Lecture Notes in Mathematics

Marc Bernot
Vicent Caselles
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# Optimal Transportation Networks

1955

## Models and Theory

Springer

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# Preface

The transportation problem can be formalized as the problem of finding the optimal paths to transport a measure $\mu^+$ onto a measure $\mu^-$ with the same mass. In contrast with the Monge-Kantorovich formalization, recent approaches model the branched structure of such supply networks by an energy functional whose essential feature is to favor wide roads. Given a flow $\varphi$ in a tube or a road or a wire, the transportation cost per unit length is supposed to be proportional to $\varphi^\alpha$ with $0 < \alpha < 1$. For the Monge-Kantorovich energy, $\alpha = 1$ so that it is equivalent to have two roads with flow 1/2 or a larger one with flow 1. If instead $0 < \alpha < 1$, a road with flow $\varphi_1 + \varphi_2$ is preferable to two individual roads $\varphi_1$ and $\varphi_2$ because $(\varphi_1 + \varphi_2)^\alpha < \varphi_1^\alpha + \varphi_2^\alpha$. Thus, this very simple model intuitively leads to branched transportation structures. Such a branched structure is observable in ground transportation networks, in draining and irrigation systems, in electric power supply systems and in natural objects like the blood vessels or the trees. When $\alpha > 1 - \frac{1}{N}$ such structures can irrigate a whole bounded open set of $\mathbb{R}^N$.

The aim of this set of lectures is to give a mathematical proof of several existence, structure and regularity properties empirically observed in transportation networks. This will be done in a simple mathematical framework (measures on the set of paths) unifying several different approaches and results due to Brancolini, Buttazzo, Devillanova, Maddalena, Pratelli, Santambrogio, Solimini, Stepanov, Xia and the authors.

The link with anterior discrete physical models of irrigation and erosion models in hydrography and with discrete telecommunication and transportation models will be discussed. It will be proved that most of these models fit in the simple model sketched above. Several mathematical conjectures and questions on the numerical simulation will be developed.

The authors thank Bernard Sapoval for introducing them to this subject and for giving them many insights on physical aspects of irrigation networks. V. Caselles acknowledges partial support by the "Departament d'Universitats, Recerca i Societat de la Informació de la Generalitat de Catalunya" and by PNPGC project, reference BFM2003-02125. J.M.Morel acknowledges many discussions with and helpful suggestions from Giuseppe Devillanova, Franco Maddalena, Filippo Santambrogio and Sergio Solimini. He also thanks UCLA for its hospitality during the revision of the manuscript.

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# 1 Introduction: The Models

The aim of this book is to give a unified mathematical theory of branched transportation (or irrigation) networks. The only axiom of the theory is a $l \times s^\alpha$ cost law ($0 \leq \alpha \leq 1$) for transporting a good with size $s$ on a path with length $l$. Let us explain first why this assumption is relevant.

Humans have designed many supply-demand distribution networks transporting goods from supply sites to widespread distribution sites. This is obviously the case with networks for ground transportation, communication [44], electric power supply, water distribution, drainage [52], or gas pipelines [16]. These networks show a striking similarity to observable natural irrigation and draining systems which connect a finite size volume to a source or to an outlet. Forests, plants, weeds, and trees together with their root systems, but also the nervous, the bronchial and the cardiovascular systems have a common morphology which seems to derive from topological constraints together with energy saving requirements. All of these systems look like spatial trees and succeed in spreading out a fluid from a volume or a source onto another volume. The associated morphology is a tree (or a union of trees) made of bifurcating vessels. Their intuitive explanation is that transport energy is saved and better protection is obtained by using broad vessels as long as possible rather than thin spread out vessels.

The above list involves a huge range of natural and artificial phenomena. So the underlying optimization problems have been treated at several complexity levels, by different communities, and with different goals. Even the names of the network optimization problems vary. The problem first emerged in the framework of operational research and graph theory. In that case the geometry of the network is fixed *a priori* and the problem is known in graph theory by the name of *Minimum Concave Cost Flows.* This model was introduced in Zangwill's article [100]. It considers graphs endowed with flows and prescribed sources and demands at certain graph nodes. The transportation of a mass along an edge has a cost, usually proportional to its length but *concave with respect to the mass.* The optimization problem is to find the minimal cost flow achieving a prescribed transport. The abundant literature dealing with this problem refers to all classical operational research applications: transportation, communication, network design and distribution, production and inventory planning, facility/plant location, scheduling and air traffic control.

M. Bernot et al., *Optimal Transportation Networks.* Lecture Notes in Mathematics 1955.

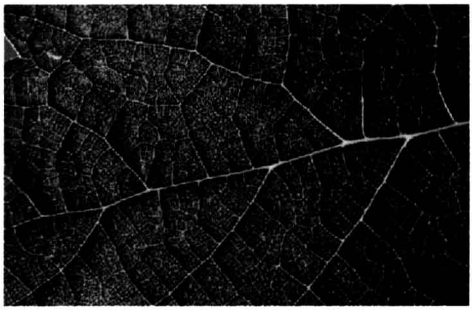

**Fig. 1.1.** The structure of the nerves of a leaf (see [97] for a model of leaves based on optimal irrigation transport).

In all of these cases, the concavity of the transport cost with respect to the flow along an edge is justified by economical arguments. In Zangwill's words: *The literature is replete with analyses of minimum cost flows in networks for which the cost of shipping from node to node is a linear function. However, the linear cost assumption is often not realistic. Situations in which there is a set-up with charge, discounting, or efficiencies of scale give rise to concave functions.*

A similar view is developed in the more recent article [99]: *Although a mathematical model with a linear arc cost function is easier to solve, it may not reflect the actual transportation cost in real operations. In practice, the unit cost for transporting freight usually decreases as the amount of freight increases. The cargo transportation cost in particular is mainly influenced by the cargo type, the loading/unloading activities, the transportation distance, and the amount. In general, each transportation unit cost decreases as the amount of cargo increases, due to economy of scale in practice. Hence, in actual operations the transportation cost function can usually be formulated as a concave cost function.*

Regarding the practical resolution of this optimization problem, the key source of complexity of the concave cost network flow problem arises from the minimization of a concave function over a convex feasible region, defined by the network constraints. As Zangwill points out [100]: *although concave functions can be minimized by an exhaustive search of all the extreme points of the convex feasible region, such an approach is impractical for all but the simplest of problems.* Indeed, there are potentially an enormous number of local optima in the search space. For this reason, concave cost network flow problems are known to be NP-hard [46] and [51]. Yet, many algorithms have been developed over the years for solving these problems [45], [46], [85], [41], [99] (this last reference thoroughly discusses the differences between existing algorithms). As an argument towards a complexity reduction, Zangwill proves in [100] that optimal flows have a tree structure. Indeed, a local optimum is necessarily an extremal point so that two flow paths connecting two points have to coincide. We shall prove the very same result, under the name of single path property, in a more general framework (see Chapter 7).

In a similar context many authors have considered the problem of optimizing branched distribution such as rural irrigation, reclaimed water distribution, and effluent disposal (see [31], [35], [77] among numerous other references). This literature is more specialized and the model is made more precise: the layout is prescribed and the decision variables include design parameters (pipe diameters, pump capacities, and reservoir elevations). The objective function to be minimized reflects the overall cost construction plus maintenance costs. The constraints are in the form of demands to be met and pressures at selected nodes in the network to be within specified limits.

A big limitation of Minimum Concave Cost Flows is that the geometry of the network is fixed. In practice the network itself has to be designed! This problem cannot really be addressed within graph theory and leads us to more geometrical considerations. In a very recent paper [53], Lejano developed a method for determining an optimal layout for a branched distribution system *given only the spatial distribution of potential customers and their respective demands.* Lejano insists on the novelty of such a problem with respect to Minimum Concave Cost Flows: *Much research has been developed around optimizing pipeline design assuming a predetermined geographical layout of the distribution system. There has been less work done, however, on the problem of optimizing the configuration of the network itself. Generally, engineers develop the basic layout through experience and sheer intuition.*

In this book, we shall retain only the essential aspect of the above problems, namely the concavity of the transport cost. The optimization problem we shall consider is the more general one, namely the optimization of the layout itself. Since the number of sources and wells is finite, most irrigation or transport practical models look at first discrete. Yet, because of the huge scale ratios and of invariance requirements, a continuous model is preferable.

In the continuous framework, the transportation or irrigation problem can be formalized as the problem of finding the optimal paths to transport any positive measure (not necessarily atomic or finite) $\mu^+$ onto another positive measure $\mu^-$ with the same mass. The first and classical statement of this transportation problem is due to Monge [62] and its formalization to Kantorovich [50]. In this original model, masses are transported on infinitely many straight routes by infinitesimal amounts. Probably the best natural phenomenon akin to this mathematical solution is the nectar gathering by bees from the fields to the hive. In the Monge-Kantorovich model the straightness of trajectories makes the transportation cost $\varphi \times l$ strictly proportional to the amounts of transported goods $\varphi$ and to the distance $l$. As we already pointed out, transportation on straight lines is neither economically nor energetically sound in most situations and does not correspond to the observed morphology of transportation networks.

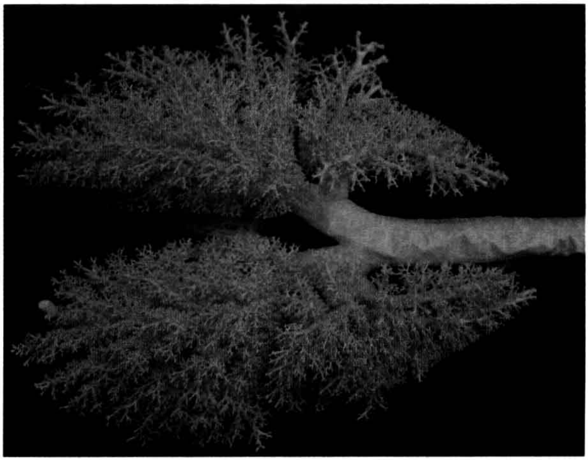

**Fig. 1.2.** A cast of a dog set of lungs. They solve the problem of bringing the air entering the trachea onto a surface with very large area (about 500 $m^2$ for the human lungs). A mathematical and physical study of the lungs efficiency is developed in [61].

The simplest mathematical model compatible with the above considerations on supply networks uses a cost function whose essential feature is to favor wide routes rather than thin ones. Given a flow $\varphi$ in a road or a tube or a wire, the transportation cost per unit length is taken proportional to $\varphi^\alpha$ with $0 < \alpha < 1$. For the Monge-Kantorovich energy $\alpha = 1$ so that it is equivalent to have two roads with flow 1/2 or a larger one with flow 1. If instead $0 < \alpha < 1$, a road with flow $\varphi_1 + \varphi_2$ is preferable to two individual roads with flows $\varphi_1$ and $\varphi_2$ because $(\varphi_1 + \varphi_2)^\alpha < \varphi_1^\alpha + \varphi_2^\alpha$. In the terms of Xia [94], the interpretation reads as follows: *In shipping two items from nearby cities to the same far away city, it may be less expensive to first bring them into a common location and put them on a single truck for most of the transport. In this case, a "Y shaped" path is preferable to "V shaped" path.* So the only axiom in this theory will be the $s^\alpha$ cost law with $0 \leq \alpha \leq 1$.

To the best of our knowledge the concave power law was first proposed in 1967 by Gilbert [44] to optimize communication networks. But the very same model has recurred in the past twenty years for the physical analysis of scaling laws in animal metabolism [88], [89], [8]. The models in the mentioned authors and in the river basins literature described extensively in the book [73] consider finite graphs $G$ made of tubes satisfying the Kirchhoff conservation law. The source and wells are modeled by finite sums of Dirac masses. The energy of the network, interpretable as a power dissipation, is

$$W(G) = \sum_{k \in G} l_k s_k^{-2} \varphi_k^2,$$

**Fig. 1.3.** A very old tree (1200 years) spans his branches towards the light. Trees and plants solve the problem of spanning their branches as much as possible in order to maximize the amount of light their leaves receive for photosynthesis. The surface of the branches is minimized for a better resistance to parasites, temperature changes, etc.

where $G$ is the set of tubes, $k$ the tube index, $s_k$ the tube section, $l_k$ the tube length, and $\varphi_k$ the flow in the tube. Thus the model is *a priori* more complex than the Gilbert model, which only considers a flow depending cost

$$\tilde{W}(G) = \sum_k l_k \varphi_k^\alpha \tag{1.1}$$

However, it will be proven in Chapter 14 that the energy $W$ reduces to a Gilbert energy $\tilde{W}$ with $\alpha = \frac{2}{3}$, under the mild and natural assumption that the network volume is fixed.

Let us give some examples. A tree (see Figure 1.3), a plant or a forest can be viewed as transportation networks from the ground (a 3-D volume) onto a 2D surface, typically a sphere in the case of an isolated tree or, at a different scale, a plane in the case of a forest. Indeed, the roots spread in the ground to attain every part of the underneath volume in search of water and nutriments. In the air, branches tend to spread out to intercept as much sunlight as possible. Thus we can roughly view the branches as means to reach by subdivision a sphere approximating the tree's foliage. The tree branches are barked bundles of fibers going from the ground to the leaves (see Figure 1.1) and transporting the sap at constant speed. Up to a multiplicative constant, the flux in a branch is equal to its section: $\varphi_k = s_k$ where $s_k$ is the section. The obvious protection requirement of the branches from external aggressions such as parasites or temperature changes leads to minimize their area, which is barked for the same reason. Thus we are led to a minimal surface problem,

$$\min_G(\tilde{W}(G) = \sum_k l_k \varphi_k^{\frac{1}{2}}) \tag{1.2}$$

with the constraints that the graph $G$ satisfies the Kirchhoff law and irrigates a stipulated volume and a stipulated surface. The cost given in (1.2) is similar to the one for pipe-lines [16]. In that case the construction cost is $W(G) = \sum_k l_k \varphi_k^{\frac{1}{2}}$ because it is proportional to the length and to the diameter of the tubes while the flow is itself roughly proportional to the size of the tubes. The discrete model is well justified in that latter case since wells and plants are indeed finite atomic masses. In the case of O.C.N. (Optimal Channel Networks, [73]) modeling river networks, the power is again $\alpha \simeq \frac{1}{2}$ and the irrigation constraints have to model the drainage of a whole region to a few river mouths. Last but not least, the human body contains irrigation networks irrigating a whole volume from the heart in the case of arteries (see Figure 1.4) and a very large surface from the throat in the case of lungs (see Figure 1.2).

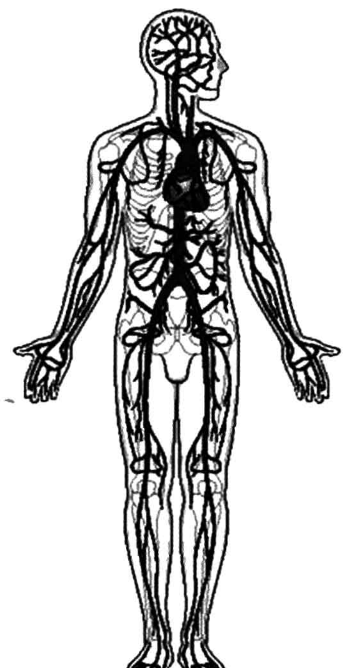

**Fig. 1.4.** Arteries of the human body. They solve the problem of transporting the blood from the heart to the whole body with very low basal metabolic rate. Attempts to demonstrate scaling laws in Nature have focused on the basal metabolic rate [89, 90]. This rate has been linked to the total blood flow.

All of these networks have a striking similarity structure. In particular, they are trees.

The Gilbert energy is simple but the devil is in the irrigation constraints which we just mentioned. In the discrete model these are just a finite set. Of course all irrigated Lebesgue measures can be approximated by finite atomic measures. Yet in the continuous setting, the feasibility of irrigation networks is no more granted. Not that there would be a geometric obstruction to the existence of infinite trees irrigating a positive volume $K$ in a strong sense, namely with a branch of the tree (a sequence of tubes) arriving at every point of $K$. Such tube trees can be constructed by rather explicit rules; they can satisfy the Kirchhoff law and can even have the fluid speed decrease and be null at the tips of capillaries. Such constructions can be found (e.g.) in [8], [67] and [27]. Figure 1.5 gives an intuitive recursive construction of a tree irrigating a cube. One of the first examples described in the literature is due to Besicovitch [15] and is precisely the construction in Figure 1.5.

Optimal irrigation networks are complex objects and some generic descriptors are needed to describe them. Among the candidates, fractal dimensions have had the preference of most authors. In geomorphology, an early study of the fractal-like behavior of natural drainage networks was started as early as 1945 by R.E. Horton [47], A.N. Strahler [82], and generalized by E. Tokunaga [84].

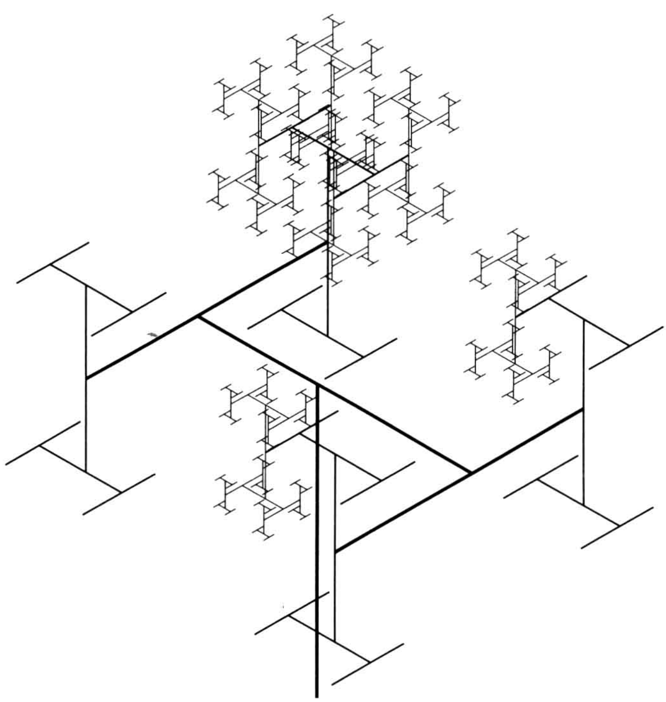

**Fig. 1.5.** An irrigating tree.

To be able to derive fractal dimensions from the variational model, a basic assumption is usually made, namely that the network has a branched tree structure made at each scale of tubes of a certain uniform length, radius and with a given branching number. In other terms, the irrigation system is a fully homogeneous tree in scales, sizes and shapes like the tree of Figure 1.5. Then, under these *ad hoc* assumptions, the authors of [88], [89], [20] prove that the network has a fractal structure with self-similar properties. The irrigation network is then characterized by the branching ratios and the ratios of radii and lengths of the tubes. Calling $n$ the branching ratio, the above assumptions permit to conclude the radii and length ratios scale as powers of $n$. This heuristic reasoning ends up with a structure described as a self-similar fractal.

The weak point of the above treatment is the very strong homogeneity assumption involved in the heuristic calculations. The self-similarity properties, if they are really true, should be deduced from first principles. The basic variational principle related to the cost of irrigation and the irrigation constraints should be the only basis for structural statements as was requested in [88]. Authors in [89] acknowledge that *In spite of the very large number of numerical and empirical studies, no general theory based on fundamental laws has yet been developed for (...) fractal behavior (...).*

Thus, the aim of these lecture notes is to go back on the foundations of optimal channel networks. We shall define a common mathematical structure to all of them and give a proof of several structure and regularity properties empirically observed in transportation networks. These results hopefully pave the way to the study of fractal properties. They already confirm that fully self-similar models are too simplistic. Actually many of the questions raised by specialists such as the regularity issues, the existence and shape of river basins, the dimensionality of the irrigated volumes, surfaces and of the irrigation network itself can only be rigorously treated once a continuous invariant model has been stated and existence and regularity proven in the general setting adopted here. For example the formation of "tubes", which is one of the assumptions of the empirical models must (and will) be demonstrated, and their regularity and branching number estimated.

The first job is to fix an adequate mathematical object for irrigation networks, simple enough to cope with the mathematical challenges but general enough to cover the variety of cases. At least three mathematical models have been proposed so far. Xia [94] modeled the networks as currents which can be approximated in the sense of currents by finite irrigation networks. This definition based on a relaxation yields easy weak existence results for minimizers but does not seem well adapted to a thorough description of the network structure.

Maddalena and Solimini [59] gave a much more explicit definition in the case of networks which start from a single source. Their definition is directly derived from the tree model as a union of fibers. The fibers are Lipschitz