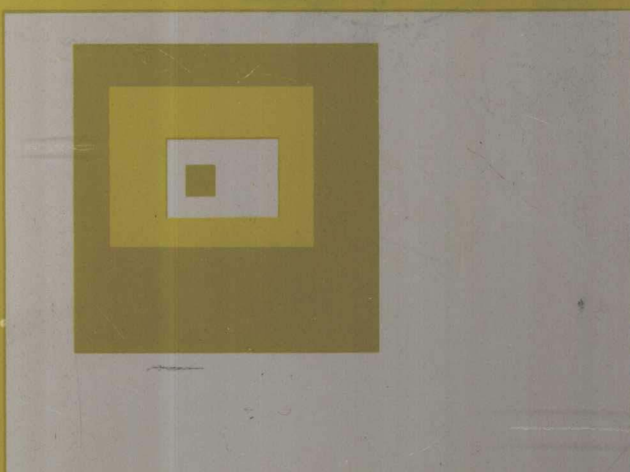
Foundations of Analysis:
A Straightforward Introduction

Book 2

Topological Ideas

K.G. BINMORE



THE FOUNDATIONS OF ANALYSIS: A STRAIGHTFORWARD INTRODUCTION

BOOK 2
TOPOLOGICAL IDEAS

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INTRODUCTION

This book is intended to bridge the gap between introductory texts in mathematical analysis and more advanced texts dealing with real and complex analysis, functional analysis and general topology. The discontinuity in the level of sophistication adopted in the introductory books as compared with the more advanced works can often represent a serious handicap to students of the subject especially if their grasp of the elementary material is not as firm as perhaps it might be. In this volume, considerable pains have been taken to introduce new ideas slowly and systematically and to relate these ideas carefully to earlier work in the knowledge that this earlier work will not always have been fully assimilated. The object is therefore not only to cover new ground in readiness for more advanced work but also to illuminate and to unify the work which will have been covered already.

Topological ideas readily admit a succinct and elegant abstract exposition. But I have found it wiser to adopt a more prosaic and leisurely approach firmly wedded to applications in the space $\mathbb{R}^n$. The idea of a relative topology, for example, is one which always seems to cause distress if introduced prematurely.

The first nine chapters of this book are concerned with open and closed sets, continuity, compactness and connectedness in metric spaces (with some fleeting references to topological spaces) but virtually all examples are drawn from $\mathbb{R}^n$. These ideas are developed independently of the notion of a limit so that this can then be subsequently introduced at a fairly high level of generality. My experience is that all students appreciate the rest from 'epsilonese' made possible by this arrangement and that many students who do not fully understand the significance of a limiting process as first explained find the presentation of the same concept in a fairly abstract setting very illuminating provided that some effort is taken to relate the abstract definition to the more concrete examples they have met before. The notion of a limit is, of course, the single most important concept in mathematical analysis. The remainder of the volume is therefore largely devoted to the application of this idea in various important special cases.

Much of the content of this book will be accessible to undergraduate students during the second half of their first year of study. This material has been indicated by the use of a larger typeface than that used for the more advanced material (which has been further distinguished by the use of the symbol †). There can be few institutions, however, with sufficient teaching time available to allow all the material theoretically accessible to first year students actually to be taught in their first year. Most students will therefore encounter the bulk of the work presented in this volume in their second or later years of study.

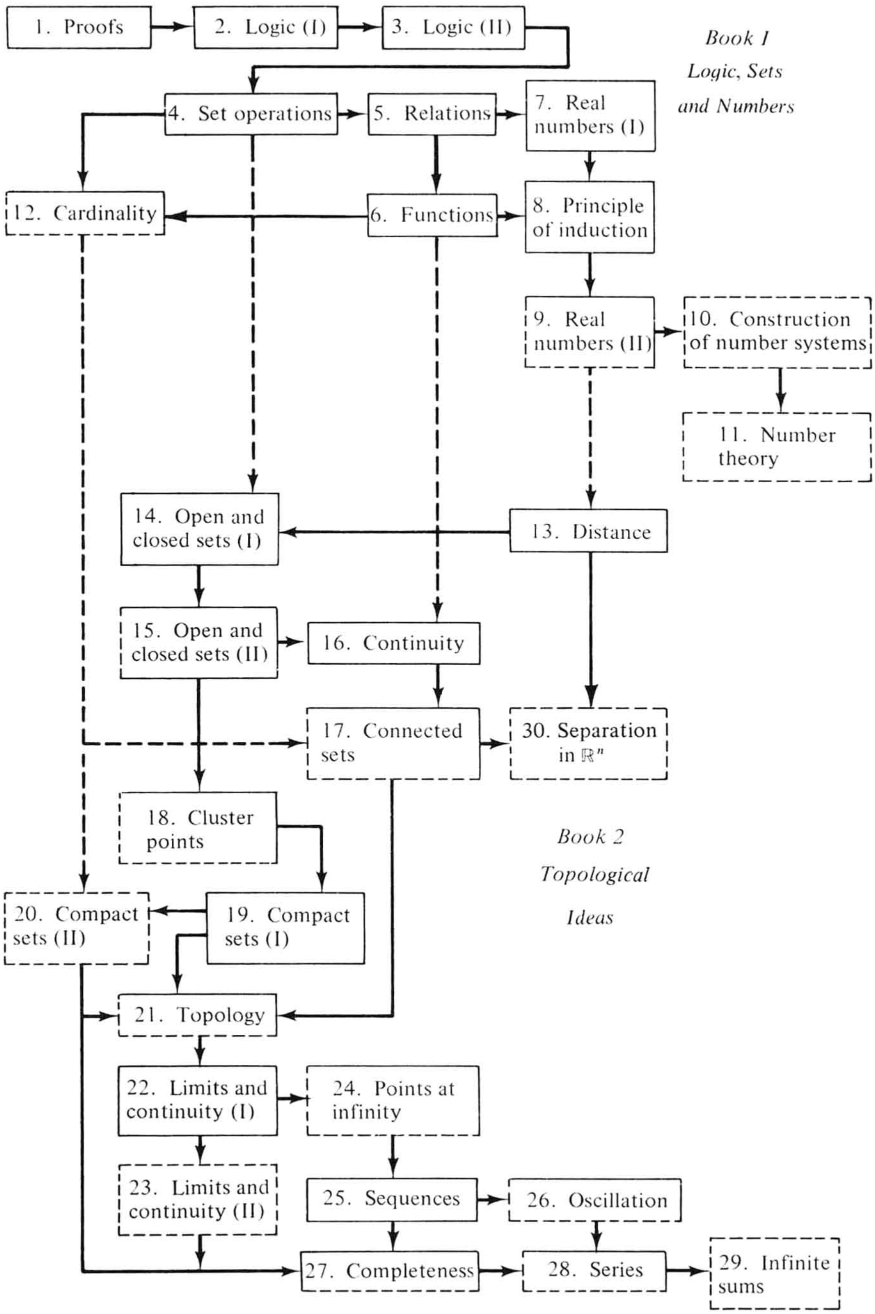
Those reading the book independently of a taught course would be wise to leave the more advanced sections (smaller typeface and marked with a †) for a second reading. This applies also to those who read the book during the long vacation separating their first and second years at an institute of higher education. Note, incidentally, that the exercises are intended as an integral part of the text. In general there is little point in seeking to read a mathematics book unless one simultaneously attempts a substantial number of the exercises given.

This is the second of two books with the common umbrella title *Foundations of Analysis: A Straightforward Introduction*. The first of these two books, subtitled *Logic, Sets and Numbers* covers the set theoretic and algebraic foundations of the subject. But those with some knowledge of elementary abstract algebra will find that *Topological Ideas* can be read without the need for a preliminary reading of *Logic, Sets and Numbers* (although I hope that most readers will think it worthwhile to acquire both).

A suitable preparation for both books is the author's introductory text, *Mathematical Analysis: A Straightforward Approach*. There is a small overlap in content between this introductory book and *Topological Ideas* in order that the latter work may be read without reference to the former.

Finally, I would like to express my gratitude to Mimi Bell for typing the manuscript with such indefatigable patience. My thanks also go to the students of L.S.E. on whom I have experimented with various types of exposition over the years. I have always found them to be a lively and appreciative audience and this book owes a good deal to their contributions.

June 1980 K. G. BINMORE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13 DISTANCE

13.1 The space $\mathbb{R}^n$

Those readers who know a little linear algebra will find the first half of this chapter very elementary and may therefore prefer to skip forward to §13.18.

The objects in the set $\mathbb{R}^n$ are the n-tuples

$$(x_1, x_2, \ldots, x_n)$$

in which $x_1, x_2, \ldots, x_n$ are real numbers. We usually use a single symbol $\mathbf{x}$ for the n-tuple and write

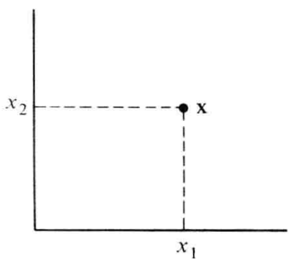
$$\mathbf{x} = (x_1, x_2, \ldots, x_n).$$

The real numbers $x_1, x_2, \ldots, x_n$ are called the *co-ordinates* or the *components* of $\mathbf{x}$.

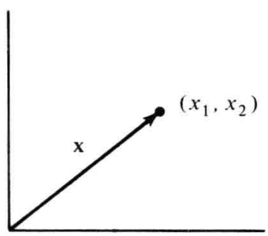
It is often convenient to refer to an object $\mathbf{x}$ in $\mathbb{R}^n$ as a *vector*. When doing so, ordinary real numbers are called *scalars*. If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are vectors and α is a scalar, we define '*vector addition*' and '*scalar multiplication*' by

$$\mathbf{x}+\mathbf{y} = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$$
$$\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n).$$

These definitions have a simple geometric interpretation which we shall illustrate in the case $n=2$. An object $\mathbf{x} \in \mathbb{R}^2$ may be thought of as a point in the plane referred to rectangular Cartesian axes. Alternatively, we can think of $\mathbf{x}$ as an arrow with its blunt end at the origin and its sharp end at the point (x_1, x_2).

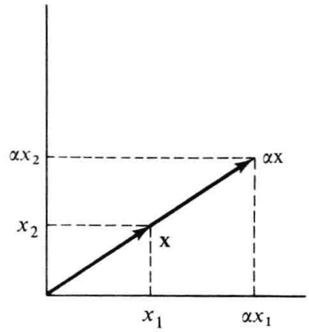


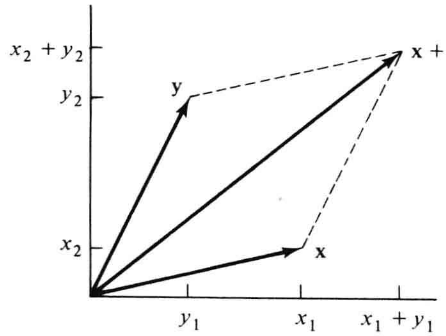
$\mathbf{x}$ as a point



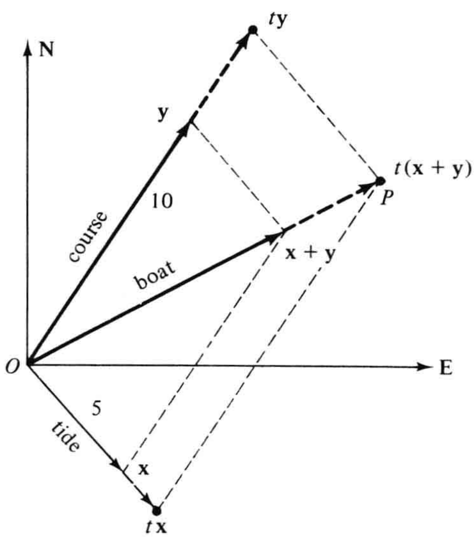
$\mathbf{x}$ as an arrow

Vector addition and scalar multiplication can then be illustrated as in the diagrams below. For obvious reasons, the rule for adding two vectors is called the *parallelogram law.*





The parallelogram law is the reason that the navigators of small boats draw little parallelograms all over their charts. Suppose a boat is at O and the navigator wishes to reach point P. Assuming that the boat can proceed at 10 knots in any direction and that the tide is moving at 5 knots in a south-easterly direction, what course should be set?



The vector $\mathbf{x}$ represents that path of the boat if it drifted on the tide for an hour (distances measured in nautical miles). The vector $\mathbf{y}$ represents the path of the boat if there were no tide and it sailed the course indicated for an hour. The vector $\mathbf{x}+\mathbf{y}$ represents the path of the boat (over the sea bed) if both influences act together. The scalar t is the time it will take to reach P.

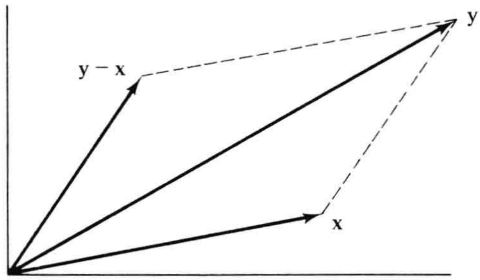
13.2 *Example* Let $\mathbf{x}=(1, 2, 3)$ and $\mathbf{y}=(2, 0, 5)$. Then

$$\mathbf{x}+\mathbf{y}=(1, 2, 3)+(2, 0, 5)=(3, 2, 8)$$
$$2\mathbf{x}=2(1, 2, 3)=(2, 4, 6).$$

It is very easy to check $\mathbb{R}^n$ is a commutative group under vector addition. (See §6.6.) This simply means that the usual rules for addition and subtraction are true. The zero vector is, of course,

$$\mathbf{0}=(0, 0, \ldots, 0).$$

The diagram below illustrates the vector $\mathbf{y}-\mathbf{x}=(y_1-x_1, y_2-x_2, \ldots, y_n-x_n)$ in the case $n=2$.



It is natural to ask about the multiplication of vectors. Is it possible to define the product of two vectors $\mathbf{x}$ and $\mathbf{y}$ as another vector $\mathbf{z}$ in a satisfactory way? There is no problem when $n=1$ since we can then identify $\mathbb{R}^1$ with $\mathbb{R}$. Nor is there a problem when $n=2$ since we can then identify $\mathbb{R}^2$ with $\mathbb{C}$ (§10.20). If $n \geqq 3$, however, there is no entirely satisfactory way of defining multiplication in $\mathbb{R}^n$. Instead we define a number of different types of 'product' none of which has all the properties which we would like a product to have.

Scalar multiplication, for example, tells us how to multiply a scalar and a vector. It does not help in multiplying two vectors. The 'inner product', which we shall meet in §13.3, tells how two vectors can be 'multiplied' to produce a scalar. In $\mathbb{R}^3$, one can introduce the 'outer product' or 'vector product' of two vectors $\mathbf{x}$ and $\mathbf{y}$. This is a vector denoted by $\mathbf{x} \wedge \mathbf{y}$ or $\mathbf{x} \times \mathbf{y}$. Unfortunately, $\mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x}$.

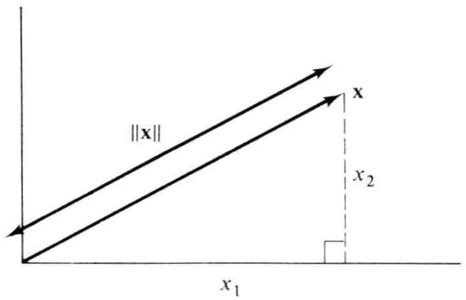
Multiplication is therefore something which does not work very well with vectors. Division is almost always *meaningless.*

13.3 Length and angle in $\mathbb{R}^n$

The *Euclidean norm* of a vector $\mathbf{x}$ in $\mathbb{R}^n$ is defined by

$$\|\mathbf{x}\| = \{x_1^2 + x_2^2 + \ldots + x_n^2\}^{1/2}.$$

We think of $\|\mathbf{x}\|$ as the *length* of the vector $\mathbf{x}$. This interpretation is justified in $\mathbb{R}^2$ by Pythagoras' theorem (13.15).



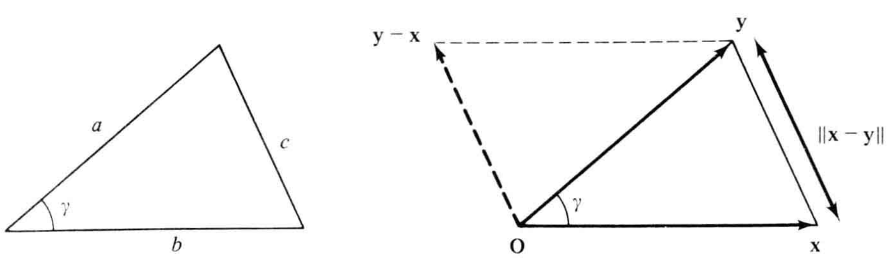
The *inner product* of two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

It is easy to check the following properties:

(i) $\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2$
(ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
(iii) $\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z} \rangle = \alpha\langle \mathbf{x}, \mathbf{z} \rangle + \beta\langle \mathbf{y}, \mathbf{z} \rangle$.

The geometric significance of the inner product can be discussed using the *cosine rule* (i.e. $c^2 = a^2 + b^2 - 2ab \cos \gamma$) in the diagram below.



Rewriting the cosine rule in terms of the vectors introduced in the right-hand diagram, we obtain that

$$\|\mathbf{x}-\mathbf{y}\|^2=\|\mathbf{x}\|^2+\|\mathbf{y}\|^2-2\|\mathbf{x}\|\cdot\|\mathbf{y}\|\cos\gamma.$$

But,

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|^2&=\langle\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{y}\rangle=\langle\mathbf{x},\mathbf{x}-\mathbf{y}\rangle-\langle\mathbf{y},\mathbf{x}-\mathbf{y}\rangle\\&=\langle\mathbf{x},\mathbf{x}\rangle-2\langle\mathbf{x},\mathbf{y}\rangle+\langle\mathbf{y},\mathbf{y}\rangle=\|\mathbf{x}\|^2+\|\mathbf{y}\|^2-2\langle\mathbf{x},\mathbf{y}\rangle\end{aligned}$$

It follows that

$$\langle\mathbf{x},\mathbf{y}\rangle=\|\mathbf{x}\|\cdot\|\mathbf{y}\|\cos\gamma.$$

Of course, this argument does not *prove* anything. It simply indicates why it is helpful to think of

$$\frac{\langle\mathbf{x},\mathbf{y}\rangle}{\|\mathbf{x}\|\cdot\|\mathbf{y}\|}$$

as the cosine of the *angle* between $\mathbf{x}$ and $\mathbf{y}$.

13.4 *Example* Find the lengths of and the cosine of the angle between the vectors $\mathbf{x}=(1, 2, 3)$ and $\mathbf{y}=(2, 0, 5)$ in $\mathbb{R}^3$.

We have that

$$\begin{aligned}\|\mathbf{x}\|&=\{1^2+2^2+3^2\}^{1/2}=\sqrt{14},\\\|\mathbf{y}\|&=\{2^2+0^2+5^2\}^{1/2}=\sqrt{29},\\\frac{\langle\mathbf{x},\mathbf{y}\rangle}{\|\mathbf{x}\|\cdot\|\mathbf{y}\|}&=\frac{1\cdot2+2\cdot0+3\cdot5}{\sqrt{14}\sqrt{29}}=\frac{17}{\sqrt{14\times29}}.\end{aligned}$$

13.5 Some inequalities

In the previous section γ was the angle between $\mathbf{x}$ and $\mathbf{y}$. The fact that $|\cos\gamma|\leq1$ translates into the following theorem.

13.6 *Theorem* (*Cauchy–Schwarz inequality*) If $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{y}\in\mathbb{R}^n$, then

$$|\langle\mathbf{x},\mathbf{y}\rangle|\leq\|\mathbf{x}\|\cdot\|\mathbf{y}\|.$$

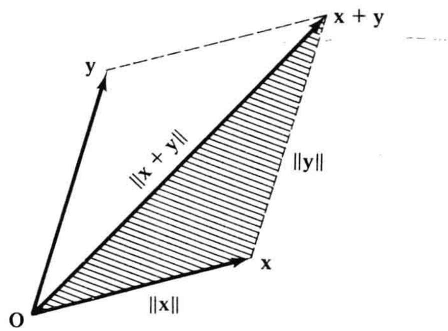
Proof Let $\alpha\in\mathbb{R}$. Then

$$\begin{aligned}0\leq\|\mathbf{x}-\alpha\mathbf{y}\|^2&=\langle\mathbf{x}-\alpha\mathbf{y},\mathbf{x}-\alpha\mathbf{y}\rangle\\&=\|\mathbf{x}\|^2-2\alpha\langle\mathbf{x},\mathbf{y}\rangle+\alpha^2\|\mathbf{y}\|^2.\end{aligned}$$

It follows that the quadratic equation $\|\mathbf{x}\|^2 - 2\alpha\langle \mathbf{x}, \mathbf{y}\rangle + \alpha^2\|\mathbf{y}\|^2$ has at most one real root (§10.10). Hence '$b^2 - 4ac \leqq 0$' – i.e.

$$4\langle \mathbf{x}, \mathbf{y}\rangle^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leqq 0.$$

It is a familiar fact in Euclidean geometry that one side of a triangle is shorter than the sum of the lengths of the other two sides.



This geometric idea translates into the following theorem.

13.7 *Theorem* (*Triangle inequality*) If $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$, then

$$\|\mathbf{x}+\mathbf{y}\| \leqq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Proof

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle \\ &= \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{y}\|^2 \\ &\leqq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|.\|\mathbf{y}\| + \|\mathbf{y}\|^2 \quad \text{(theorem 13.6)} \\ &= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.\end{aligned}$$

13.8 *Corollary* If $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$, then

$$\|\mathbf{x}-\mathbf{y}\| \geqq \|\mathbf{x}\| - \|\mathbf{y}\|.$$

Proof It follows from the triangle inequality that

$$\|\mathbf{x}\| = \|(\mathbf{x}-\mathbf{y}) + \mathbf{y}\| \leqq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}\|.$$
