Lecture Notes in Mathematics

Edited by A. Dold, B. Eckmann and F. Takens

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Proof Theory

An Introduction

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Preface

This book contains the somewhat extended lecture notes of an introductory course in proof theory I gave during the winter term 1987/88 at the University of Münster, FRG. The decision to publish these notes in the Springer series has grown out of the demand for an introductory text on proof theory. The books by K.Schütte and G.Takeuti are commonly considered to be quite advanced and J.Y.Girard's brilliant book also, is too broad to serve as an introduction.

I tried, therefore, to write a book which needs no previous knowledge of proof theory at all and only little knowledge in logic. This is of course impossible, so the book runs on two levels - a very basic one, at which the book is self-contained, and a more advanced one (chiefly in the exercises) with some cross-references to definability theory. The beginner in logic should neglect these cross-references.

In the presentation I have tried not to use the 'cabal language' of proof theory but a language familiar to students in mathematical logic.

Since proof theory is a very inhomogeneous area of mathematical logic, a choice had to been made about the parts to be presented here. I have decided to opt for what I consider to be the heart of proof theory - the ordinal analysis of axiom systems. Emphasis is given to the ordinal analysis of the axiom system of the impredicative theory of elementary inductive definitions on the natural numbers. A rough sketch of the 'constructive' consequences of ordinal analysis is given in the epilogue.

Many people helped me to write this book. *J.Columbus* suggested and checked nearly all the exercises. *A.Weiermann* made a lot of valuable suggestions especially in the section about alternative interpretations for Ω. *A.Schlüter* did the proof-reading, drew up the subject index and the index of notations and suggested many corrections especially in the part about the autonomous ordinals of $\mathbf{Z}_\infty$.

I am also indebted to the students of the workshop on proof theory in Münster who suggested many more corrections. Last but not least I want to thank all the students attending my course of lectures during the winter term 1987/88. It was their interest in the topic that encouraged me to write this book.

A first version of the typescript was typed by my secretary Mrs. *J.Pröbsting* using the Signum text system. She also wrote the table of contents. Many thanks to all these persons.

July 19, 1989 W. P.

Münster

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INTRODUCTION

The history of proof theory begins with the foundational crisis in the first decades of our century. At the turn of the century, as a reaction to the explosion of mathematical knowledge in the last two centuries, endeavours began to provide the growing body of mathematics with a firm foundation. Some of the notions used then seemed to be quite problematic. This was especially true of those which somehow depended upon that of infinity. On the one hand there was the notion of infinitesimals which embodied 'infinity in the small'. The elimination of infinitesimals by the introduction of limit processes represented a great progress in foundational work (although one may again find a justification for infinitesimals as it is done today in the field of nonstandard analysis). But on the other hand there were also notions which, at least implicitly, depended on 'infinity in the large'. *G.Cantor* in his research about trigonometrical series was repeatedly confronted with such notions. This led him to develop a completely new mathematical theory of infinity, namely set theory. The main feature of set theory is the comprehension principle which allows to form collection of possibly infinitely many objects (of the mathematical universe) as a single object. Cantor called the objects of the mathematical universe 'Mengen' usually translated by 'sets'. Set theory, however, soon turned out to be a source of doubt itself. Since Cantor's comprehension principle allows the collection of all sets x sharing an arbitrary property E(x) into the set $\{x: E(x)\}$ one easily runs into contradictions.[1] For instance if we form the set $M := \{x: x \notin x\}$, then we obtain the well-known Russellian antinomy: $M \in M$ if and only if $M \notin M$. It is easy to construct further antinomies of a

1) Cantor himself was well aware of the distinction between sets and other collections which may lead to contradictions. See his letter to Dedekind from 27.7.1899 [Purkert et al. 1987]

similar sort. Another annoying fact was that the plausible looking axiom of choice

(AC) For any family $(S_k)_{k \in I}$ of non empty sets there is a choice function $f: I \to \bigcup\{S_k : k \in I\}$ such that $f(k) \in S_k$ for all $k \in I$

had as a consequence the apparently paradoxical possibility of wellordering any set. Nobody could imagine what a wellordering of the reals could look like and D.Hilbert, in his famous list of mathematical problems presented in Paris in 1900, stated in his remarks concerning problem one (the Continuum Hypothesis) that it would be extremely desirable to have a direct proof of this mysterious statement.

Today we know that there is no elementary construction of a wellordering of the reals. Any wellordering of the reals has the same degree of constructiveness as the choice function itself. The existence of a choice function, however, is not even provable from the Zermelo Fraenkel axioms for set theory.

All these facts contributed to a feeling of uncertainty among members of the mathematical society about the notion of a set that they were opposed to set theory in general. But it was of course not possible to simply ignore Cantor's discoveries. *Hermann Weyl* in his paper 'Über die neue Grundlagenkrise der Mathematik' [Weyl 1921] tried to convince his contemporaries that the foundational problems arising in set theory were not just exotic phenomena of an isolated branch of mathematics but also concerned analysis, the very heart of mathematics. It was he who introduced the term 'foundational crisis' into the discussion. In his book 'Das Kontinuum' [Weyl 1918] he had already suggested a development of mathematics which avoided the use of unrestricted set constructions. In more modern terms one could say that he proposed a predicative development of mathematics. Others, like L.E.J.Brouwer, already doubted the logical basis of mathematics. Their point of attack was the law of the excluded middle. With the help of the law of the excluded middle it becomes possible to prove the existence of objects without constructing them explicitly. Brouwer suggested developing mathematics on the basis of alternative intuitive principles which excluded the law of the excluded middle. Their formalization – due to *Heyting* – now is known as intuitionistic logic. Both approaches, Weyl's as well as Brouwer's, meant rigid restrictions on mathematics. *D.Hilbert*, then one of the most prominent mathematicians, was not willing to accept any foundation of mathematics which would mutilate existing mathematics. To him the foundational crisis was a nightmare haunting mathematics. In his opinion mathematics was *the* science, the model for all sciences, whose 'truths had been proven on the basis of definitions via infallible inferences' and therefore were 'valid overall in reality'. He felt that this position of mathematics was in danger and therefore wanted to preserve it as it was. He was especially unwilling to give up Cantor's set theory, a paradise from which no one would expel him. In his opinion Cantor's treatment of transfinite ordinals was one of the supreme achievements of human thought. Therefore he planned a program to save mathematics in its existing form. He charted his program in a couple of writings and debated it in several talks (cf. [Hilbert 1932–1935]). Therefore it would be inadequate to try to sketch Hilbert's program in only a few sentences. For a serious evaluation of the status of Hilbert's program today deeper considerations are necessary (cf. JSL 53 (1988)). The part of Hilbert's program, however, which was essential for the development of the kind of proof theory we want to give an introduction to in this lecture may be roughly characterized by the following steps:

I. Axiomatize the whole of mathematics

II. Prove that the axioms obtained in step I are consistent.

Hilbert proposed that step II of his program, the consistency proof, should be carried out within a new mathematical theory which he called '*Beweistheorie*', i.e. *Proof Theory*. According to Hilbert, proof theory should use contentual reasoning in contrast to the formal inferences of mathematics. Hilbert himself was aware of the fact that the reasoning of proof theory must itself not become the subject of criticism. He therefore required proof theory to obtain its results by methods beyond the shadow of a doubt. He suggested using only finitistic methods. By finitistic methods he understood those methods 'without which neither reasoning nor scientific action are possible'. In my personal opinion, finitistic reasoning may be interpreted as combinatorial reasoning over finite domains. Some of Hilbert's students (e.g. Ackermann, J.v.Neumann, P.Bernays) soon obtained concrete results. Following Hilbert's maxim of first developing the mathematical tools necessary for the solution of a general problem by studying special cases of the problem they first tackled subsystems of elementary arithmetic. In fact they succeeded in obtaining consistency proofs for subsystems not containing the scheme of complete induction. It thus seemed to be just a matter of technical refinement to extend these consistency proofs to systems containing the full induction scheme. However, the systems containing complete induction stubbornly resisted all attempts to prove their consistency. That this failure was neither an accident nor was due to the incompetence of the researchers, became clear after the publication of *Kurt Gödel's* paper 'Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme' [Gödel 1931]. In this paper Gödel proved his famous theorems which, roughly speaking, say the following:

I. In any formal system, satisfying certain natural requirements, it is possible to formulate sentences which are true in the intended structure but are also undecidable within the formal system (i.e. neither the sentence nor its negation are provable in the formal system).

II. The consistency proof for any formal system, again satisfying canonical requirements, may not be formalized in the system itself.

One might think that Gödel's theorems meant a sudden end to Hilbert's program. The first theorem shows that step I in Hilbert's program is indeed impossible. This, however, might be remedied by the oberservation that in fact it is not necessary to formalize all possible mathematics. It would suffice just to axiomatize existing mathematics. Today we know that nearly everything in everyday's mathematics (and, except for the Continuum Hypothesis, probably all which Hilbert may have thought of) is formalizable in one single formal system, namely Zermelo Fraenkel set theory with the axiom of choice (ZFC). Most parts are even formalizable in much weaker systems. Gödel II, however, is a lethal blow to Hilbert's program. Since the methods 'without which neither reasoning nor scientific action are possible' (combinatorial reasoning over finite domains, in our interpretation) should itself be available in mathematics, any reasonable axiomatization of mathematics should allow the formalization of Hilbert's finitistic methods. Therefore there is no finitistic consistency proof for an axiomatization of stronger fragments of mathematics (i.e. essentially those containing the scheme of complete induction). Luckily for the development of proof theory, the researchers in the thirties did not interpret these results als having such drastic consequences. It is hard to say why. Gödel's results were known to the Hilbert school. For instance Bernays mentions them in [Bernays 1935a] but although he expresses doubts about the feasibility of finitistic consistency proofs he denies that Gödel's results imply their impossibility. I conjecture that the true reasons were Hilbert's authority as well as the vagueness of his program. Since he gave no precise definition of what he meant by finitistic methods one could hope that these methods comprised a kind of contentual reasoning which cannot be mathematically formalized. As a matter of fact mathematicians did not stop searching for consistency proofs and in 1936 *Gerhard Gentzen* succeeded in proving the consistency of elementary number theory. According to Gödel's second theorem Gentzen's proof had to use nonfinitistic means. Gentzen succeeded in concentrating all nonfinitistic means in one single point – induction along a wellordering of transfinite ordertype. This result confirmed the Hilbert school's opinion that just a slight modification of the finitistic standpoint (i.e. accepting a weak form of transfinite induction) would suffice to make the whole program feasible. In §16 we will discuss the consequences of this 'slight modification' for Hilbert's program. There we will try to argue, in the spirit of Hilbert's program, that Gentzen's proof is of little help. This, however, does not mean that Gentzen's proof and his results are of no importance. Quite on the contrary, in our opinion Gentzen's proof is one of the deepest results in logic. To see why, we propose a reinterpretation of his results.

In point of fact it is very easy to prove the consistency of pure number theory. One simply has to show that there exists a model for it. So what is the advantage of Gentzen's consistency proof? The construction of the model itself needs a certain framework, e.g. set theory. Thus what is obtained by a consistency proof via a model construction in set theory (or some even weaker theory) is that the consistency of set theory also entails the consistency of pure number theory. Gentzen's proof, however, gives much more information. It has already been mentioned that Gentzen's proof is finitistic apart from his use of induction along a wellordering of transfinite ordertype. In our opinion this is the essential contribution of Gentzen's proof. Its consequences are twofold:

1. The induction in Gentzen's proof need only be applied to formulas of a very restricted complexity. In addition the consistency proof never uses the law of the excluded middle. Thus it may be formalized within a system T based on intuitionistic logic with induction along a wellordering of transfinite ordertype where this induction scheme is restricted to formulas of a very low complexity. So the problem of the consistency of pure number theory may be decided within the system T. Although the wellordering is of transfinite order type it can easily be visualized. So it seems to be completely plain that the system T is consistent. By Gödel's second theorem the proof theoretic strength of the system T, as it will be defined later in this lecture, has to exceed that of pure number theory. But the subsystem T_0 of T which is obtained from T by restricting induction to initial segments of the wellordering only can be shown to be equiconsistent with elementary number theory. Thus Gentzen's proof provides a reduction of the consistency problem for elementary number theory to that of a theory T_0, which from a conceptual point of view may be regarded as 'safer' than elementary number theory itself.

This is an example of *reductive proof theory*. In reductive proof theory one generally tries to reduce the consistency problem of a theory T_1 to that of a theory T_2. For a clever choice of T_2 both systems will have the same proof theoretic strength. The principles used in T_2, however, may be easier to visualize and therefore a justification of the system T_2 seems more plausible. This type of proof theory is of great foundational importance (cf. the introduction to [BFPS] by S.Feferman). One important feature of Hilbert's program we did not mention is the 'elimination of ideal elements'. In this sense reductive proof theory contributes to Hilbert's program by eliminating complicated unperspicuous principles. Since both systems (T_1 and T_2 in the above example) are of the same proof theoretical strength reductive proof theory is in full accordance with Gödel's second theorem.

2. The fact that induction along the wellordering is the only nonfinitistic means in Gentzen's proof also suggests using this wellordering as a measure for the transfinite content of pure number theory. Pursuing this idea one had defined the *proof theoretic ordinal* of a formal theory T as the ordertype of the smallest wellordering which is needed for a consistency proof of T. This definition, however, is somehow vague since it says nothing about the means used besides the induction along this wellordering (one tacitly has to assume that these at least have to be formalizable in T). To obtain a more precise definition one calls an ordinal α provable in T if there is a primitive recursively definable wellordering $\prec$ of ordertype α such that the wellordering of $\prec$ is provable in T. It is a consequence of Gödel's second theorem, that the proof theoretic ordinal of T (in the previous sense) cannot be a provable ordinal of T. Therefore one may define the proof theoretic ordinal of T as the least ordinal which is not provable in T. This is the common definition today. The computation of the proof theoretic ordinal of T is called the *ordinal analyis of* T. Gentzen's paper ' Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie' [Gentzen 1943] indicates that he himself already interpreted his result as an ordinal analysis (and not just as a consistency proof).

The intention of this lecture is to give an introduction to the techniques of ordinal analysis. We suppress the aspects of reductive proof theory. Only in the epilogue it will be indicated how the results and methods of ordinal analysis may be used in reductive proof theory. To get acquainted with the basic notions and techniques we reprove Gentzen's result in the first chapter. The second chapter will discuss the limits of Gentzen's methods. There we will reprove S.Feferman's and K.Schütte's results on the limits of predicativity. The emphasis, however, is on the ordinal analysis of impredicative formal systems. To demonstrate this method we will give in chapter III an ordinal analysis for one of the simplest impredicative formal systems, the system ID_1 for noniterated inductive definitions by the *method of local predicativity.* A discussion on the foundational significance of ordinal analysis will be added in the epilogue.

CHAPTER I

ORDINAL ANALYSIS OF PURE NUMBER THEORY

To begin with we follow Hilbert's program and, in a first step, try to axiomatize - if not the whole of mathematics - but the theory of natural numbers. To obtain a feeling how this might be done we start by some heuristic considerations.

The aim of the 'working' mathematician interested in the theory of a certain structure is to discover the 'mathematical facts' which hold in this structure. In order to do this he first has to be able to formulate the 'mathematical facts'. This means that he needs a language in which he may talk about this structure. The mathematical facts which possibly may hold in the structure will then be expressed by sentences in this language. The problem then is to figure out which of the sentences are the true ones. This may be done by pure intuition. But to be really sure about the truth of a sentence it needs a proof. The only way to prove a sentence, however, is to show that it is a logical consequence of some other sentences which already are known to be true in the structure. Tracking back this procedure we finally end up with a set of sentences, the mathematical axioms of the structure, which cannot be proved themselves but either are true by definition or by common agreement. Showing that a sentence is a logical consequence of other sentences usually is done by deriving the sentence from those others through a series of inferences. A set of inference rules will be called a *proof procedure.* Some of the inferences in a proof procedure may have no premises. Those inferences will be called the *logical axioms* of the proof procedure. The choice of the axioms and of the proof procedure is of course not arbitrary. As a first requirement the truth of every mathematical axiom really has to be indubitable and it also must be clear that the truth of the premises of an inference undoubtedly entails the truth of its conclusion (if there is no premise, then the conclusion must be true in every structure, i.e. logically valid.). This will guarantee that all proven sentences really are true. But the 'working' mathematician does not only want to ensure himself about his theorem but he also wants to convince his colleagues about its truth. Therefore there must be a way of checking a proof. Thus the second requirement is, that it must be decidable whether a given sentence is an axiom or not, and it also has to be decidable whether an inference is a correct application of an inference rule or not. Otherwise we had no possibility to check the correctness of a given proof. A proof procedure meeting these requirements will be called *decidable*.

This little heuristic teaches us the following facts about axiomatization:

In order to axiomatize the theory of a structure we

- first need a *formalization of the language of the structure*. The formal language of the structure has to be given in such a way that it becomes decidable whether a symbol string is a wellformed expression or not;
- second need a decidable set of sentences in this language which undoubtedly are true. The sentences in this set are the *axioms of the structure*;
- third need a decidable *proof procedure* which produces logical consequences of the axioms.

A decidable formal language together with a decidable set of mathematical axioms and a decidable proof procedure will be called a *formal system* or sometimes also a *formal theory* for the structure. From this it immediately follows that the set of sentences which are provable in one formal system always is a recursively enumerable set.

By results of mathematical logic there are complete proof procedures for first order languages, i.e. there are proof procedures which produce all logical consequences of a given set of mathematical axioms. This of course must not be mistaken in that way that the proof procedure together with the mathematical axioms produce all true first order sentences of the structure. In general the set of true sentences of a structure is not recursively enumerable but of higher complexity. Thus in general we cannot expect a complete axiomatization even for the first order theory of a structure. Since we have to abandon completeness anyway we may as well regard the second order language of the structure although there is not even a complete proof procedure for second order logic. The only important thing is that there are sound proof procedures. It will then be the task of proof theoretical research to determine the limits of a formal system.

In the present lecture we will not use full second order logic but first order logic with free set variables. We will introduce the notion of a Π^1_1-sentence and then examine the power of formal systems with respect to their provable Π^1_1-sentences.
In the first sections of the following chapter we are going to develop a quite simple formal system for the structure of natural numbers which in the later sections will be analyzed proof theoretically.

§1. The language $\mathscr{L}$ of pure number theory

A structure usually is given by a non void set together with collections of constants, of functions and of relations on that set. In order to obtain a formal language for the structure of natural numbers we first need to specify our picture of this structure. The set of natural numbers is characterized by the facts that every natural number either is zero or the successor of another natural number and that every natural number possesses a uniquely determined successor. Using this characterization we obtain a name (or constant as we are going to call it) for every natural number. We start with $\underline{0}$ as a name for the natural number zero and a symbol $\underline{S}$ for the successor function. Then a constant for every natural number is obtained by successively applying the successor function to the symbol $\underline{0}$. So it should be clear that we at least need a constant for zero and the successor function in our language (and then as well may assume that we already have a constant $\underline{n}$ for every natural number n). The next question to be answered is which functions and relations besides the successor function on the natural numbers we should consider. The most general answer is of course "all possible functions and relations on the set of natural numbers". Since there are uncountably many such functions and relations this already would lead to a language with uncountably many basic symbols. In a formal system only those constants for which there are defining axioms contribute to the power of the formal system. Therefore we would need an uncountable set of axioms which is outside the scope of a formal system since every decidable set already is countable. If we dispense with defining axioms for function or relation constants we may as well treat them as variables. In fact we will introduce a language which has such second order variables. In our framework it will suffice just to introduce set variables. The introduction of bare set variables (or function variables) will in general also not raise the power of a formal system (cf. exercise 3.15.4). But if we add the defining axioms for set variables, i.e. the comprehension scheme, we will obtain a system which is so strong that up to now we have not been able to do its proof theoretic analysis. Therefore we will be more modest and in a first step will restrict ourselves to a system which we are going to call the system of *pure number theory*. The most important functions in number theory are *'plus'* and *'times'*. 'Plus' and 'times' are primitive recursive functions and it is possible to obtain all primitive recursive functions from 'plus' and 'times' (cf. remark 3.12.). Therefore we are going to introduce a seemingly stronger system in which we have a constant for every primitive recursive function and relation. In order to do this we first will introduce names for all primitive recursive functions. In definition 1.1. we will give the syntactical definition of the primitive recursive function terms, while the meaning of those terms becomes clear from definition 1.2. in which we define the evaluation of an n-ary primitive recursive function term f on an n-tuple $t_1,\dots,t_n$ of natural numbers.

1.1. Primitive recursive function terms

(i) S (the symbol for the successor function) is an unary primitive recursive function term.

(ii) P^n_k (the symbol for the k-th projection of an n-tuple) and C^n_k (the symbol for the n-ary constant function with value k) are n-ary primitive recursive function terms, where in the case of P^n_k we require $1 \le k \le n$.

(iii) If $h_1,\dots h_m$ are n-ary primitive recursive function terms and g is an m-ary primitive recursive function term, then $Sub(g,h_1,\dots,h_m)$ is an n-ary primitive recursive function term. (Substitution of functions).

(iv) If g is an n-ary and h an n+2-ary primitive recursive function term, then Rec(g,h) is an n+1-ary primitive recursive function term. (Primitive recursion).

1.2. Inductive definition of $f(t_1,\dots,t_n) = t$ for an n-ary primitive recursive function term f and natural numbers $t_1,\dots,t_n,t$

(i) $S(t_1)=t$ if t is the successor of t_1,

(ii) $C^n_k(t_1,\dots,t_n) = t$ if $t=k$,

(iii) $P^n_k(t_1,\dots,t_n) = t$ if $t=t_k$,

(iv) $Sub(g,h_1,\dots,h_m)(t_1,\dots,t_n)=t$ if there are natural numbers $u_1,\dots,u_m$ such that $h_i(t_1,\dots,t_n) = u_i$ and $g(u_1,\dots,u_m) = t$.

(iv) $Rec(g,h)(t_1,...,t_n,k) = t$ holds if $k=0$ and $g(t_1,...,t_n) = t$ or if k is the successor of k_0 and $h(t_1,...,t_n,k_0,Rec(g,h)(t_1,...,t_n,k_0)) = t$.

$f(t_1,...,t_n) = t$ is to be read as: "The *evaluation* of the n-ary primitive recursive function f on the n-tuple $t_1,...,t_n$ of natural numbers yields the value t".

1.3. Definition

The graph of an n-ary primitive recursive function term f is the n+1-ary relation $\{f\}$ given by $\{f\}(t_1,...,t_n,t) :\Leftrightarrow f(t_1,...,t_n) = t$.

1.4. Definition

An n-ary relation R on $\mathbb{N}$ is *primitive recursive* if its characteristic function χ_R defined by

$$\chi_R(t_1,...,t_n) := \begin{cases} 1, & \text{if } R(t_1,...,t_n) \\ 0, & \text{otherwise} \end{cases}$$

is primitive recursive.

We do not want to go deeper into the theory of primitive recursive functions. This is the topic of another lecture. The aim of the preceeding definitions was to emphasize that it is possible to name every primitive recursive function by a term. This also means that, via its characteristic function, we have a name for every primitive recursive relation. We now are prepared to introduce the formal language $\mathscr{L}$ for the structure of natural numbers.

1.5 Basic symbols of the language $\mathscr{L}$

1. *Logical symbols*

(i) Countably many number variables denoted by u,v,w,x,y,z,...

(ii) Countably many set variables denoted by U,V,W,X,Y,Z,...

(iii) The *sentential connectives* $\neg, \wedge, \vee$, the *quantifiers* $\forall, \exists$ and the *membership relation* symbol $\in$.

2. *Nonlogical symbols*

(i) A constant $\underline{n}$ for every natural number n.

(ii) An n-ary function constant $\underline{f}$ for every n-ary primitive recursive function term f.

(iii) An n-ary relation symbol $\underline{R}$ for every primitive recursive relation R.