \alpha'(\bar{G}) \geq n(2n - \Delta(G))$. However this necessary condition in general is not a sufficient condition for G to be Type 1. Lemma 6.4 provides such examples.

The maximum cardinality of an independent set of elements in $VE(G)$ is called the <u>total independence number</u> of G and is denoted by $\alpha_T(G)$. Suppose S is a maximum independent set of vertices in G and M is a perfect matching in $G - S$ or a near perfect matching of $G - S$, then clearly

$$\alpha_T(G) = |S| + \left\lfloor \frac{|G| - |S|}{2} \right\rfloor.$$

We observe that each element in an independent set $I \subseteq VE(G)$ consists of either a vertex or an edge, and if an edge is exchanged for two independent vertices, then the size of I increases. This simple observation can be stated as a lemma.

Lemma 2.9 If G contains a maximum independent set of vertices S and $G - S$ contains an independent set of edges E' such that $|E'| = \left\lfloor \frac{|G|-|S|}{2} \right\rfloor$, then

$$\alpha_T(G) = |S| + \left\lfloor \frac{|G| - |S|}{2} \right\rfloor.$$

Hence, in general, $\alpha_T(G) \leq \alpha(G) + \left\lfloor \frac{|G| - \alpha(G)}{2} \right\rfloor$.

试读结束，需要全本PDF请购买 www.ertongbook.com