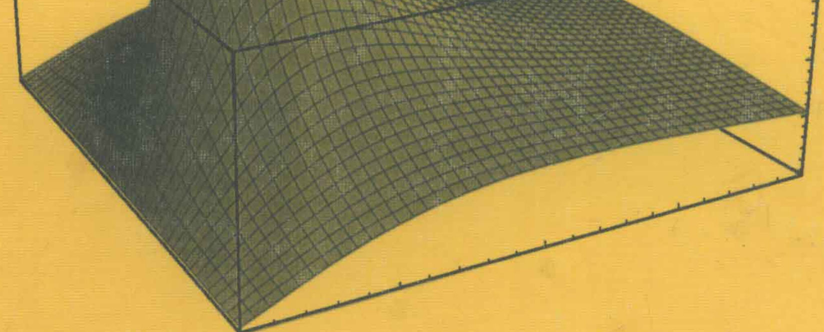
Lecture Notes in Mathematics 1633

Hiroaki Aikawa   Matts Essén

Potential Theory – Selected Topics



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Authors

Hiroaki Aikawwa
Department of Mathematics
Shimane University
Matsue 690, Japan
E-mail: haikawa@riko.shimane-u.ac.jp

Matts Essén
Department of Mathematics
Uppsala University
Box 480
75106 Uppsala, Sweden
E-mail: matts.essen@math.uu.se

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Contents

POTENTIAL THEORY PART I

Matts Essén

1. Preface

The first part of these notes were written to prepare the audience for lectures by H. Aikawa on recent developments in potential theory and by A. Volberg on harmonic measure. It was assumed that the participants were familiar with the theory of integration, distributions and with basic functional analysis.

Section 2 to 8 give an introduction to potential theory based on the books [7] and [29]. We begin by discussing two definitions of capacity. In the first case, the capacity of a set is the supremum of the mass which can be supported by the set if the potential is at most one on the set (cf. Section 5; also the remarks at the end of Section 8.3). This definition is taken from L. Carleson's book [7]: the book deals with general kernels, takes us quickly to interesting problems but is short on details. In the second case, the inverse of the capacity of a set is the infimum of the energy integral if the total mass is one (cf. Section 8.1). This definition is taken from the book of Landkof [29]. Landkof considers α–potentials and the corresponding α–capacity and gives many details.

When discussing α–potentials in Section 8, we can often use results from the general theory in the first six sections: it applies without change in the case $0 < \alpha \leq 2$. When $2 < \alpha < N$, where N is the dimension of the space, it is no longer possible to use the definition in Section 5, since there is no strong maximum principle in this case (cf. Theorem 4.3 and the remarks preceding Theorem 5.11). We can always use results which do not depend on the strong maximum principle: we have therefore tried to make clear what can be proved without applying this result.

In Sections 9 – 16, there is a survey of minimal thinness and rarefiedness. Minimal thinness of a set E at infinity in a half-space is defined in terms of properties of the regularized reduced function $\hat{R}_h^E$ of a minimal harmonic function with pole at infinity (cf. Section 11). A set E is defined to be rarefied at infinity in a half-space D if certain positive superharmonic functions in D dominate $|x|$ on E (cf. Definition 12.4). Characterizations of these exceptional sets in terms of conditions of Wiener type involving Greeen energy and Green mass appear here as Theorems 11.3 and 13.1.

In [2], Aikawa uses singular integral techniques to study problems in potential theory. In Section 14, we consider Green potentials and Poisson integrals in a half-space and carry through the program of Aikawa for these kinds of potentials.

In Section 15, we show that Green capacity and Green mass can in the Wiener-type conditions be replaced by ordinary capacity. This is one of the starting points of the work of Aikawa on "quasi-additivity" of capacity (cf. Section 16). There are also other interesting consequences (see the remark at the end of Section 15).

It was a pleasure to give these lectures. The participants were very active and our discussions led to many improvements in the final version of these notes. I am particularly grateful to Torbjörn Lundh for typing these notes using the $\mathcal{A}_\mathcal{M}\mathcal{S}$-LaTeX--system. There has been a lot of interaction between us, many preliminary versions have been circulated in the class and Torbjörn has with enthusiasm done a tremendous amount of work.

These lecture notes are dedicated to the memory of Howard Jackson of McMaster University who died in 1986 at the age of 52. Together, we tried to understand exceptional sets of the type discussed in Sections 11–15.

2. Introduction

Let us start with some "hard analysis" following Carleson's book [7]. Let our universe be $\mathbb{R}^N$, unless otherwise specified. We begin with a discussion of a family of sets that we will use.

2.1. Analytic sets. Analytic sets can be seen as a generalization of Borel sets. A situation where analytic sets arise is when you have a Borel set of a product space $X \times Y$ and project this set on X. This new subset of X is not necessarily a Borel set but it is always an analytic set.

Analytic sets have two properties in common with Borel sets: they are closed under countable intersections and countable unions, but unlike Borel sets, analytic sets are not closed under complementation.

A Borel set is always analytic, but an analytic set does not have to be a Borel set. If the complement of an analytic set is analytic then it is a Borel set.

One property of analytic sets which is of importance in connection with capacities is that for every finite Borel measure μ and every analytic set A we have that the outer measure of A is equal to the inner measure of A, i.e.

$$\mu^*(A) := \inf_{\mathcal{O} \supset A} \mu(\mathcal{O}) = \sup_{F \subset A} \mu(F) =: \mu_*(A).$$

Here the $\mathcal{O}$ and F denote open and closed sets, respectively. This convention is often used in these notes. For a thorough discussion of analytic sets we refer to [8] or [10, Appendix I].

2.2. Capacity. Let f be a non-negative set function, defined on compact sets such that $f(F_1) \leq f(F_2)$ if $F_1 \subset F_2$, where F_1, F_2 are compact.

For E bounded we define,

(1) $f(E) := \sup_{F \subset E} f(F)$, F compact. (f is an interior "measure"[1].)
It follows that
(2) $f(E_1) \leq f(E_2)$ if $E_1 \subset E_2$.
Furthermore, we assume that
(3) $f^*(E) = \lim_{n \to \infty} f^*(E_n)$, $E_n \nearrow E$, where f^* is the outer "measure" defined by $f^*(E) := \inf_{\mathcal{O} \supset E} f(\mathcal{O})$, $\mathcal{O}$ open.

The following definitions can now be made,

DEFINITION 2.1. *Capacity is a set function satisfying the above conditions 1,2 and 3.*

DEFINITION 2.2. *A set E is capacitable (with respect to f) if $f^*(E) = f(E)$.*

THEOREM 2.3 ([7] PAGE 3). *If compact sets are capacitable, then all analytic sets are capacitable.*

As an example, let us study the following set function:

$$f(E) = \begin{cases} 1 & \text{if } E^o \neq \emptyset, \\ 0 & \text{if } E^o = \emptyset. \end{cases}$$

We now choose $E = \{0\}$, and will have $f^*(E) = 1$ but $f(E) = 0$; telling us that E is not capacitable with respect to f.

[1] f need not be additive.

2.3. Hausdorff measures. Let h be an increasing, continuous function from $[0,\infty)$ to $[0,\infty)$ and assume further that $h(0) = 0$. The classical choice is $h(r) = r^s, s > 0$. If E is a bounded set in $\mathbb{R}^N$ we can cover it by a sequence of balls, $\{B_\nu\}$, where $B_\nu = B(x_\nu, r_\nu)$, i.e. a ball in $\mathbb{R}^N$ centered at x_ν with radius r_ν. Having $E \subset \bigcup B_\nu$ we define,

DEFINITION 2.4 ($M_h(E)$). *$M_h(E) = \inf \sum h(r_\nu)$, taken over all such coverings of E.*

With an extra condition on the size of the r_ν's we also state,

DEFINITION 2.5 ($\Lambda^{(\rho)}(E)$). *$\Lambda^{(\rho)}(E) = \inf \sum h(r_\nu)$, taken over all coverings of E such that, $E \subset \bigcup B_\nu$* and $\sup r_\nu \leq \rho$.

We see that $\Lambda^{(\rho)}(E)$ increases as ρ decreases to 0. This will finally lead to

DEFINITION 2.6 (THE CLASSICAL HAUSDORFF MEASURE).

$$\Lambda_h(E) = \lim_{\rho\to 0} \Lambda^{(\rho)}(E).$$

While studying Hausdorff measures it is perhaps worthwhile mentioning Hausdorff dimension, which is a very important concept in the theory of fractals. If we let $h(r) = r^s$ then $\Lambda_s(E) := \Lambda_h(E)$ is the outer s-dimensional Hausdorff measure (restricted to Borel sets E). We note that $\Lambda_s(E)$ decreases as s increases. There exists a s_0 such that,

$$\Lambda_s(E) = \begin{cases} \infty & \text{if } s \in (0, s_0), \\ 0 & \text{if } s > s_0. \end{cases}$$

s_0 is called the Hausdorff dimension of E, i.e. $s_0 = \dim(E)$. The Hausdorff dimension coincides with the Euclidean dimension for the cases when $s_0 \in \mathbb{N}$; e.g. dim(line) = 1 , dim(plane) = 2 and so on.

A non trivial example is the $\frac{1}{3}$–Cantor set, which has Hausdorff dimension $\frac{\log(2)}{\log(3)} \approx 0.6309$. See [18] for further details on this subject. It is not always suitable to cover our set E by balls; sometimes a net will do the work better , or at least differently. To give a good definition of a net we first need to define the notion of a cube.

DEFINITION 2.7 (CUBE). *A set of the form $\{x \in \mathbb{R}^N : a_i \leq x_i \leq b_i\}$ where $b_i - a_i$ is constant over the indices i.*

DEFINITION 2.8 (NET). *A net is a division of $\mathbb{R}^N$ into cubes $\{Q\}$ all of the same side-length L with sides parallel to the coordinate axis, such that the cube $\{x \in \mathbb{R}^N : 0 \leq x_i \leq L\}$ is in the net. Moreover we have $\bigcup \bar{Q} = \mathbb{R}^N$ and $Q_i^0 \bigcap Q_j^0 = \emptyset$ if $i \neq j$.*

The dyadic refinement is a useful way to construct nets. Let G_p be a net with $L = 2^{-p}$. The next generation, G_{p+1} is obtained from G_p by dividing every cube in G_p into 2^N subcubes each of side-length 2^{-p-1}. We can now form the family of all dyadic cubes, $G = \{G_p\}_{p=-\infty}^{\infty}$.

Consider now a set E covered by $\bigcup Q_\nu, Q_\nu \in G$. The side-length of Q_ν is δ_ν and we define:

DEFINITION 2.9. *$m_h(E) = \inf \sum_\nu h(\delta_\nu)$, for all such coverings $\bigcup Q_\nu$ of E.*

Remark

- It does not matter if the balls $\{B_\nu\}$, which were used in the definition of M_h, are closed or open.
- It is a convention to let the cubes $\{Q_\nu\}$ be closed.

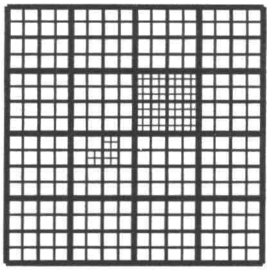

FIGURE 2.1. A dyadic net under construction.

LEMMA 2.10. *There are constants, C_1 and C_2 dependent only on the dimension, N, such that*

$$C_1 M_h(E) \leq m_h(E) \leq C_2 M_h(E).$$

PROOF. (of the last inequality.) Let us first consider the planar case $(N = 2)$. To cover a ball of radius r by cubes, or in our case squares, we need at most 5×5 squares of side-length 2^{-p} when $2^{-p} \leq r \leq 2^{-p+1}$. See figure 2.2 for the geometrical argument. If we now pick a sufficiently good ball covering of E, i.e. $\sum h(r_\nu) \leq M_h(E) + \epsilon$, then we have

$$m_h(E) \leq 25 \sum h(r_\nu) \leq 25 M_h(E) + 25\epsilon.$$

So we conclude that C_2 can be chosen as 25 in the two–dimensional case. For $N > 2$ we just exchange 25 for 5^N. The other inequality is treated in an analogous way. □

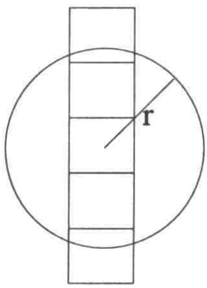

FIGURE 2.2. To cover a ball by squares.

THEOREM 2.11 (FROSTMAN 1935). *Let μ be a non–negative and sub–additive set function such that*

(1) $$\mu(B) \leq h(r) \text{ for every ball of radius } r,$$

then

(2) $$\mu(E) \leq M_h(E).$$

Conversely, there exists a constant a, depending only on the dimension, such that for every compact set F, there exists a measure μ such that

$$\mu(F) \geq a M_h(F)$$

and μ satisfies equation (1).

PROOF. First, we show the easy part, (1) $\Rightarrow$ (2) . If $E \subset \bigcup B_\nu$ then by covering by balls we have $\mu(E) \leq \sum \mu(B_\nu) \leq \sum h(r_\nu)$, the last inequality is just condition (1). Now we can take infimum over such coverings and get $\mu(E) \leq \inf \sum h(r_\nu)$, which is (2).

The second part of the theorem will be proved for the planar case ($N = 2$). Fix a large integer n and consider a net G_n, see figure (2.3). Define a measure, μ_n to have

FIGURE 2.3. F is captured in a net, G_n.

constant density on each $Q \in G_n$ and the value:

$$\mu_n(Q) = \begin{cases} h(2^{-n}) & \text{if } Q \bigcap F \neq \emptyset, \\ 0 & \text{if } Q \bigcap F = \emptyset. \end{cases}$$

Let us look at a lower level in the net, a larger square Q in G_{n-1}. We will then have two possibilities: either $\mu_n(Q) \leq h(2^{-n+1})$ or $\mu_n(Q) > h(2^{-n+1})$. In the latter case we change the measure by multiplying by a constant times the characteristic function of Q, i.e. $c\chi_Q$, where c is defined by $c\mu_n(Q) = h(2^{-n+1})$. This is done for every Q in G_{n-1} and the resulting measure is called μ_{n-1}.

Continuing this way "down" in the net, we arrive after n steps at the measure μ_0. We do know that the relation $\mu_n(Q) \leq h(2^{-\nu})$ holds for every Q in G_ν for every ν in $[0, n]$. Since our resulting μ_0 depends on the starting level n, we indicate this by writing $\mu_0^{(n)}$ instead of μ_0. It is then possible to find a weakly converging subsequence of the sequence $(\mu_0^{(n)})_{n=1}^\infty$, i.e.

$$\mu_0^{(n_k)} \rightharpoonup \mu.$$

It only remains to check that the resulting measure μ has the right properties. It is fairly easy to see that *supp* $\mu \subset F$ and $\mu(Q) \leq h(2^{-\nu}), \forall Q \in G_\nu$, telling us that condition (1) is valid. Furthermore, let $\epsilon > 0$ be given, we can then choose a covering $\{Q_j\}, Q_j \in G$ so that $F \subset \bigcup Q_j$ and $m_h(F) + \epsilon \geq \sum h(\delta_j)$. Now, let n be large. For each $Q \in \{Q_j\}$ with side-length $= \delta$, we have two possibilities,

$$\text{either } \mu_0^{(n)}(Q) = h(\delta) \text{ or } \sum^{*} \mu_0^{(n)}(Q_k) = h(\delta).$$

(The $*$ indicates summation over all $Q_k \subset Q$.) The total mass of $\mu_0^{(n)}$ is then bounded below:

$$\mu_0^{(n)}(\mathbb{R}^N) \geq \inf \sum^{*} h(\delta_j).$$

The same inequality holds for the limit measure μ. That is,

$$\mu(\mathbb{R}^N) \geq \inf \sum^{*} h(\delta_j),$$

and it follows that

$$\mu(\mathbb{R}^N) \geq m_h(F).$$

By the fact that *supp* $\mu \subset F$ one concludes that $\mu(F) \geq m_h(F)$. Using lemma 2.10 we finally observe

$$\mu(F) \geq m_h(F) \geq aM_h(E),$$

ending the proof. □

Claim. $M_h(E)$ is an outer measure. That is, $M_h(E) = \inf_{\mathcal{O}\supset E} M_h(\mathcal{O})$, $\mathcal{O}$ is open.

This is easy to see. Given an $\epsilon > 0$ there is a collection of (open) balls, $\{B_\nu\}$ such that $\mathcal{O} = \bigcup B_\nu$, $\mathcal{O} \supset E$ and $\sum h(r_\nu) \leq M_h(E) + \epsilon$ giving,

$$M_h(\mathcal{O}) + \epsilon \geq M_h(E) + \epsilon \geq \sum h(r_\nu) \geq M_h(\mathcal{O}).$$

Thus, $M_h(E)$ is an outer measure. What about $m_h(E)$? It turns out that $m_h(E) = \inf_{\mathcal{O}\supset E} m_h(\mathcal{O})$ is true when $N = 1$, but not necessarily for $N \geq 2$. The problem in higher dimensions occurs due to the fact that it is necessary to add many more cubes than we can control by an estimate of the contributed terms in the defining sum of $m_h(E)$.

The following lemma is closely related to the above questions.

LEMMA 2.12 ($N = 1$). $\lim_{n\to\infty} m_h(E_n) = m_h(E), E_n \nearrow E$.

PROOF. Take a sequence $\{\epsilon_n\}$ to be defined later and let $\{\omega_{\nu n}\}$ be closed dyadic intervals that cover E_n such that $\sum_\nu h(\delta_{\nu n}) \leq m_h(E_n) + \epsilon_n$. Then we pick, for every $x \in E$, the largest interval in the sequence, $\{\omega_{\nu n}\}$, containing x and call it ω. We may also assume that $\frac{k}{2^m}$ is not in E for any $m, k \in \mathbb{Z}$ (just remove a countable set). The different "non–intersecting" intervals ω —countably many— that we obtain this way, are denoted $\{\omega_\mu\}$ and have lengths $\{\delta_\mu\}$.

An alternative way of constructing the sequence $\{\omega_\mu\}$ is the following; if $\omega', \omega'' \in \{\omega_{\nu n}\}$ and $\omega' \subset \omega''$, then throw away ω'. Keep on until we have denumerably many intervals ω_μ such that,

$$\bigcup_{n=1}^{\infty}\bigcup_{\nu=1}^{\infty} \omega_{\nu n} = \bigcup \omega_\mu.$$

One also notes that ω_μ^o and $\widetilde{\omega_\mu}^o$ are disjoint if $\omega_\mu \neq \widetilde{\omega_\mu}$. We have obtained a new sequence by throwing away the smaller, already covered, sets. Obviously $E_n \subset \bigcup \omega_\mu$.

Choose an integer n and consider those ω_μ's that are taken from $\{\omega_{\nu 1}\}$. They cover a certain subset Q_1 of E_n. The same subset is covered by a certain subsequence of $\{\omega_{\nu n}\}$ denoted $\{\omega_{\nu n}\}^{(1)}$ which is a sequence of subintervals of the chosen ω_ν's. We now claim that

$$\sum^{(1)} h(\delta_\mu) \leq \sum^{(1)} h(\delta_{\nu n}) + \epsilon_1.$$

To show this we assume the contrary and observe

$$\sum h(\delta_{\nu 1}) = \sum^{(1)} h(\delta_\mu) + (\text{"the rest"}),$$

where "the rest" could be estimated from below by $m_h(E_1 \setminus Q_1)$. By the assumption, we know that

$$\sum^{(1)} h(\delta_\mu) > \sum^{(1)} h(\delta_{\nu n}) + \epsilon_1.$$

Since $\{\omega_{\nu n}\}^{(1)}$ covers Q_1 and therefore the smaller set $Q_1 \bigcap E_1$ we have the estimate

$$\sum^{(1)} h(\delta_{\nu n}) \geq m_h(Q_1 \bigcap E_1).$$

We obtain, after putting it all together,

$$\sum h(\delta_{\nu 1}) > m_h(Q_1 \bigcap E_1) + \epsilon_1 + m_h(E_1 \setminus Q_1) = \epsilon_1 + m_h(E_1),$$

which is the desired contradiction.

The next step is to consider the ω_ν's taken from $\{\omega_{\nu 2}\}$ but not from $\{\omega_{\nu 1}\}$. Again, we find $\sum^{(2)} h(\delta_\mu) \leq \sum^{(2)} h(\delta_{\nu n}) + \epsilon_2$. Repeat this argument for all coverings $\{\omega_{\nu k}\}_{k=1}^n$! We obtain n inequalities that can be added, giving

$$\sum_{k=1}^{n} \sum^{(k)} h(\delta_\mu) \leq \sum h(\delta_{\nu n}) + \sum_{\nu=1}^{n} \epsilon_\nu \leq m_h(E_n) + \sum_{\nu=1}^{n} \epsilon_\nu + \epsilon_n.$$

If we now let n tend to infinity we will have

$$m_h(E) \leq \sum_1^\infty h(\delta_\mu) \leq \lim_{n\to\infty} m_h(E_n) + \sum_{\nu=1}^{\infty} \epsilon_\nu.$$

Since $\sum_{\nu=1}^\infty \epsilon_\nu$ can be chosen arbitrarily small we find

$$m_h(E) \leq \lim_{n\to\infty} m_h(E_n)$$

Trivially, we also have $m_h(E_n) \leq m_h(E)$ concluding the proof. □

<u>**Remark**</u> The same argument holds for $N > 1$.

2.4. Is m_h a capacity? Let us now specialize to the case when E_n is compact and E is open and bounded. Lemma 2.12 gives us then

$$m_h(\mathcal{O}) = \sup_{F \subset \mathcal{O}} m_h(F) \tag{3}$$

where F is compact and $\mathcal{O}$ is open and bounded.

What about the other relation, the outer relation?

$$m_h(E) = \inf_{\mathcal{O} \supset E} m_h(\mathcal{O}),\ \mathcal{O} \text{ open}, \tag{4}$$

where E is arbitrary. We know it is true for $N = 1$; but what happens otherwise?

EXERCISE 2.1. *Find sufficient conditions on h so that equation (4) holds for each set E in $\mathbb{R}^N$. What can we say when $h(r) = r^{N-\alpha}, N \geq 2$, or when $h(r) = (\log^+ \frac{1}{r})^{-1}, N = 2$?*

Let us repeat the conditions for the set function f to be a capacity.

(1) $f(E) := \sup f(F)$ for all compact F that are subsets of E.
(2) $f(E_1) \leq f(E_2)$ if $E_1 \subset E_2$
(3) $f^*(E) = \lim_{n\to\infty} f^*(E_n)$, where $E_n \nearrow E$.

So, is $f_0 := m_h$ a capacity? We have only to check the last condition, the first is clear due to lemma 2.12 and equation (3) and the second condition follows immediately from the definition of m_h.

Given $E_n \nearrow E$ find nested open sets $\mathcal{O}_n$ such that $\mathcal{O}_n \supset E_n$ and $f_0^*(\mathcal{O}_n) \leq f_0^*(E_n) + \epsilon$. Thus $E = \bigcup E_n \subset \bigcup \mathcal{O}_n$. We then have $f_0^*(E) \leq f_0^*(\bigcup \mathcal{O}_n)$ but since $\bigcup \mathcal{O}_n$ is open $f_0^*(\bigcup \mathcal{O}_n) = f_0(\bigcup \mathcal{O}_n)$. Using lemma 2.12 gives us $f_0(\bigcup \mathcal{O}_n) = \lim_{n\to\infty} f_0(\mathcal{O}_n)$ but this is bounded above by the assumptions on $\mathcal{O}_n$ by $\lim_{n\to\infty} f_0^*(E_n) + \epsilon$, giving us

$f_0^*(E) \le \lim_{n\to\infty} f_0^*(E_n)$ The opposite inequality is trivial. Hence the answer is yes, $f_0 := m_h$ is a capacity.

We turn now, impatiently, to the next question: "Are compact sets capacitable?" Or in other words: "Is $f^*(F) = f(F)$?" or, equivalently :"Does (4) hold?" The equivalence follows immediately from the definitions of capacity $f^*(F) = \inf_{\mathcal{O}\supset F} m_h(\mathcal{O})$ and $f(F) = m_h(F)$.

Remark: We can modify m_h by slightly changing the meaning of "cover". Let $\{Q_\nu\}$ "cover" a set E if every $x \in E$ is an interior point of $\bigcup Q_\nu$. Then we define,

$$m_h'(E) := \inf \sum h(\delta_\nu),$$

where infimum is taken over all such coverings.

EXERCISE 2.2. *Is equation (4) true with m_h replaced by m_h'?*
Is $\lim m_h'(E_n) = m_h'(E)$ as $E_n \nearrow E$?

3. The Physical background of Potential theory

Behind the theory there are old and interesting questions about the physical reality around us. The classical examples are the theories of gravity and electrostatics. More information on this can be found in [35].

3.1. Electrostatics in space. Consider an electrically charged body, say negatively charged. If we now look at a test particle nearby the body, we will find that there is a force acting on the particle due to the presence of the charged body, see picture 3.1. The force, $\vec{F}$, on the test-charge is $\vec{F} = e\vec{\mathcal{E}}(x,y,z)$, where $\vec{\mathcal{E}}(x,y,z)$ is

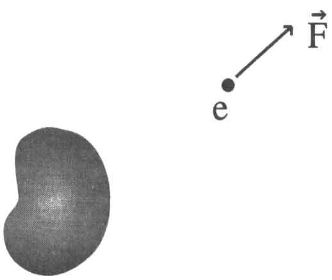

FIGURE 3.1. A test particle, e, in an electric field

a vector field generated by the charged body. If we consider the special case where the body is just a point–charge, with charge $e > 0$, at the location $\bar{x} \in \mathbb{R}^3$, then the electric field is given by the law of Coulomb's.

$$\vec{\mathcal{E}}(x) = \frac{e}{r^2}\frac{x-\bar{x}}{r} = \frac{e}{r^3}(x-\bar{x}).$$

At the end of the 18th century it was observed by Lagrange that there exists a scalar function, φ, with

$$\vec{\mathcal{E}} = -\nabla\varphi,$$

where $\varphi(x) = \frac{e}{|x-\bar{x}|}$.

If we have several charges, with charge e_i at x_i the scalar function becomes $\varphi(x) = \sum \frac{e_i}{|x-x_i|}$. The electrical field will still be $\vec{\mathcal{E}}(x) = -\nabla\varphi(x)$. Suppose now we have charges continuously distributed over a body, rather than a finite set of point charges. That is, suppose there is a continuous function ρ defined on the body Ω such that for every portion B of Ω, the charge in B is given by $\int_B \rho\, dV$. Then, $\varphi = \int_\Omega \frac{\rho(\xi)\, dV}{|x-\xi|}$ for all $x \in \mathbb{R}^3 \setminus \Omega$ and as before, $\vec{\mathcal{E}} = -\nabla\varphi$. The Laplacian can, in distributional sense, be computed

$$\Delta\varphi = -4\pi\rho.$$

The scalar function φ is the potential function of $\vec{\mathcal{E}}$ and these functions will be studied in the next section.

4. Potential theory

We need the fundamental solution of Laplace's equation

$$\Phi(x) = \phi(r) = \begin{cases} \log\frac{1}{r} & \text{if } N = 2 \\ r^{2-N} & \text{if } N \geq 3. \end{cases}$$

Here $r = |x|$ and $\Delta\Phi = -c_N\delta$ with the constant $c_N > 0$. Let now $H : \mathbb{R} \to [0, \infty)$ be a continuous, increasing and convex function. (The convexity is not always essential and this condition will later be removed in some cases.) We shall study kernels of the form

$$K(r) = H(\phi(r))$$

and we will also assume integrability

$$\int_0 K(r) r^{N-1}\, dr < \infty.$$

We allow ourselves to write $K(x) = K(|x|)$ letting us write the above condition

$$\int_{|x|<a} K(x)\, dx < \infty.$$

With respect to $K(r)$, we form the *potential* of the set function σ

$$u_\sigma(x) := \int K(|x-y|)\, d\sigma(y),$$

and the *energy integral*

$$I(\sigma) := \iint K(|x-y|)\, d\sigma(y)\, d\sigma(x)$$

or, using the newly defined potential,

$$I(\sigma) = \int u_\sigma(x)\, d\sigma(x).$$

The potential and the energy of course have physical origins; see [35].

Remarks:

- There is a problem when $N = 2$ because $\log\frac{1}{|z|}$ changes sign , where $z \in \mathbb{C}$. This will be studied later.
- If we allow $K(r) = r^{\alpha-N}$ we will get something called an α–potential.

Let us look at an example of "strange" behavior of a potential in $\mathbb{R}^N$. Let $u(x) = \sum \frac{a_i}{|x-y_i|}$ where $\{y_i\}$ is a dense set in $\mathbb{R}^N, a_i > 0, \sum a_i < \infty$. We have then $u \in L^1_{loc}$ but $u(y_i) = \infty$! What are the sets $\{x : u(x) > A\}$?

From now on we will restrict ourselves to non–negative measures, μ, with bounded support. We have then the following lemma on the property of semi–continuity. First, the definition,