Instantons and Four-Manifolds

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PREFACE

This book is the outcome of a seminar organized by Michael Freedman and Karen Uhlenbeck (the senior author) at the Mathematical Sciences Research Institute in Berkeley during its first few months of existence. Dan Freed (the junior author) was originally appointed as notetaker. The express purpose of the seminar was to go through a proof of Simon Donaldson's Theorem, which had been announced the previous spring. Donaldson proved the nonsmoothability of certain topological four-manifolds; a year earlier Freedman had constructed these manifolds as part of his solution to the four dimensional Poincaré conjecture. The spectacular application of Donaldson's and Freedman's theorems to the existence of fake $\mathbb{R}^4$'s made headlines (insofar as mathematics ever makes headlines). Moreover, Donaldson proved his theorem in topology by studying the solution space of equations -- the Yang-Mills equations -- which come from ultra-modern physics. The philosophical implications are unavoidable: we mathematicians need physics!

The seminar was initially very well attended. Unfortunately, we found after three months that we had covered most of the published material, but had made little real progress towards giving a complete, detailed proof. After joint work extending over three cities and 3000 miles, this book now provides such a proof. The seminar bogged down in the hard analysis (§6 – §9), which also takes up most of Donaldson's paper (in less detail). As we proceeded it became clear to us that the techniques in partial differential equations used in the proof differ strikingly from the geometric and topological material. The latter can be obtained from basic information in standard references and graduate courses, while no standard accessible set of references exists for all the nonlinear analysis. We have attempted to remedy this by including background material in all subjects, but particularly in analysis (meaning nonlinear elliptic partial differential equations).

Specific mathematical debts are owed. First of all, our proof does follow Donaldson in most essential matters, although we provide much greater detail. On the other hand, we give a more concrete proof of the transversality theorems (§3-§4), a slightly different proof of the orientability theorem (§5), and a completely new proof of Taubes' existence theorem using noncompact manifolds (§7). As a byproduct we obtain a new, easy proof of the Removable Singularities Theorem (Appendix D). We are also able to include the newer important technique of Fintushel and Stern (§10). Our second debt is to Michael Freedman. The seminar was his idea. He has also been our Chief Topological Consultant throughout the entire project. Chapter One follows his first lecture, and large parts of the introduction are due to him. Also, we thank the original speakers in the seminar: Michael Freedman, as well as Andreas Floers, Steve Sedlacek, and Andrejs Treibergs. Many other mathematicians contributed ideas, suggestions, and references. We list a few here, extending to them our heartfelt appreciation, and pray that we have not insulted anyone by inadvertent omission: Bob Edwards, Rob Kirby, Richard Lashof, John Lott, Mark Mahowald, Ken Millett, Tom Parker, Mark Ronan, Rick Schoen, Ron Stern, Cliff Taubes, and John Wood. Dan would particularly like to thank his advisor, Iz Singer, for his advice, information, inspiration, and perspective. The bulk of the proofreading was carried out by David Groisser, and the reader will want to join us in praising David and Louis Crane, who have caught several mysterious statements and incomplete proofs.

MSRI has cheerfully and generously provided us many services, from office space and typing on up; their support even covered some airfares. Larry Castro deserves a special award for enduring all of our corrections and revisions -- thank God for the word processor! Both Harvard and Northwestern provided short-term office space. Evy Kavaler drew the creative illustrations. Finally, thanks from Dan to Raoul Bott for his warm hospitality and continued guidance.

Berkeley, California
January, 1984

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INTRODUCTION

Topologists study three types of manifolds -- topological or continuous (TOP), piecewise linear (PL), differentiable (DIFF) -- and the relationships among them. A basic problem is to ascertain when a topological manifold admits a PL structure and, if it does, whether there is also a compatible smooth structure. By the early 1950's it was known that every topological manifold of dimension less than or equal to three admits a unique smooth structure. In 1968 Kirby and Siebenmann determined that for a topological manifold M of dimension at least five, there is a single obstruction $\alpha(M) \in H^4(M;\mathbb{Z}_2)$ to the existence of a PL structure. There are further discrete obstructions to lifting from PL to DIFF; these have coefficients in groups of homotopy spheres. Fortunately, a simplification in dimension four absolves us from having to consider the piecewise linear category again: every PL 4-manifold carries a unique compatible differentiable structure. Now the Kirby-Siebenmann obstruction $\alpha(M)$, which lives on the 4-skeleton of an n-manifold M, relates in special cases to a result of Rohlin dating back to 1952. Rohlin's Theorem states that the signature of a smooth spin 4-manifold is divisible by 16. The arithmetic of quadratic forms shows that the signature of a topological "spin" (= almost parallelizable) 4-manifold M is divisible by 8, and $\alpha(M) \in \mathbb{Z}_2 = 8\mathbb{Z}/16\mathbb{Z}$ is the signature mod 16. If M is not spinable, the Kirby-Siebenmann invariant is an extra piece of information not related to the intersection form.

Recently, a new type of "obstruction" to the smoothability of 4-manifolds was discovered by Simon Donaldson. He proved that if the intersection form ω of a compact, simply connected *smooth* 4-manifold is definite, then ω is equivalent over the integers to the standard diagonal form $\pm\mathrm{diag}(1,1,\ldots,1)$. One year earlier Michael Freedman had classified all compact, simply connected *topological* 4-manifolds, and he found that every unimodular symmetric bilinear form is realized as the intersection form of some topological 4-manifold. Together these results give many examples of nonsmoothable 4-manifolds with vanishing Kirby-Siebenmann invariant.

Freedman and others saw that Donaldson's Theorem, in view of work done by Andrew Casson and others in the early 1970's, implies an even more striking result: the existence of exotic differentiable structures on $\mathbb{R}^4$. At this time it is not known how many such fake $\mathbb{R}^4$'s exist, although several have been found. According to Freedman, topologists speculate that there may be an uncountable number. If this turns out to be true, then the classification of smooth structures, which in higher dimensions is accomplished with characteristic classes and is therefore a discrete problem, could stray into the realm of geometry; just as there are (continuous) moduli spaces of complex structures on Riemann surfaces, so too there may be the moduli spaces of smooth structures on 4-manifolds! Regardless, Donaldson's Theorem makes clear the impossibility of characterizing smooth structures in four dimensions in terms of bundle lifting (i.e. characteristic classes). As concrete examples where bundle lifting fails, we cite $|E_8 \oplus E_8|$ and fake $\mathbb{R}^4$. It is striking that Rohlin's Theorem and Donaldson's Theorem can both be proved by studying a class of decidedly nondiscrete objects: elliptic operators on smooth 4-manifolds. In fact, it remains a challenge for topologists to find a proof of Donaldson's Theorem which does not rely so heavily on geometry and analysis.

The study of elliptic operators on compact manifolds often leads to theorems relating the geometry of the manifold to its topology. We begin with the cornerstone of linear elliptic theory, the Hodge-de Rham Theorem. A smooth n-manifold M comes equipped with a natural elliptic complex of differential operators

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \longrightarrow 0,$$

where at the q^{th} stage $d\colon \Omega^q(M) \longrightarrow \Omega^{q+1}(M)$ is exterior differentiation from q-forms to (q+1)-forms. For compact M this de Rham complex has finite dimensional cohomology groups

(1) $$H^q_{DR}(M) = \frac{\operatorname{Ker} d\colon \Omega^q(M) \longrightarrow \Omega^{q+1}(M)}{\operatorname{Im} d\colon \Omega^{q-1}(M) \longrightarrow \Omega^q(M)}$$

which are isomorphic to the real singular cohomology groups $H^q(M;\mathbb{R})$.

Hence these spaces $H^q_{DR}(M)$, which a priori depend on the differentiable structure, are actually invariants of the topological structure. When M has a Riemannian metric, there is a canonical representative of each cohomology class. This is chosen by minimizing the energy

$$\mathcal{E}(\alpha) = \int_M |\alpha|^2$$

over a given cohomology class (i.e. over $\alpha = \alpha_o + d\beta$ where $\beta \in \Omega^{q-1}(M)$ and α_o is any closed q-form in the given class). The Hodge–de Rham Theorem states that in each cohomology class there is a unique minimizing α, which satisfies the Euler–Lagrange equations

$$(2) \qquad d^*\alpha = 0.$$

Since we also have $d\alpha = 0$, equation (2) is equivalent to

$$(3) \qquad \Delta\alpha = (dd^* + d^*d)\alpha = 0.$$

Here $\Delta = dd^* + d^*d$ is the Laplace-Beltrami operator on forms. Any α satisfying (3) is called harmonic. Applications of Hodge–de Rham Theory to global differential geometry often obtain by expressing the difference of the Laplace operator on forms, dd^*+d^*d, and a differential operator formed from the full covariant derivative, $\nabla^*\nabla$, as an algebraic operator involving curvature. Applying this to 1-forms, for example, Bochner proved in 1946 that $H^1(M;\mathbb{R}) = 0$ for compact M which carry a metric of positive Ricci curvature.

Hodge–de Rham Theory extends to more general linear elliptic operators. An elliptic complex is a finite sequence of (first order) operators

$$0 \longrightarrow C^\infty(\xi_o) \xrightarrow{D_1} C^\infty(\xi_1) \xrightarrow{D_2} \cdots \xrightarrow{D_r} C^\infty(\xi_r) \longrightarrow 0$$

between vector bundles ξ_i over M such that (i) $D_{i+1} \circ D_i = 0$, and (ii) on the symbol level

$$0 \longrightarrow \xi_0 \xrightarrow{\sigma(D_1)(\theta)} \xi_1 \xrightarrow{\sigma(D_2)(\theta)} \cdots \xrightarrow{\sigma(D_r)(\theta)} \xi_r \longrightarrow 0$$

is exact for nonzero $\theta \in T^* M$. Generalized cohomology groups $H^q(\xi)$ are defined as in (1), and for compact M these are finite dimensional. If metrics on ξ_i and a volume form on M are given, the (formal) L^2 adjoints D_q^* are defined. The generalized Hodge-de Rham Theorem says that again there is a unique canonical representative f in each cohomology class satisfying

$$D_{q-1}^* f = 0,$$

or equivalently, since $D_q f = 0$ also,

$$(D_{q-1} D_{q-1}^* + D_q^* D_q) f = 0.$$

The Atiyah-Singer Index Theorem expresses the alternating sum $\sum_q (-1)^q \dim H^q(\xi)$ in terms of characteristic classes of M and ξ constructed from the symbol sequence. (To determine a particular $\dim H^q(\xi)$, one usually combines this with vanishing theorems.)

Elliptic complexes can be used to explore the relationship between differential geometry, algebraic geometry, and topology. Of immediate interest is a particular application involving only topology: Rohlin's Theorem. On spin manifolds M there is a natural elliptic operator, the Dirac operator, whose index is the $\hat{A}$-genus of M. This is a certain characteristic class of M evaluated on the fundamental cycle, and for 4-manifolds it turns out to be $\frac{1}{8}$ times the signature. Since the index of an elliptic complex is an integer, the $\hat{A}$-genus of M is integral. (It was precisely this problem -- to explain the integrality of $\hat{A}(M)$ for spin manifolds M -- which led Atiyah and Singer to the Index Theorem.) Furthermore, the spin representation in four dimensions is symplectic, and thus the space of harmonic spinors (the kernel and cokernel of the Dirac operator) is quaternionic. It follows that $\hat{A}(M)$ is an *even* integer, and the signature of M is divisible by 16.

In four dimensions there is an important twisted Dirac operator

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obtained by tensoring with one of the half-spin bundles. We mention it here as it is essentially a linearized version of the nonlinear operator Donaldson studies to deduce his topological result. This operator can be described explicitly in terms of self-duality and differential forms. Namely, if M is an oriented Riemannian 4-manifold, then the six dimensional bundle $\Lambda^2 M$ splits canonically into the sum of three dimensional bundles $\Lambda^2 M = \Lambda^2_+ M \oplus \Lambda^2_- M$. This corresponds to the Lie algebra decomposition $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. We get a new elliptic complex

$$(4) \qquad 0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d_-} \Omega^2_-(M) \longrightarrow 0$$

by composing $d: \Omega^1(M) \longrightarrow \Omega^2(M)$ with the projection $P_-: \Omega^2(M) \longrightarrow \Omega^2_-(M)$. Then the twisted Dirac operator is $d^* \oplus \sqrt{2}\, d_-: \Omega^1(M) \longrightarrow \Omega^0(M) \oplus \Omega^2_-(M)$.

Nonlinear analysis has had as great an impact on geometry and topology as linear analysis. Basic results come from the Morse Theory of geodesics on Riemannian manifolds, a variational theory for nonlinear ordinary differential equations. One of the first applications is the Hadamard-Cartan Theorem (1898/1928) which asserts that the universal cover of a complete Riemannian n-manifold of nonpositive curvature is diffeomorphic to $\mathbb{R}^n$. For positively curved manifolds we have Sumner Byron Myer's Theorem (1941): A complete Riemannian manifold with positive Ricci curvature is compact and has finite fundamental group. This is a stronger result than is obtained from linear theory, since Bochner's Theorem assumes M compact and only concludes $H^1(M;\mathbb{R}) = 0$. More spectacular is the use of Morse Theory by Bott in 1956 to study geodesics on Lie groups, which led him to his celebrated Periodicity Theorem.

Important applications of nonlinear elliptic partial differential equations to geometry and topology lagged behind until very recently. In the last five years a number of results have appeared, many using techniques involving minimal surfaces. For example, Schoen and Yau's proof of the positive mass conjecture in general relativity, which relies on properties of the minimal surface equation, yields the following geometric by-product: if the fundamental group of a 3-manifold M contains a subgroup isomorphic to the fundamental group of a compact surface with genus $\geqslant 1$, then M admits no metric of positive scalar curvature. At about the same time Meeks and Yau used minimal surfaces to give new proofs of Dehn's Lemma (the Loop Theorem) and the Sphere Theorem, two fundamental results in 3-manifold topology. More importantly, they proved a new theorem -- the Equivariant Loop Theorem -- which, added to work of Thurston, Bass, and others, completed a proof of the Smith Conjecture, a longstanding open problem about $\mathbb{Z}_n$ actions on S^3. Recent work of Freedman and Yau examine more general group actions on S^3 using minimal surface techniques. Alan Edmunds has recently given a purely topological proof of the Equivariant Loop Theorem. However, for a theorem of Meeks, Simon, and Yau of the same vintage -- if a 3-manifold has no fake cell (counterexample to the Poincaré conjecture), then its universal cover has no fake cell -- there is stiil no purely topological proof. Of all applications of analysis to topology via geometry, the Equivariant Loop Theorem and its consequences in 3-manifold topology bear the closest relationship to Donaldson's Theorem in 4-manifold topology. The same low dimensional topologists who were learning about minimal surfaces in 3-manifolds a few years ago are now studying the Yang-Mills equations on 4-manifolds.

Even with hindsight afforded by the passage of time, it is difficult to find a pattern in the important applications of analysis to topology, and to make predictions for the future would be foolhardy. Nevertheless, our brief historical survey omitted applications of partial differential equations to the geometry and topology of complex manifolds, which are even more numerous than applications to differentiable manifolds. In fact, an extension of the self-dual equations Donaldson uses can be used to study stable holomorphic vector bundles over complex Kähler manifolds.

We can formulate Yang-Mills as a nonlinear generalization of Hodge Theory. In addition to a Riemannian 4-manifold M, we also start with a normed vector bundle η. We set up a variational problem for connections D on η by taking as action the energy (L^2 norm) of the curvature F_D:

(5) $$\mathcal{YM}(D) = \int_M |F_D|^2.$$

A critical point of this Yang-Mills functional satisfies the Euler-Lagrange equations

(6) $$D^*F_D = 0,$$

a nonlinear generalization of (2). (Recall that curvature is a quadratic expression in the connection, so the nonlinearity is mild.) In view of the Bianchi identity $DF_D = 0$, we also get a Laplace-like equation

$$(DD^* + D^*D)F_D = 0.$$

The second order Yang-Mills equations (6) are automatically satisfied by solutions to first order equations which yields absolute minima of (5). These are the self-dual (anti-self-dual) equations

(7) $$*F_D = \pm F_D.$$

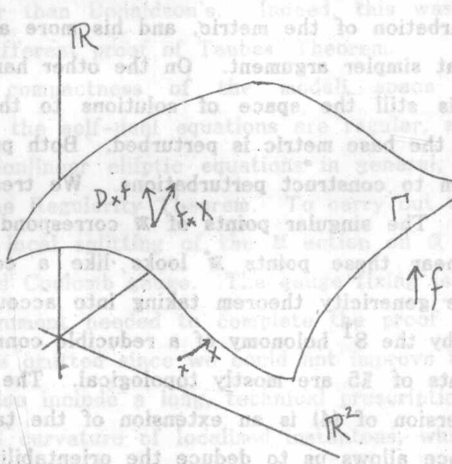
Donaldson's Theorem, stated above, gives a restriction on the topology of a compact, simply connected smooth four-manifold M. The theorem is proved by studying the solutions of the nonlinear semi-elliptic system of equations (7). The operators involved in the equations are nonlinear generalizations of the four-manifold Dirac operator described earlier, and as such are special to four dimensions. The space of solutions is divided out by a natural equivalence to produce the "moduli space" $\mathcal{M}$. As with linear elliptic systems, we learn about the topology of M by studying the geometry of the solution space, only now that study is much more involved -- in the linear case the solutions form a vector space, and the geometry is completely determined by its dimension. For the self-dual equations, roughly speaking, the moduli space $\mathcal{M}$ is an oriented five-manifold with point singularities, neighborhoods of the singular points are cones on $\mathbb{CP}^2$, and M appears as the boundary of $\mathcal{M}$. Now the argument proceeds using cobordism. Remarkable is how neatly each bit of topological information on M fits the analysis! The positivity of the intersection form is necessary for Taubes' existence theorem. Our proof that $\mathcal{M}$ is orientable and the fact that dim $\mathcal{M} = 5$ both require that the first Betti number of M vanish. The ends of $\mathcal{M}$ can be identified as $\mathbb{R} \times M$, and postulating $\pi_1(M) = 0$ ensures that there is only one end. The proof works for exactly the hypotheses given, and basically for no other.

Due to this fine tuning between the analysis and topology, the directions in which Donaldson's Theorem can be extended are very limited, although there are possibilities open for treating 4-manifolds with singularities or with boundary. Nevertheless, all the evidence indicates that gauge theory is here to stay, both in mathematics and in physics. There are several quite different reasons why gauge theory is important in mathematics, aside from the application discussed here. One is the beautiful dichotomy between the algebraic twistor description of self-dual fields over self-dual 4-manifolds and the nonlinear analysis. Here $\mathcal{M}$ can be studied with tools from algebraic geometry, quaternionic linear algebra, and nonlinear PDE. In a similar vein, holomorphic bundles over complex Kähler manifolds of all dimensions can be examined using an extension of the self-dual equations. Atiyah and Bott have already investigated the topology of the moduli space of stable vector bundles over Riemann surfaces in this framework. The three dimensional Yang-Mills equations remain a challenge. Although abstract existence theorems guarantee solutions, their geometrical significance has yet to be determined. Finally, the equations themselves, particularly when coupled with an external "matter field" (the Yang-Mills-Higgs equations), are really interesting PDE's. Not only is there motivation from physics to study them, but their topological and geometric features are both conceptually and technically fascinating.

Because our exposition draws on three branches of mathematics -- topology, geometry, and analysis -- we have endeavored to supply background material whenever possible. The following chapter by chapter description will enable the reader to make his own roadmap through the book.

In §1 we discuss both topological and differentiable four-manifolds. Three equivalent definitions of the intersection form are given. At the end of this chapter we sketch Freedman's argument for the existence of a fake $\mathbb{R}^4$.

The basic geometry of gauge theory is set up in §2. We choose to work with vector bundles rather than principal bundles in order that concrete formulas be expressed. Perhaps some geometric insight into connections is lost, though, and we take this opportunity to explain the covariant derivative with pictures. Consider the simplest case of real-valued functions f on $\mathbb{R}^2$. A basic principle of modern differential geometry is simply this: we understand functions (or sections of bundles) by studying the geometry of their graphs. In this spirit the directional derivative $D_X f$ of f in the direction X can



be computed by first lifting X to a tangent vector f_*X to Γ = graph f. Then the vertical part of f_*X measures the rate of change of f in the direction X. By identifying the tangent space to $\mathbb{R}$ with $\mathbb{R}$, we have determined $D_X f$. In this example the vertical projection, fixed by specifying its kernel, the horizontal subspace at f(x), is given canonically by the product structure of $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$. Over topologically nontrivial manifolds there are vector bundles which are not products, and then the horizontal distribution, or connection, must be chosen as an additional piece of geometric data.

The obstruction to a local basis of flat sections is the curvature of the connection, and global properties of the curvature reflect the twisting of the bundle.

We study connections satisfying a particular system of differential equations. The set of all connections on a bundle forms an affine space $\mathfrak{A}$ (the difference of two connections is a tensor field on the base), and the group $\mathfrak{G}$ of bundle automorphisms acts naturally on $\mathfrak{A}$. The Yang-Mills equations are invariant under this action. Therefore, our moduli space $\mathfrak{M}$ is taken to be a subset of $\mathfrak{A}/\mathfrak{G}$, where it is finite dimensional. At the end of §2 we prove Donaldson's Theorem modulo the topological properties of $\mathfrak{M}$ demonstrated in later chapters.

For a generic metric on M, the moduli space is a smooth 5-manifold with a finite number of singular points. Our approach in §3 and §4 differs from Donaldson's. His perturbation of $\mathfrak{M}$ is not induced by a perturbation of the metric, and his more abstract setup leads to a somewhat simpler argument. On the other hand, the space we end up with is still the space of solutions to the Yang-Mills equations, but now the base metric is perturbed. Both proofs use the Sard-Smale Theorem to construct perturbations. We treat irreducible connections in §3. The singular points of $\mathfrak{M}$ correspond to reducible connections, and near these points $\mathfrak{M}$ looks like a cone on $\mathbb{CP}^2$. In §4 we redo the genericity theorem taking into account the extra symmetry provided by the S^1 holonomy of a reducible connection.

The arguments of §5 are mostly topological. The index bundle of our nonlinear version of (4) is an extension of the tangent bundle $T\mathfrak{M}$, and its existence allows us to deduce the orientability of $\mathfrak{M}$ from the simple connectivity of $\mathfrak{A}/\mathfrak{G}$. This, in turn, follows from the connectedness of $\mathfrak{G}$. The path group of $\mathfrak{G}$ turns out to be the set of homotopy classes $[M,S^3]$, and this can be computed from the Steenrod Classification Theorem. A more geometric argument based on Pontrjagin's Construction is given in Appendix B.

§6 is an odd mix of analysis and geometry. Only the grafting procedure is part of Taubes' Theorem; the rest is background material. We begin with a geometric description of the moduli space of instantons on S^4. Because the conformal group preserves the

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