Lecture Notes in Mathematics

1614

Alexander Koshelev

Regularity Problem for Quasilinear Elliptic and Parabolic Systems

Springer

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Cataloging-in-Publication Data.

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Koševel, Aleksandr I.:
Regularity problem for quasilinear elliptic and parabolic systems / Alexander Koshelev. - Berlin ; Heidelberg ; New York ; Barcelona ; Budapest ; Hong Kong ; London ; Milan ; Paris ; Tokyo : Springer, 1995
(Lecture notes in mathematics ; 1614)
ISBN 3-540-60251-8
NE: GT

Mathematics Subject Classification (1991): 35D10, 35K55

ISBN 3-540-60251-8 Springer-Verlag Berlin Heidelberg New York

Typesetting: Camera-ready TEX output by the author
SPIN: 10479536 46/3142-543210 - Printed on acid-free paper

In memory of my mother Rebecca.

Preface

This book was written while the author was visiting professor at Heidelberg University, supported by SFB 123 in 1990-1992 and at Stuttgart University Mathematisches Institut A supported by the Deutsche Forschungsgemeinschaft (DFG).

It contains material based on the lectures given to graduate and PhD students at these Universities and the author's most current works concerning regularity of weak solutions for quasilinear elliptic and parabolic systems. Some of the author's earlier results are also presented. The author took this opportunity to include his most recent results concerning the regularity of solutions for some special systems e.g. the Navier-Stokes system. Most of the results are based on coercivity estimates with explicit, sometimes sharp, constants.

I would like give my thanks to Dr. S. Chelkak for all of his help in preparing this book. I also would like to express my gratitude to Prof. W. Jäger and Prof. W. Wendland who made it possible to write and publish this book. The author will never forget the friendly support and fruitful discussions with Prof. S. Hildenbrandt which were a great help. I am thankful to Prof. F. Tomi who encouraged me to present this book to Springer-Verlag for publication. I would like to express my appreciation to Mr. R. Show for taking the time to review my limited English.

St.Petersburg – Stuttgart, 1995 A.Koshelev

Introduction

The smoothness of solutions for quasilinear systems is one of the most important problems in modern mathematical physics. It is impossible to overestimate the significance of these questions not only for theoretical purposes, but for different applications as well. It is clear that the problem of regularity is a part of a more general problem concerning the existence and uniqueness of solutions for systems of partial differential equations.
In this book we concentrate on the second order elliptic and parabolic systems. The last chapter is devoted to the Navier - Stokes system.
The problem of regularity of solutions for elliptic systems was originally formulated by Hilbert as one of his famous problems which at first, was concerned only with the analyticity of solutions for a single general elliptic analytic differential equation of second order with two variables

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \tag{0.0.1}$$

The problem was to prove that any twice continuously differentiable solution must be analytic. If we formulate the same statement in a modern way, it sounds as follows: the ellipticity and analyticity of equation (0.0.1) guarantees that a weak (twice continuously differentiable) solution must be a strong (analytical) one.
The problem was solved by S.Bernstein [1] for at least quasilinear equations. His method was based on the following main principles: 1) maximum principle, which was known earlier for analytic functions, 2) method of continuation of parameter; later this method was generalized by Leray and Schauder [1] and formulated as a homotopy principle, 3) method of apriori estimates. The apriori estimates belong to the so-called class of coercivity.
We can illustrate these estimates for the Poisson equation in a bounded domain Ω

$$\Delta u = f \tag{0.0.2}$$

with the condition

$$u \mid_{\partial\Omega} = 0. \tag{0.0.3}$$

Let X be a space of functions which are defined on Ω and let the derivatives D^2u of these functions also be defined. Suppose that f belongs to X. The estimate

$$\|D^2u\|_X \leq C_X\|f\|_X + C\|u\|_X \tag{0.0.4}$$

where C_X and C are positive constants which do not depend on f and u we shall call a coercivity estimate. S.Bernstein was the first to show that inequality (0.0.4) holds true for $X = \mathcal{L}_2$ and $C_{\mathcal{L}_2} = 1$. Of course, it was supposed by S.Bernstein that f is at least continuous and the second derivatives of the solution of the problem (0.0.2),(0.0.3) are also continuous.
As we shall see later, C_X has a crucial influence on the regularity of solutions for the systems which we consider in this book.
As we have mentioned earlier, S.Bernstein obtained his results for the two-dimensional case. It took about fifty years to find the solution of the Hilbert problem considered for one second-order elliptic equation with an arbitrary number of independent variables.

In 1930, Petrovsky [1] proved that the smooth solution of an analytic elliptic system is also analytic. The paper by Oleinik [1] gives a survey of results relating to the smoothness of solutions for boundary value problems of elliptic equations and systems. The results of Schauder [1], Leray, Schauder [1], De Giorgi [1], Moser [1], Nirenberg [1], Campanato [1] and many other mathematicians are of great importance with regard to this problem. The most complete results concerning this problem can be found in the book by Ladyzhenskaya, Ural'tseva [1].

The situation with parabolic equations of second order is even more complex. Here the breakthrough result was obtained by Nash [1]. Important results and surveys can be found in the monographs of Ladyzhenskaya, Solonnikov,Ural'tseva [1] and Krylov [1]. It is not the purpose of this book to give a description of complete results obtained in the problem of regularity for second order elliptic and parabolic equations. The reader can get a more or less complete picture from the monographs mentioned.

At the present time, the problem of regularity can be formulated as follows. Let a $2l$ order system in m-dimensional space have a weak solution belonging to some Sobolev space W. The question is: what should the additional restrictions on the coefficients of the systems, boundary surface and boundary conditions be in order for the weak solution to become regular, i.e. Hölder continuous, differentiable, etc?

In recent years, a significant number of papers have been devoted to the study of smoothness of solutions for the above mentioned systems.

In 1930-1940, Morrey [1]-[3] obtained the most complete results for the second-order elliptic systems ($l = 1$ and $m = 2$), concerning the analyticity and differentiability of weak solutions. In Giaquinta [2] regularity for the solutions of some general variational problems, so-called minimizers, is proved.

In his paper [1], Frehse showed that if $u \in W_s^{(l)}$ and $ls = m$, then the weak solution of the problem is bounded. Under the same or slightly different conditions bounding l, s and m, Widman [1]-[2] proved the solution is Hölder continuous. A somewhat more general result was obtained by Solonnikov [1].

In 1968, Almgren [1] proved for second-order systems that if the data of the boundary value problem are smooth, the weak solution can lose its smoothness only on a set with the Hausdorff $(m-1)$-dimensional measure equal to zero. Later these results were extended by Morrey [4] to arbitrary ordered systems. Significant results in this direction obtained by Giusti [1], Giaquinta [1],[2] and other mathematicians formed the so-called theory of partial regularity of solutions.The partial regularity approach for the Navier-Stokes systems was implemented by Caffarelli, Kohn and Nirenberg [1].

The situation for parabolic systems was more uncertain. For example, Campanato [2] proved in the two-dimensional case that under certain natural conditions the weak bounded solution of some class of parabolic systems will be Hölder-continuous, both in space and time. An additional asumption concerning weak solution boundedness in $\mathcal{L}_\infty(Q)$ makes this result different from the analogous theorem proved by Morrey for elliptic systems in the two-dimensional case.

However, Giaquinta and Giusti [1] proved earlier that the weak solution for the parabolic system is regular with the possible exception of a singular closed set. They studied its Hausdorff measure.

Recently in an article by Nečas and Sverak [1] the regularity of solutions was proved for parabolic systems of small dimensions ($m \leq 4$) where the coefficients depended only on the gradient of solutions. The Hölder continuity of the first derivatives was also obtained for the two-dimensional case with respect to space variables. The situation for systems is considerably different from the case of a single second-order quasilinear elliptic or parabolic equation with natural smoothness conditions on the coefficients and the domain in which the boundary value problem is solved.

In 1968, it was established in a series of examples for the case when the dimension m of the space in which the problem is solved is sufficiently large ($m \geq 3$), that there exists a nonsmooth solution for the smooth boundary value problem (Mazja [1], Giusti and Miranda [1]). This fact holds true even for a linear elliptic system of divergent form (De Giorgi [2]). For parabolic systems a similar result was obtained by Stara, John, Mali [1].

Hence, there arises the question of singling out the more or less precise class of systems for which the weak solution is at least Hölder continuous. Such an approach was applied by Cordes [1] for a singular second-order linear elliptic non-divergent form equation with bounded coefficients. He proved that if the dispersion of the spectrum for the ellipticity matrix is bounded by some explicit constant, then the solution will be Hölder continuous.

We shall also consider systems for which the spectrum of the so-called ellipticity or parabolicity matrix satisfies stronger conditions than positiveness and boundedness. The dispersion of the spectrum should be connected with the rate of asymmetry of the matrix mentioned above. For this class of systems, we prove the weak solution is regular (Hölder continuous, differentiable with Hölder continuous derivatives, ...). We have shown that the conditions which single out this class are sharp for elliptic systems and are at least unavoidable for parabolic systems.

It is well known that the maximum principle is not valid for general elliptic and parabolic systems. Hence, this most fruitful and strong analytical method, which has been applied to scalar second order equations, cannot be used here.

The results obtained in this book are based on two main ideas:

1) The universal iterative method which converges to the solution not only in weak ("energetic") norms, but also in strong ones. 2)Coercivity estimates of the type (0.0.4) in singular weighted Sobolev spaces with explicit, sometimes precise constants C_X.

The sharpness of some results indicates this approach may in some sense provide an optimal method for investigation. However, the author was not able to prove the conditions and estimates obtained lead to precise results in parabolic systems and the Navier-Stokes system. On the other hand, the estimates which are presented in this book allow us to find explicit constants for the norms of some singular integral operators and to prove the Liouville theorem for both elliptic and parabolic systems.

Below, we outline in brief the contents of each chapter.

This monograph consists of six chapters. Chapter 1 is devoted to principal definitions and some results concerning the existence of solutions and convergence of the universal iterative methods in "energetic" spaces.

We consider a bounded domain Ω in the m-dimensional Euclidean space $R^m (m \geq 2)$, whose boundary is a sufficiently smooth closed surface Γ. Inside Ω a system of equations with respect to the vector function $u = (u^{(1)}, \dots, u^{(N)})$

$$L(u) \equiv \sum_{0 \leq |\beta| \leq l} (-1)^{|\beta|} D^{\beta} a_{\beta}(x; D^{\widetilde{\beta}} u) = 0 \quad (0 \leq |\widetilde{\beta}| \leq l), \tag{0.0.5}$$

is given. Here $\beta = (\beta_1, \dots, \beta_m)$ and $\widetilde{\beta} = (\widetilde{\beta_1}..., \widetilde{\beta}_m)$ are multi-indices,

$D^\beta = D_1^{\beta_1} \ldots D_m^{\beta_m}$, D_i is the operator of differentiation with respect to x_i, D^0 is the identity operator and $|\beta| = \beta_1 + \ldots + \beta_m$.
With regard to N-dimensional coefficients $a_\beta(x, p_{\tilde{\beta}})$, we assume that certain conditions of smoothness are satisfied. We assume that for any collection of N-dimensional real vectors $\xi_\beta = (\xi_\beta^{(1)}, \ldots, \xi_\beta^{(N)})$, any $x \in \overline{\Omega}$, and $p_{\tilde{\beta}}$, the inequalities

$$\sum_{i,k=1}^{N} \sum_{0 \le |\beta|, |\tilde{\beta}| \le l} \frac{\partial a_\beta^{(i)}}{\partial p_{\tilde{\beta}}^{(k)}} \xi_\beta^{(i)} \xi_{\tilde{\beta}}^{(k)} \ge \mu_0 (1 + |p|^2)^{\frac{s-2}{2}} \sum_{0 \le |\beta| \le l} |\xi_\beta|^2, \tag{0.0.6}$$

$$\|\frac{\partial a_\beta}{\partial p_{\tilde{\beta}}}\| \le \nu_0 (1 + |p|^2)^{\frac{s}{2}}$$

with $s > 1$, $\mu_0, \nu_0 = \text{const} > 0$ and $|p|^2 = \sum_{i=1}^{N} \sum_{0 \le |\beta| \le l} |p_\beta^{(i)}|^2$ hold true. This means the system is strong elliptic. In addition to the latter conditions, certain conditions relating to the behaviour of functions $a_\beta, \frac{\partial a_\beta}{\partial x}$ are necessary when $p \to \infty$. They provide the existence and uniquieness of the weak solution for the system (0.0.5) with the following boundary conditions

$$(u - \varphi)|_\Gamma = \frac{\partial (u - \varphi)}{\partial \nu}|_\Gamma = \ldots = \frac{\partial^{l-1}(u - \varphi)}{\partial \nu^{l-1}}|_\Gamma = 0, \tag{0.0.7}$$

where ν is the normal to Γ and φ is a trace of some function from $W_p^{(l)}(\Omega)$. The existence of a weak solution of problem (0.0.5), (0.0.7) for any dimension m was first proved in a number of papers (Vishik [1] , Browder [1], Minty [1] ...).
Among these papers, the most important for us is the paper by Vishik in which the existence of a weak solution, belonging to the space of $W_p^{(l+1)}(\Omega')$ (with Ω' essentially contained in Ω) is proved.
Taking into account the numerical applications, the existence of the weak solution can be obtained by the application of the following iterative process

$$\sum_{k=0}^{l} (-1)^k \Delta^k u_{n+1} = \sum_{k=0}^{l} (-1)^k \Delta^k u_n - \varepsilon L(u_n) \ (\varepsilon = \text{const} > 0), \tag{0.0.8}$$

where the iterations satisfy conditions (0.0.7). This process was proposed by the author (Koshelev [3], [4]).
This process converges for $s = 2$ in the energetic norm under simple natural restrictions beginning with an arbitrary initial iteration $u_0 \in W_2^{(l)}(\Omega)$. Therefore, we call process (0.0.8) a universal one.
For $l = 1, 1 < s \le 2$, when the system (0.0.5) can degenerate, we also consider an iterative process with penalty

$$\Delta u_{n+1,\delta} = \Delta u_{n,\delta} - \varepsilon L_\delta(u_{n,\delta}), \quad \delta = const > 0, \tag{0.0.9}$$

where $L_\delta = L + \delta\Delta$. In the author's paper (Koshelev [4]), it is proved that there exists a subsequence δ_n such that if $n \to +\infty$ then u_{n,δ_n} tends to the solution of (0.0.5),(0.0.7) in the energetic norm. Further, this allows us to obtain the regularity of solutions for the problem under consideration from the boundedness of strong norms for u_{n,δ_n}.
In chapter 1 we also discuss the parabolic system

$$\partial_t u - L(u) = 0 \tag{0.0.10}$$

in a cylinder $Q = (0,T) \times \Omega$ with boundary conditions (0.0.10) and initial conditions $u|_{t=0} = 0$. (The coefficients of $L(u)$ can also depend on t). Under general assumptions, an iterative method

$$\varepsilon \partial_t u_{n+1} - \Delta u_{n+1} = - \Delta u_n + \varepsilon L(u_n) \tag{0.0.11}$$

with the above-mentioned boundary and initial conditions converges. We provide the proof of convergence of this method in the energetic norm. Initially, it was done by Chistyakov[1].

The second chapter is devoted mainly to Hölder continuity of weak solutions for nondegenerate second-order elliptic systems with bounded nonlinearities in divergence form where A is the matrix of the left-hand side quadratic form in (0.0.5), i.e. the matrix

$$A = \left\{ \frac{\partial a_k^{(i)}}{\partial p_l^{(j)}} \right\} \quad (i,j = 1,...,N; k,l = 0,...,m). \tag{0.0.12}$$

We assume A^+ and A^- are respectively the symmetric and skew-symmetric parts of A. Denote the eigenvalues of A^+ by λ_i and the infimum and supremum of λ_i respectively by λ and Λ. We also denote the upper boundary of the eigenvalues of the matrix $C = A^+A^- - A^-A^+ - (A^-)^2$ by σ and suppose that $\lambda > 0$ and $\Lambda < \infty$. Let

$$K^2 = \begin{cases} \sigma(\lambda^2 + \sigma)^{-1}, & \sigma \geq \frac{\lambda(\Lambda - \lambda)}{2}, \\ \frac{(\Lambda-\lambda)^2 + 4\sigma}{(\Lambda+\lambda)^2}, & \sigma \leq, \frac{\lambda(\Lambda-\lambda)}{2}, \end{cases} \tag{0.0.13}$$

$\alpha = 2 - m - 2\gamma (0 < \gamma < 1)$ and

$$1 - \frac{\alpha(\alpha + m - 2)}{2(m-1)} > 0. \tag{0.0.14}$$

In chapter II, we prove that if some natural conditions concerning the coefficients $a_k(x,p)$, domain Ω and boundary conditions are satisfied, and

$$K\sqrt{1 - \frac{\alpha(m-2)}{m-1}} \left[1 - \frac{\alpha(\alpha + m - 2)}{2(m-1)} \right]^{-1} < 1 \tag{0.0.15}$$

is true, then the weak solution of (0.0.5), (0.0.7) ($l = 1$) is Hölder continuous with exponent

$$\gamma = \frac{2 - m - \alpha}{2}. \tag{0.0.16}$$

This was proved by the author in [11] and [15]. The above mentioned result follows from the coercivity estimate (0.0.4) for $X = \mathcal{L}_{2,\alpha}(B_\delta)$ where $\mathcal{L}_{2,\alpha}(B_\delta)$ is the space of square summable functions with the weight $|x - x_0|^\alpha$ and B_δ is a ball in R^m with the center x_0 and radius δ. More precisely, the solution of the problem

$$\Delta u = div f, \quad u|_{\partial B_\delta} = 0 \tag{0.0.17}$$

satisfies the inequality

$$(0.0.18)\quad \int_{B'} |\nabla u|^2 |x-x_0|^\alpha dx \le \left[1-\frac{\alpha(m-2)}{m-1}+\eta\right]\left[1-\frac{\alpha(\alpha+m-2)}{2(m-1)}\right]^{-2}\times$$
$$\times \int_{B_\delta} |f|^2|x-x_0|^\alpha dx + C\int_{B_\delta}|f|^2 dx$$

where η is an arbitrary small positive constant (C is as usualy an unessential nonnegative constant). From this inequality follows the analogous estimate for the singular operator

$$(0.0.19)\qquad J(f)=\frac{\partial}{\partial x_k}\frac{1}{(m-2)|S|}\int_{R^m} f^{(k)}(y)|y-x|^{2-m}dy,$$

where $|S|$ is the surface of a unit sphere. More precisely

$$(0.0.20)\qquad J(f)\le \left[1-\frac{\alpha(m-2)}{m-1}+\eta\right]\left[1-\frac{\alpha(\alpha+m-2)}{2(m-1)}\right]^{-2}\times$$
$$\times\int_{B_\delta}|f|^2|x-x_0|^\alpha dx + C\int_{B_\delta}|f|^2dx.$$

This and analogous estimates can be obtained using the Stein's result [1], but his method does not provide us with an explicit constant in either (0.0.18) or (0.0.20). It should also be mentioned that the estimate (0.0.15) is sharp for a small γ. In fact, for a symmetric A ($A^- = 0$) we have

$$K=\frac{\Lambda-\lambda}{\Lambda+\lambda}$$

and the condition (0.0.15) gives

$$(0.0.21)\qquad \frac{\Lambda-\lambda}{\Lambda+\lambda}\sqrt{1+\frac{(m-2)^2}{m-1}}<1.$$

The example given in 2.5 shows this condition to be sharp, which means that if (0.0.21) is false, then there exists such a system for which the weak solution is discontinuous. Chapter III is devoted to the applications of the results obtained in chapters I and II. Here we consider some elasto-plastic problems for media with hardening. It is proved that the method of elastic solutions converges both in energetic and strong norms if the anisotropy of the material is small enough. This is guaranteed by an inequality of type (0.0.15). It is shown that if this condition is false, there exists a system, whose solution has a finite energy and a discontinious displacement. Analytically, the result is based on the so-called Korn inequality for weighted spaces

$$(0.0.22)\quad \sum_{i,k=1}^m\int_\Omega\left[D_k u^{(i)}+D_i u^{(k)}\right]^2 r^\alpha dx \ge \min\left\{2\frac{(\alpha+m)^2}{(\alpha+m)^2-4\alpha},\frac{2m+\alpha}{m}\right\}\times$$
$$\times\sum_{k=1}^m\int_\Omega|\nabla u^{(k)}|^2 r^\alpha dx - C\int_\Omega\left(\sum_{k=1}^m|\nabla u^{(k)}|^2+|u|^2\right)dx.$$

The main results of this Chapter are published in the author's monograph [15] and in [13],[14]. Chapter III also contains an exact form of the Liouville theorem, which was proved by the author in [15] (Chapter 4).
In chapter IV, we consider additional regularity properties of a weak solution for second-order elliptic systems, for example, Hölder continuity of the first derivatives up to the boundary of the domain.
The results , which can be found in 4.5 and 4.6, are based on the explicit constants for coercivity estimates of the type (0.0.4) with $C = 0$ in $\mathcal{L}_{2,\alpha}$, $\alpha = 2 - m - \gamma$ $(0 < \gamma < 1)$. For $m \geq 2$ and $B_R(x_0)$ with boundary condition $u|_{\partial B_R} = 0$ the estimate is as follows:

$$\int\limits_{B_R} |D^2 u|^2 |x - x_0|^\alpha dx \leq C_\alpha \int\limits_{B_R} |\triangle u|^2 |x - x_0|^\alpha dx \tag{0.0.23}$$

and C_α is given by formula (4.2.35).
In spite of the explicit form of C_α , which is given in 4.2, it is impossible to apply (0.0.23) to particular cases. Therefore, we have the following additional inequalities

$$\int\limits_{B_R} |D'^2 u|^2 r^\alpha \zeta dx \leq (1 + M_\gamma^2 + \eta) \int\limits_{B_R} |\triangle u|^2 r^\alpha \zeta dx + \tag{0.0.24}$$

$$+C\left\{\left(\int\limits_{B_R} |D'^2 u|^2 r^\alpha \zeta dx\right)^{\frac{m}{m+2\gamma}} \left(\int\limits_{B_R} |Du|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} + \int\limits_{B_R} (|Du|^2 + |u|^2) dx\right\},$$

where

$$M_\gamma^2 = \frac{(m - 2 + 2\gamma)\{(1+\gamma)^2 + [2 - (1-\gamma)^2]\, m\}}{(m + 1 + \gamma)^2 (1 - \gamma)^2} \tag{0.0.25}$$

and ζ is a smooth cut-off function.
The estimate (0.0.24) is based on the multiplicative inequality

$$|u(0)|^2 \leq C \left(\int\limits_{B_R} |D' u|^2 r^\alpha dx\right)^{\frac{m}{m+2\gamma}} \left(\int\limits_{B_R} |u|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}}$$

and can be applied to obtain the regularity of weak solutions. In section 4.6 of chapter IV, we obtain the following statement: let the eigenvalues λ_i of the symmetrix matrix

$$A = \left\{\frac{\partial a_i}{\partial p_j}\right\} \ (i, j = 0, ..., m)$$

satisfy the following inequalities

$$\frac{\lambda}{1 + |p|^s} \leq \lambda_j \leq \frac{\Lambda}{1 + |p|^s}$$

with $\Lambda, \lambda = \text{const} > 0$ and $0 \leq s \leq 1$; if the relation

$$\frac{\left(1 + \frac{m-2}{m+1}\right)[1 + (m-2)(m-1)]}{\left(1 + \frac{m-2}{m+1}\right)[1 + (m-2)(m-1)] - 1} \frac{\lambda}{\Lambda} > 1$$

holds, then under some conditions of smallness, the weak solution of system (0.0.5) with condition (0.0.3) satisfies the Hölder condition. These results were published by the author in [18].
Chapter V is devoted to second-order parabolic systems which we consider in cylinder $Q = (0,T) \times \Omega$ with a finite $T > 0$. The results which are proved there are based primarily on two lemmas for a function $w(t,x)$ satisfying the parabolic equation

$$\varepsilon\partial_t w + \Delta w = f \tag{0.0.26}$$

in $Q_R = (0,T) \times B_R$ with boundary conditions

$$w|_{\partial B_R} = w|_{t=T} = 0. \tag{0.0.27}$$

Lemma 5.2.1 Let β' be an arbitrary number, $\beta + m - 4 > 0$ and $0 \leq \beta < m$. Then the weak solution of the problem (0.0.26), (0.0.27) satisfies the following estimate

$$\int_{Q_R} |\Delta w|^2 r^\beta \zeta dxdt \leq \frac{m}{m-\beta} \int_{Q_R} |f|^2 r^\beta \zeta dxdt + C(R) \int_{Q_R} |D^2 w|^2 r^{\beta'} dxdt. \tag{0.0.28}$$

Lemma 5.2.2 If $0 \leq \beta < m$ and $\beta + m - 4 > 0$, then for the weak solution of problem (0.0.26), (0.0.27) the inequality

$$\int_{Q_R} |\Delta w|^2 r^\beta dxdt \leq \frac{m}{m-\beta} \int_{Q_R} |f|^2 r^\beta dxdt$$

holds.

In chapter V, we prove the Hölder continuity of the weak solution for both t and x for the parabolic system (0.0.10) under additional assumptions concerning the differentiability of the system's coefficients with respect to x. It is also assumed that the boundary conditions

$$u|_{\partial\Omega} = u|_{t=0} = 0 \tag{0.0.29}$$

and the inequality analogous to (0.0.15) for $m \geq 3$

$$\frac{m}{2(1-\gamma)} \left[1 - \frac{(2-m-2\gamma)(m-1)}{(1-\gamma)^2}\right] M_\gamma K < 1 \tag{0.0.30}$$

are satisfied.
If we take a small γ, then for $m \geq 3$ the inequality (0.0.30) has the form

$$\frac{m}{2}[1 + (m-2)(m-1)]\left(1 + \frac{m-2}{m+1}\right) K < 1 \tag{0.0.31}$$

So for $m \geq 3$, the condition (0.0.31) guarantees the Hölder continuity of weak solutions both in x and in t for the problem (0.0.10), (0.0.29). It is also proved here for $m = 2$, the condition (0.0.31) takes the form

$$\sqrt{2}K < 1. \tag{0.0.32}$$

This inequality was obtained with the help of S. Chelkak. The last relations were proved by the author in [19].
As we see in contrast to the elliptic case, under conditions (0.0.31) and (0.0.32) parabolicity does not guarantee the Hölder continuity of a weak solution in the parabolic case.
In chapter V, some coercivity inequalities are proved. We shall mention here only one inequality of this type.
Theorem 5.3.2 Suppose that $u \in L_2\{0,T;W^{(2)}_{2,\alpha}(B_R)\}$ satisfies only the second of the conditions (0.0.29), ($u=0$ when $t=0$). Then the estimate

$$(0.0.33)\quad \int\limits_{Q_R} |D'^2u|^2 r^\alpha \zeta dxdt \le \left(1+M_\gamma^2+\eta\right) A^2_{\alpha,m} \int\limits_{Q_R} |\varepsilon\partial_t u - \Delta u|^2 r^\alpha \zeta dxdt +$$

$$+C\left\{\left(\int\limits_{Q_R} |D'^2u|^2 r^\alpha \zeta dxdt\right)^{\frac{m}{m+2\gamma}} \left(\int\limits_{Q_R} |Du|^2 dxdt\right)^{\frac{2\gamma}{m+2\gamma}} + \int\limits_{Q_R} |Du|^2 dxdt\right\}$$

holds, where C does not depend on $\varepsilon>0$, $\alpha = 2-m-2\gamma$ $(0<\gamma<1)$ and

$$(0.0.34)\qquad A^2_{\alpha,m} = \begin{cases} 1-\frac{\alpha}{2+\alpha}-\frac{2+\alpha}{(1+\alpha/4)^2\alpha}, & m=2,\\ \frac{m}{(m+\alpha)}, & m>2. \end{cases}$$

In chapter V, we provide applications for problems related to the 'blow-up' problem for some coupled systems. The Liouville theorem for parabolic systems is also proved in this chapter.
Chapter VI, the last chapter in the book, is devoted to Stokes and the Navier-Stokes system in a bounded domain Ω. In this chapter, we consider mainly the problem of the existence of strong solutions for the nonstationary Navier-Stokes system. From the results of Ladyzhenskaya [1] and Solonnikov [3] it follows that for small Reynolds numbers and some smoothness assumptions concerning the boundary of the domain and the massive forces, there exists for the first boundary problem a continuous regular (for example, Hölder continuous, etc.) solution. In these results the constants, which estimate the solution pointwise, have an implicit form. With the help of the coercivity inequalities with explicit constants, we obtain some explicit estimates of the strong solution for finite time.
The first two sections contain coercivity estimates for both stationary and nonstationary Stokes systems. We shall now give two examples of the inequalities which are proved in chapter VI . Some of these results were obtained with the help of A.Wagner and published in the paper of Chelkak and Koshelev[1].
We begin by considering the stationary Stokes system

$$(0.0.35)\qquad \begin{cases} \Delta u + \nabla p = f,\\ \operatorname{div} u = 0 \end{cases}$$

with condition (0.0.3). Suppose $\int\limits_\Omega p dx = 0$ and x_0 is an arbitrary point of Ω with $\operatorname{dist}(x_0,\partial\Omega) > 0$ (R_0 =const and $R<R_0$). If $u \in L_2\{\delta,T;W^{(1)}_{2,\alpha}(B_R)\}$ then the

following estimates for weak solution u, p

(0.0.36)
$$\int\limits_{B_R(x_0)} |\nabla p|^2 |x-x_0|^\alpha dx \le \left[1+\frac{(m-2)^2}{m-1}+O(\gamma)\right] \int\limits_{B_R(x_0)} |f|^2|x-x_0|^\alpha dx + +C\int\limits_{B_R(x_0)} |p|^2 dx$$

and

(0.0.37)
$$\int\limits_{B_R(x_0)} |D'^2u|^2|x-x_0|^\alpha \zeta dx \le \left\{1+\left[1+\frac{(m-2)^2}{m-1}\right]^{1/2}\right\}^2 \times \times\left[1+\frac{(m-2)}{m-1}+O(\gamma)\right] \int\limits_{B_R(x_0)} |f|^2|x-x_0|^\alpha dx + +C\left[\Big(\int\limits_\Omega |D'^2u|^2|x-x_0|^\alpha\zeta dx\Big)^{\frac{m}{m+2\gamma}}\Big(\int\limits_\Omega |D'u|^2dx\Big)^{\frac{2\gamma}{m+2\gamma}}+\int\limits_\Omega |D'u|^2+|f|^2)dx\right]$$

are true and C is independent of x_0 (theorem 6.1.1).
Further on, we consider the nonstationary Stokes system

(0.0.38)
$$\begin{cases} \partial_t u - \nu \triangle u + \nabla p = f \\ \operatorname{div} u = 0 \end{cases}$$

with conditions (0.0.29). We prove if $f \in L_2\{0,T; L_{2,\alpha}(\Omega)\}$ with $\alpha = 2-m-2\gamma$ $(0<\gamma<1)$ satisfying (2.3.20), then the solution of the system (0.0.36) with the boundary condition (0.0.27) satisfies the estimates

(0.0.39)
$$\int\limits_{Q_R} |\nabla p|^2 r^\alpha \zeta dxdt \le (N_\gamma^2+\eta)\int\limits_{Q_R}|f|^2r^\alpha\zeta dxdt + +C\left[\left(\int\limits_{Q_R}|\nabla p|^2r^\alpha\zeta dxdt\right)^{\frac{m}{m+2\gamma}}\left(\int\limits_{Q_R}|f|^2dxdt\right)^{\frac{2\gamma}{m+2\gamma}}+\int\limits_{Q_R}|f|^2dxdt\right]$$

and

(0.0.40)
$$\int\limits_{Q_R}|D'^2u|^2r^\alpha\zeta dxdt \le \le \frac{2A^2_{\alpha,m}}{\nu^2}(1+M_\gamma^2+\eta)(1+N_\gamma)^2\int\limits_{Q_R}|f|^2r^\alpha\zeta dxdt + +C\left(\int\limits_{Q_R}|D'^2u|^2r^\alpha\zeta dxdt\right)^{\frac{m}{m+2\gamma}}\left(\int\limits_{Q_R}|Du|^2dxdt\right)^{\frac{2\gamma}{m+2\gamma}} + +\left[\left(\int\limits_{Q_R}|\nabla p|^2r^\alpha\zeta dxdt\right)^{\frac{m}{m+2\gamma}}\left(\int\limits_{Q_R}|f|^2dxdt\right)^{\frac{2\gamma}{m+2\gamma}}+\int\limits_{Q_R}|f|^2dxdt\right],$$

where

$$N_\gamma^2 = \max\{[1 - \frac{\alpha(m-2)}{m-1}][1 - \frac{\alpha(\alpha+m-2)}{2(m-1)}]^{-2}, 2m^2(1+M_\gamma^2)\} \tag{0.0.41}$$

and $1 - 2^{-1}(\alpha+m-2)\alpha(m-1)^{-1} > 0$. The third and the fourth sections are devoted to the Navier-Stokes system

$$\begin{aligned} &\partial_t u - \nu \triangle u + u^{(k)} D_k u + \nabla p = f(x,t), \\ &\mathrm{div} u = 0 \end{aligned}$$

with homogeneous boundary conditions (0.0.29) in Q_R. It is proved that if $\alpha = 2 - m - 2\gamma$ $(0 < \gamma < 1)$ and the Reynolds number

$$\nu^{-1}(\sup_{x_0} \int_Q |f|^2 |x - x_0|^\alpha dx dt)^{1/2}$$

is sufficiently small, then there exists a Hölder continuous solution in both t and x of the problem, satisfying the estimate

$$\sup_{x_0 \in \Omega} \left[\int_{Q_R} (|\partial_t u|^2 + |D'^2 u|^2) r^\alpha dx dt \right]^{1/2} \leq \tag{0.0.42}$$

$$\leq \frac{2}{\nu^2} A_{\alpha,m} \left((1 + M_\gamma^2 + \eta)(1 + N_\gamma)\right)^{1/2} \sup_{x_0 \in \Omega} \left[\int_{Q_R} |f|^2 r^\alpha dx dt + o(\frac{1}{\nu^2}) \right],$$

where $A_{\alpha,m}$, M_γ and N_γ are defined by (0.0.24), (0.0.34) and (0.0.41).
It follows from (0.0.42) that $\max|u(x,t)|$ is finite.

List of Notation

C: unessential nonnegative constant.
η : sufficiently small positive constant.
R^m: m-dimensional Euclidean space.
$R^m_+(R^m_-) = R^m \cap (x_m > 0)((x_m < 0))$.
$x(x_1, \ldots, x_m)$-vector in R^m with components x_i.
xy-scalar product in R^m.
$|x|$: length of x.
$B_\delta(x_0)$ ball in R^m with center x_0 and radius δ.
$B = B_1(0)$.
$B^+_\delta(x_0)(B^-_\delta(x_0)) = B_\delta(x_0) \cap (x_m > 0)((x_m < 0))$.
$B_{\delta_1,\delta_2}(x_0) = B_{\delta_1}(x_0) \backslash B_{\delta_2}(x_0)(\delta_2 < \delta_1)$.
$B^+_{\delta_1,\delta_2}(x_0)(B^-_{\delta_1,\delta_2}(x_0)) = B_{\delta_1,\delta_2} \cap (x_m > 0)((x_m < 0))$.
S: unit sphere in R^m.
Ω: bounded domain in R^m.
$\Omega_\delta(x_0) = \Omega \cap B_\delta(x_0)$.
$u(x) = \{u^{(1)}(x), \ldots, u^{(N)}(x)\}$: vector-function with N scalar functional components $u^{(j)}(x)$ defined on Ω.
$D_i = \frac{\partial}{\partial x_i}(i = 1, \ldots, m)$.
$D^\alpha = \prod_{k=1}^m D_k^{\alpha_k}(\alpha = (\alpha_1, \ldots, \alpha_m)$: multi-index.
$D_0 = D^0 = I$: unit operator.
$D^\ell u D^\ell v = \sum_{0 \le |\alpha| \le \ell} D^\alpha u D^\alpha v (|\alpha| = \sum_{i=1} \alpha_i)$.

$$|D^\ell u|^2 = D^\ell u D^\ell u.$$

$$DuDv = D^1 u D^1 v.$$

$$D'^\ell u D'^\ell v = \sum_{|\alpha| = \ell} D^\alpha u D^\alpha v.$$

$$|D'^\ell u|^2 = D'^\ell u D'^\ell u.$$

$$D'uD'v = D'^1 u D'^1 v.$$

$W_p^{(\ell)}(\Omega)$: Sobolev space of vector functions, defined on Ω with all weak p-summable derivatives up to the order ℓ; the norm is defined by the equality

$$\|u\|_{W_p^{(\ell)}(\Omega)} = (\int_\Omega |D^\ell u|^p dx)^{\frac{1}{p}}.$$

Analogously

$$\|u\|_{W_p^{(\ell)}(\Omega)} = (\int_\Omega |D'^\ell u|^p dx)^{\frac{1}{p}}$$

and

$$\|u\|'_{W_p^{(\ell)}(\Omega; x_0)} = (\int_\Omega |D'^\ell u|^p dx)^{\frac{1}{p}}$$

(with sufficient homogeneous conditions).
$H_{p,\ell,\alpha}(\Omega) \subset W^{(\ell)}_{p,\alpha}(\Omega; x_0)$ for $\forall x_0 \in \Omega$ with finite norm

$$\|u\|_{H_{p,\ell,\alpha}(\Omega)} = \sup_{x_0 \in \Omega} \|u\|_{W^{(\ell)}_{p,\alpha}(\Omega;x_0)}$$

or

$$\|u\|'_{H_{p,\ell,\alpha}(\Omega)} = \sup_{x_0 \in \Omega} \|u\|_{W^{(\ell)\prime}_{p,\alpha}(\Omega,x_0)}.$$

$$H_{2,\ell,\alpha} = H_{\ell,\alpha}.$$

$$H_{2,1,\alpha} = H_{\alpha}.$$

$C^{k,\alpha}(\Omega)$: space of functions defined on Ω with all derivatives of order k satisfying Hölder condition with the exponent α.

$$C^{0,\alpha}(\Omega) \equiv C^{\alpha}(\Omega)$$

$$C^{0,0}(\Omega) \equiv C(\Omega).$$

$Q = (0,T) \times \Omega$: cylinder with $\forall T = const > 0$.

$$(t,x) \in Q.$$

$$Q_\delta(x_0) = Q \cap ((0,T) \times B_\delta(x_0)).$$

$u(t;x)$: functions defined on Q.
$\partial_t u = \dot{u}$: derivative with respect to t.
$W^{k,\ell}_p(Q)$: Sobolev space of functions, defined on Q, possessing all derivatives up to the order k with respect to t and up to the order ℓ with respect to x, which are p-summable; the norm is defined by the equality

$$\|u\|_{W^{k,\ell}_p(Q)} = \left(\int_Q (|\partial_t^k u|^p + |D^\ell u|^p) dx\right)^{\frac{1}{p}}.$$

Analogously to $W^{(\ell)}_{p,\alpha}(\Omega; x_0)$ and $H_{\ell,p,\alpha}(\Omega)$ are defined the spaces $W^{k,\ell}_p(Q; x_0)$ with the norm

$$\|u\|_{W^{k,\ell}_{p,\alpha}(Q;x_0)} = \left(\int_Q (|\partial_t^k u|^p + |D^\ell u|^p)|x - x_0|^\alpha dx\right)^{\frac{1}{p}}$$

and

$$\|u\|_{H^{k,\ell}_{p,\alpha}(Q)} = sup_{x_0 \in \Omega} \|u\|_{W^{k,\ell}_{p,\alpha}(Q;x_0)}.$$

A: matrix of ellipticity (parabolicity).
A^+: symmetric part of A.
A^-: skew-symmetric part of A.
matrix $C = A^+A^- - A^-A^+ - (A^-)^2$.
$\{\lambda_i\}$: eigenvalues of A^+.