Lecture Notes in Mathematics

Brooks Roberts
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Local Newforms for GSp(4)

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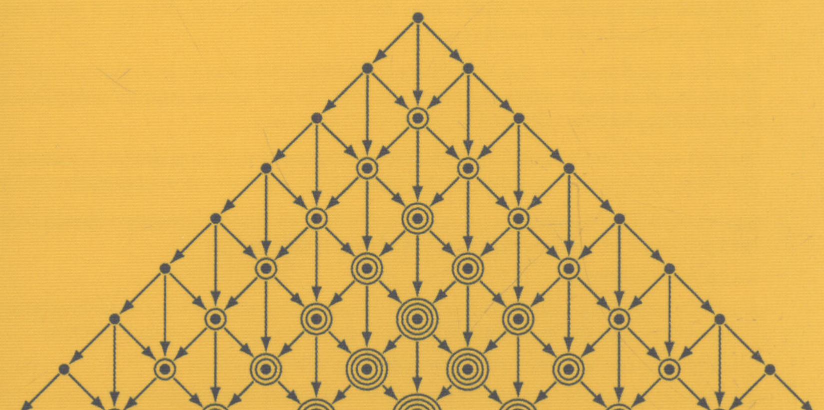

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Local Newforms for GSp(4)

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About the diagram. The diagram illustrates natural bases for the new- and oldforms in a generic representation π of $\mathrm{GSp}(4,F)$ with trivial central character. The solid dot in the first row is the newform at level N_π. The solid dots and circles of the k-th row represent vectors in a natural basis for the oldforms at level $N_\pi + k$. Thus, the dimension of the paramodular vectors at level N_π is 1, the dimension at level $N_\pi + 1$ is 2, the dimension at level $N_\pi + 2$ is 4, and so on. The basis at a particular level is obtained from the newform by application of the commuting level raising operators θ, θ', and the self-dual operator η. The arrows pointing down and to the left correspond to θ, the arrows pointing down and to the right correspond to θ', and the vertical arrows correspond to η. The black dots represent oldforms obtained solely by application of θ's and θ''s. The inner-most circles represent oldforms obtained by a single application of η, the circles immediately around the inner-most circles represent oldforms obtained by two applications of η, etc.

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Contents

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1

A Summary

The local theory of new- and oldforms for representations of GL(2) is a tool for studying automorphic forms on GL(2) and their applications. This theory singles out, in infinite-dimensional representations, certain vectors which encode information. Thus, this theory lies at the intersection of representation theory, modular forms theory, and applications to number theory. See the work of Casselman [Ca2]; for more information and references, see [Sch1]. The paper [JPSS] generalized some aspects of the theory for GL(2) to GL(n) for generic representations.

This work presents a local theory of new- and oldforms for representations of GSp(4) with trivial central character. This theory resembles the GL(2) theory, but also has some new features. Our theory considers vectors fixed by the paramodular subgroups $\mathrm{K}(\mathfrak{p}^n)$ as defined below. Paramodular groups, their modular forms, and their application to abelian surfaces with polarizations of type $(1, N)$ have been considered for about fifty years. At the same time, the literature perhaps shows a greater emphasis on Siegel modular forms defined with respect to the Siegel congruence subgroup $\Gamma_0(N)$. Nevertheless, in hindsight, it seems clear that the paramodular subgroups are good analogues of the congruence subgroups underlying the new- and oldforms theory for GL(2) and GL(n). In combination with the structure of the discrete spectrum of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$, the results of this work lead to a satisfactory theory of new- and oldforms for paramodular Siegel modular forms of genus 2. This is discussed in our paper [RS]. We intend to consider this topic again in a later work. This introduction is divided into three parts. The first part briefly reviews the GL(2) theory, the second part summarizes our main results, and the final part delineates the three methods used to prove the main theorems.

Before beginning, we mention some comments that apply to the entire body of this work. First, as far as we know, our theory of new- and oldforms is novel and is unanticipated by the existing framework of general conjectures. Second, as concerns methods and assumptions, this work contains complete proofs of all results, does not depend on any conjectures, and does not use global methods. And finally, this work makes no assumptions about the residual characteristic of the underlying non-archimedean local field.

The GL(2) Theory

The purpose of this work is to demonstrate the existence of a new- and oldforms theory for GSp(4). We begin by recalling the relevant new- and oldforms theory for GL(2), since this is the archetype for our collection of theorems.

First we require some definitions. Let F be a nonarchimedean local field of characteristic zero with ring of integers $\mathfrak{o}$, let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$, and let q be the number of elements of $\mathfrak{o}/\mathfrak{p}$. Fix a generator ϖ for $\mathfrak{p}$. Let ψ be a non-trivial character of F with conductor $\mathfrak{o}$. For each non-negative integer n let $\Gamma_0(\mathfrak{p}^n)$ be the subgroup of k in $\mathrm{GL}(2,F)$ such that $\det(k)$ is in $\mathfrak{o}^\times$ and

$$k \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} \end{bmatrix}.$$

The group $\Gamma_0(\mathfrak{p}^n)$ is normalized by the *Atkin–Lehner element* of level $\mathfrak{p}^n$

$$u_n = \begin{bmatrix} & 1 \\ -\varpi^n & \end{bmatrix}.$$

Note that u_n^2 lies in the center of $\mathrm{GL}(2,F)$.

Next, we consider representations. Recall that an irreducible, admissible representation of $\mathrm{GL}(2,F)$ with trivial central character is either generic, in which case it is infinite-dimensional, or non-generic, in which case it is one-dimensional. The theory of new- and oldforms is mainly about generic representations, and we consider them first. However, since our goal is to provide motivation for the case of GSp(4), and since non-generic representations play a much larger role in the local and global representation theory of GSp(4), we will also treat the case of non-generic, i.e., one-dimensional, representations at the end of this section.

Let π be a generic, irreducible, admissible representation of $\mathrm{GL}(2,F)$ with trivial central character. Let $\mathcal{W}(\pi,\psi)$ be the Whittaker model of π with respect to ψ, and let

$$Z(s,W) = \int\limits_{F^\times} W(\begin{bmatrix} a & \\ & 1 \end{bmatrix})|a|^{s-1/2}\,d^\times a$$

be the zeta integral of $W \in \mathcal{W}(\pi,\psi)$. Zeta integrals satisfy a functional equation involving the element u_0, and the theory of zeta integrals assigns to π an L-factor $L(s,\pi)$ and an ε-factor $\varepsilon(s,\pi)$; see Chapter 6 of [G] for a summary. Let W_F' be the Weil–Deligne group of F. If $\varphi : W_F' \to \mathrm{GL}(2,\mathbb{C})$ is the L-parameter of π, then $L(s,\pi) = L(s,\varphi)$ and $\varepsilon(s,\pi) = \varepsilon(s,\varphi)$. For the definitions of $L(s,\varphi)$ and $\varepsilon(s,\varphi)$ see the end of Sect. 2.4. The following is the main theorem about newforms for GL(2) that is relevant for our purposes.

Theorem (GL(2) Generic Newforms Theorem). *Let (π, V) be a generic, irreducible, admissible representation of* GL(2, F) *with trivial central character. For each non-negative integer n, let $V(n)$ be the subspace of V of vectors W such that $\pi(k)W = W$ for all k in $\Gamma_0(\mathfrak{p}^n)$. Then the following statements hold:*

i) For some n the space $V(n)$ is non-zero.
ii) If N_π is the minimal n such that $V(n)$ is non-zero, then $\dim V(N_\pi) = 1$.
iii) Assume $V = \mathcal{W}(\pi, \psi)$. There exists W_π in $V(N_\pi)$ such that

$$Z(s, W_\pi) = L(s, \pi).$$

If (π, V) is a generic, irreducible, admissible representation of GL(2, F) with trivial central character, then we call N_π the *level* of π; in some references, N_π is called the conductor of π. Any non-zero element of the one-dimensional space $V(N_\pi)$ is called a *newform*, and the elements of the spaces $V(n)$ for $n > N_\pi$ are called *oldforms*.

A corollary of the GL(2) Generic Newforms Theorem is the computation of the ε-factor of a generic representation. Let π be a generic, irreducible, admissible representation of GL(2, F) with trivial central character. Since $u_{N_\pi}^2$ lies in the center of $F^\times$, and since the space $V(N_\pi)$ is one-dimensional, any non-zero element of $V(N_\pi)$ is an eigenvector of $\pi(u_{N_\pi})$ with eigenvalue $\varepsilon_\pi = \pm 1$. As a consequence of the functional equation for zeta integrals and the GL(2) Newforms Theorem we obtain the following corollary.

Corollary (ε-factors of Generic GL(2) Representations). *Let (π, V) be a generic, irreducible, admissible representation of* GL(2, F) *with trivial central character. Then $\varepsilon(s, \pi) = \varepsilon_\pi q^{-N_\pi(s-1/2)}$.*

This result computes the ε-factor of a generic representation in terms of invariants of a newform that make no reference to a specific kind of model: can $L(s, \pi)$ also be computed in this way? This is possible using a Hecke operator. Let n be a non-negative integer, and let $\mathcal{H}(\Gamma_0(\mathfrak{p}^n))$ be the Hecke algebra of $\Gamma_0(\mathfrak{p}^n)$, i.e., the vector space of left and right $\Gamma_0(\mathfrak{p}^n)$-invariant, compactly supported functions on GL(2, F) with product given by convolution. Let (π, V) be a smooth representation of GL(2, F) with trivial central character; we do not assume V is the Whittaker model of π. Then $\mathcal{H}(\Gamma_0(\mathfrak{p}^n))$ acts on $V(n)$ by

$$\pi(f)v = \int\limits_{\mathrm{GL}(2,F)} f(g)\pi(g)v\, dg,$$

where the Haar measure on GL(2, F) assigns $\Gamma_0(\mathfrak{p}^n)$ volume 1. We will use the operator $\pi(f)$ on $V(n)$ corresponding to the characteristic function f of

$$\Gamma_0(\mathfrak{p}^n)\begin{bmatrix}\varpi & \\ & 1\end{bmatrix}\Gamma_0(\mathfrak{p}^n).$$

We will write $T_1 = \pi(f)$.

Theorem (GL(2) Hecke Eigenvalues and L-functions). *Let (π, V) be a generic, irreducible, admissible representation of* GL$(2, F)$ *with trivial central character, and let $W \in V(N_\pi)$ be a newform, i.e., a non-zero element of the one-dimensional space $V(N_\pi)$. Then W is an eigenform for T_1; let*

$$T_1 W = \lambda_\pi W.$$

i) Assume $N_\pi = 0$, so that π is unramified. Then

$$L(s, \pi) = \frac{1}{1 - \lambda_\pi q^{-1/2} q^{-s} + q^{-2s}}.$$

ii) Assume $N_\pi = 1$. Then

$$L(s, \pi) = \frac{1}{1 - \lambda_\pi q^{-1/2} q^{-s}}.$$

iii) Assume $N_\pi \geq 2$. Then $\lambda_\pi = 0$, and $L(s, \pi) = 1$.

The last result of the theory for generic representations asserts that vectors in the spaces $V(n)$ for $n > N_\pi$ are obtained by repeatedly applying two level raising operators to a newform and taking linear combinations. For n a non-negative integer, define $\beta' : V(n) \to V(n+1)$ by $\beta'(v) = v$. Also, define $\beta : V(n) \to V(n+1)$ to be the Atkin–Lehner conjugate of β', i.e., define $\beta = \pi(u_{n+1}) \circ \beta' \circ \pi(u_n)$, so that

$$\beta = \pi\left(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}\right).$$

Theorem (GL(2) Oldforms Theorem). *Let (π, V) be a generic, irreducible, admissible representation of* GL$(2, F)$ *with trivial central character. Then, for any integer $n \geq N_\pi$,*

$$\dim V(n) = n - N_\pi + 1.$$

If $W \in V(N_\pi)$ is non-zero, then the space $V(n)$ for $n \geq N_\pi$ is spanned by the linearly independent vectors

$$\beta'^i \beta^j W, \quad i, j \geq 0, \ i + j = n - N_\pi.$$

In particular, all oldforms can be obtained by applying level raising operators to the newform and taking linear combinations.

Finally, similar results hold for non-generic representations which admit non-zero vectors fixed by some $\Gamma_0(\mathfrak{p}^n)$. Again, any non-generic, irreducible, admissible representations of GL$(2, F)$ with trivial central character is one-dimensional, and is thus of the form $\alpha \circ \det$ for some character α of $F^\times$. Let $\pi = \alpha \circ \det$, where α is a character of $F^\times$. Then π admits a non-zero vector fixed by $\Gamma_0(\mathfrak{p}^n)$ for some non-negative integer n if and only if α is unramified. Assume α is unramified. Obviously, $V(n)$ is non-zero and one-dimensional for all non-negative integers n. The quantities N_π, ε_π and λ_π from above are all defined since they are model-independent. We have

$$N_\pi = 0, \quad \varepsilon_\pi = 1, \quad \lambda_\pi = (q+1)\alpha(\varpi).$$

Though the theory of zeta integrals for generic representations does not apply, the Langlands correspondence assigns ε-factors and L-factors to all irreducible, admissible representations of $\mathrm{GL}(2,F)$. These assignments coincide with the assignments made by the theory of zeta integrals for generic representations. If $\varphi_\pi : W_F' \to \mathrm{GL}(2,\mathbb{C})$ is the L-parameter assigned to π, then a computation shows that $\varepsilon(s,\varphi_\pi)$ and $L(s,\varphi_\pi)$ can be expressed by exactly the same formulas as in the generic setting:

$$\varepsilon(s,\varphi_\pi) = \varepsilon_\pi q^{-N_\pi(s-1/2)}, \quad L(s,\varphi_\pi) = \frac{1}{1-\lambda_\pi q^{-1/2}q^{-s}+q^{-2s}}.$$

It is trivial that the elements of $V(n)$ for $n \geq N_\pi = 0$ are obtained from a newform by applying the level raising operator β'. Though it is obvious, we note also that, in contrast to the case of generic representations, β and β' do not produce linearly independent vectors.

Main Results

In analogy to the GL(2) theory, this work considers vectors in irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character that are fixed by the paramodular groups $\mathrm{K}(\mathfrak{p}^n)$, as defined below. Such vectors are called paramodular, as are representations which admit non-zero paramodular vectors. Briefly summarized, our work has three main results. First, a theory of new- and oldforms exists for generic representations of GSp(4) with trivial central character, and this theory strongly resembles the GL(2) theory described above. In particular, all generic representations with trivial central character are paramodular. Second, the two essential aspects of the generic theory also hold for arbitrary paramodular representations π of GSp(4): there is uniqueness at the minimal paramodular level, and all oldforms are obtained from a newform by applying certain level raising operators and taking linear combinations. Third, newforms in paramodular representations encode important canonical information. If the language of the conjectural Langlands correspondence is used, then our results, which do not depend on or use any conjectures, indicate that a newform in a paramodular representation π determines the ε_π-factor $\varepsilon(s,\varphi_\pi)$ and the L-factor $L(s,\varphi_\pi)$ of the L-parameter φ_π of π. In this section we will discuss the main results in the order mentioned, beginning with the theorems about generic representations. Readers desiring to see additional data should consult the tables in the appendix. These tables explicitly describe important objects and quantities for each irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. The basis

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for these tables is the Sally-Tadić classification [ST] of non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ in the form of Table A.1. Methods and proofs will be discussed in the next section.

First we need some general definitions. Throughout this work, $\mathrm{GSp}(4,F)$ is the group of g in $\mathrm{GL}(4,F)$ such that ${}^tgJg=\lambda J$ for some λ in $F^\times$, where

$$J=\begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix}.$$

The element λ is unique, and will be denoted by $\lambda(g)$. If $n\geq 0$ is a non-negative integer, then the *paramodular group* $\mathrm{K}(\mathfrak{p}^n)$ of level $\mathfrak{p}^n$ is the subgroup of $k\in\mathrm{GSp}(4,F)$ such that $\lambda(k)$ is in $\mathfrak{o}^\times$ and

$$k\in\begin{bmatrix} \mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{p}^{-n}\\ \mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o} \end{bmatrix}.$$

The first paramodular group $\mathrm{K}(\mathfrak{p}^0)$ is just $\mathrm{GSp}(4,\mathfrak{o})$, a maximal compact, open subgroup of $\mathrm{GSp}(4,F)$. The second paramodular group $\mathrm{K}(\mathfrak{p}^1)$ is the other maximal compact, open subgroup of $\mathrm{GSp}(4,F)$, up to conjugacy. Note that, in contrast to the case of the Hecke subgroups in $\mathrm{GL}(2,F)$, $\mathrm{K}(\mathfrak{p}^n)$ is not contained in $\mathrm{K}(\mathfrak{p}^m)$ for any pair of distinct non-negative integers n and m. The paramodular group $\mathrm{K}(\mathfrak{p}^n)$ is normalized by the *Atkin–Lehner element*

$$u_n=\begin{bmatrix} &&1&\\ &&&-1\\ \varpi^n&&&\\ &-\varpi^n&& \end{bmatrix}.$$

Suppose that (π,V) is an irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. If n is a non-negative integer, then we define $V(n)$ to be the subspace of vectors v in V such that $\pi(k)v=v$ for $k\in\mathrm{K}(\mathfrak{p}^n)$. The elements of $V(n)$ are called *paramodular vectors.* We say that π is *paramodular* if $V(n)\neq 0$ for some n. If π is paramodular, then we define N_π to be the minimal n such that $V(n)$ is non-zero, and we call N_π the *paramodular level* of π.

Generic Representations. Now we will discuss our results for generic representations. Let (π,V) be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. Again, let ψ be a non-trivial character of F with conductor $\mathfrak{o}$, fix $c_1,c_2\in\mathfrak{o}^\times$, and define the Whittaker model $\mathcal{W}(\pi,\psi_{c_1,c_2})$ of π with respect to a certain character ψ_{c_1,c_2} of the upper-triangular subgroup of $\mathrm{GSp}(4,F)$ as in Section 2.1. A theory of zeta integrals, introduced by Novodvorsky [N], exists for generic representations of $\mathrm{GSp}(4,F)$. If $W\in\mathcal{W}(\pi,\psi_{c_1,c_2})$, then the zeta integral of W is

$$Z(s,W) = \int\limits_{F^\times}\int\limits_{F} W(\begin{bmatrix} a & & & \\ & a & & \\ & x & 1 & \\ & & & 1 \end{bmatrix})|a|^{s-3/2}\,dx\,d^\times a.$$

See Section 2.6 for more theory and references. As regards basic facts, this theory of zeta integrals is similar to the theory of zeta integrals for generic representations of $\mathrm{GL}(2,F)$. In particular, zeta integrals satisfy a functional equation involving u_0, and the theory associates to π an L-factor $L(s,\pi)$ and an ε-factor $\varepsilon(s,\pi)$. The work [Tak] computed the factors $L(s,\pi)$ for all generic, irreducible, admissible representations π of $\mathrm{GSp}(4,F)$. The factor $L(s,\pi)$ is sometimes called the spin L-function of π, and is of the form $1/Q(q^{-s})$, where $Q(X) \in \mathbb{C}[X]$ is a polynomial of at most degree four such that $Q(0) = 1$. If the conjectural Langlands correspondence for $\mathrm{GSp}(4,F)$ exists, and if $\varphi_\pi : W'_F \to \mathrm{GSp}(4,\mathbb{C})$ is the L-parameter of π according to this correspondence, then it is conjectured that $L(s,\pi) = L(s,\varphi_\pi)$ and $\varepsilon(s,\pi) = \varepsilon(s,\varphi_\pi)$. Here, φ_π is regarded as a four-dimensional representation of the Weil–Deligne group W'_F; we have $\varepsilon(s,\varphi_\pi) = \varepsilon_{\varphi_\pi} q^{-a(\varphi_\pi)(s-1/2)}$, where $\varepsilon_{\varphi_\pi} = \pm 1$, and $a(\varphi_\pi)$ is a non-negative integer. We call $a(\varphi_\pi)$ the *conductor* of φ_π. The following theorem is an analogue of the corresponding $\mathrm{GL}(2)$ result described above.

Theorem 7.5.4 (Generic Main Theorem). *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Then the following statements hold:*

i) There exists an n such that $V(n) \neq 0$, i.e., π is paramodular.
ii) If N_π is the minimal n such that $V(n) \neq 0$, then $\dim V(N_\pi) = 1$.
iii) Assume $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. There exists $W_\pi \in V(N_\pi)$ such that

$$Z(s, W_\pi) = L(s,\pi).$$

One immediate consequence of this theorem is that paramodular representations exist and include generic representations. If π is a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, then we call the non-zero elements of $V(N_\pi)$ *newforms*; the above theorem asserts that a newform for π is essentially unique. The elements of $V(n)$ for $n > N_\pi$ are called *oldforms*.

Just as for $\mathrm{GL}(2)$, if π is a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, then the ε-factor and L-factor of π can be computed in terms of universal invariants of a newform, i.e., invariants of a newform that do not depend on a specific model for π. These formulas for $\varepsilon(s,\pi)$ and $L(s,\pi)$ involve the level N_π, the Atkin–Lehner eigenvalue of a newform, and the Hecke eigenvalues of a newform. The formula for $\varepsilon(s,\pi)$, and its derivation from Theorem 7.5.4, are identical to those of the $\mathrm{GL}(2)$ theory.

Corollary 7.5.5 (ε-factors of Generic Representations). *Let (π, V) be a generic, irreducible, admissible representation of* GSp$(4, F)$ *with trivial central character. Let N_π be the paramodular level of π as in Theorem 7.5.4, and let ε_π be the eigenvalue of the Atkin–Lehner involution $\pi(u_{N_\pi})$ on the one-dimensional space $V(N_\pi)$. Then*

$$\varepsilon(s,\pi) = \varepsilon_\pi q^{-N_\pi(s-1/2)}.$$

The formula for $L(s,\pi)$ in terms of model-independent invariants of a newform requires two Hecke operators. Let n be a non-negative integer, and let $\mathcal{H}(\mathrm{K}(\mathfrak{p}^n))$ be the Hecke algebra of $\mathrm{K}(\mathfrak{p}^n)$, i.e., the vector space of left and right $\mathrm{K}(\mathfrak{p}^n)$-invariant, compactly supported functions on GSp$(4, F)$ with product given by convolution. Suppose that (π, V) is a smooth representation of GSp$(4, F)$ with trivial central character; no assumption is made about V. Then $\mathcal{H}(\mathrm{K}(\mathfrak{p}^n))$ acts on $V(n)$ via the formula

$$\pi(f)v = \int\limits_{\mathrm{GSp}(4,F)} f(g)\pi(g)v\,dg.$$

Here the Haar measure on GSp$(4, F)$ gives $\mathrm{K}(\mathfrak{p}^n)$ volume 1. We will use the operators on $V(n)$ induced by the characteristic functions of

$$\mathrm{K}(\mathfrak{p}^n)\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix}\mathrm{K}(\mathfrak{p}^n) \quad\text{and}\quad \mathrm{K}(\mathfrak{p}^n)\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}\mathrm{K}(\mathfrak{p}^n).$$

These operators will be called $T_{0,1}$ and $T_{1,0}$, respectively. Motivation for the consideration of these Hecke operators is provided below in the section on methods and proofs.

Theorem 7.5.3 (Hecke Eigenvalues and L-functions). *Let (π, V) be a generic, irreducible, admissible representation of* GSp$(4, F)$ *with trivial central character. Let W be a newform of π, i.e., a non-zero element of the one-dimensional space $V(N_\pi)$. Let*

$$T_{0,1}W = \lambda_\pi W, \quad T_{1,0}W = \mu_\pi W,$$

where λ_π and μ_π are complex numbers.

i) Assume $N_\pi = 0$, so that π is unramified. Then

$$L(s,\pi) = \frac{1}{1 - q^{-3/2}\lambda_\pi q^{-s} + (q^{-2}\mu_\pi + 1 + q^{-2})q^{-2s} - q^{-3/2}\lambda_\pi q^{-3s} + q^{-4s}}.$$

ii) Assume $N_\pi = 1$, and let $\pi(u_1)W = \varepsilon_\pi W$, where $\varepsilon_\pi = \pm 1$ is the Atkin Lehner eigenvalue of W. Then

$$L(s,\pi) = \frac{1}{1 - q^{-3/2}(\lambda_\pi + \varepsilon_\pi)q^{-s} + (q^{-2}\mu_\pi + 1)q^{-2s} + \varepsilon_\pi q^{-1/2}q^{-3s}}.$$

iii) Assume $N_\pi \geq 2$. *Then*

$$L(s,\pi) = \frac{1}{1 - q^{-3/2}\lambda_\pi q^{-s} + (q^{-2}\mu_\pi + 1)q^{-2s}}.$$

This theorem exhibits two new phenomena not present in the GL(2) theory. First, when $N_\pi = 1$, the formula for $L(s,\pi)$ involves not just a Hecke eigenvalue, but also the Atkin–Lehner eigenvalue ε_π. Second, in contrast to the GL(2) theory, it is not true that $L(s,\pi) = 1$ if N_π is sufficiently large. There are examples of π such that N_π is arbitrarily large and $\mu_\pi = 0$; for such π we have $L(s,\pi) \neq 1$ by iii) of Theorem 7.5.3.

Oldforms in generic representations of $\mathrm{GSp}(4,F)$ also exhibit a new phenomenon. Just as in the GL(2) case, oldforms for GSp(4) are obtained from a newform via level raising operators; however, the GSp(4) case requires an extra operator, and the spaces of oldforms have an additional summand. Let (π, V) be a smooth representation of $\mathrm{GSp}(4,F)$ with trivial central character. The first two level raising operators, called θ' and θ, are analogues of the GL(2) operators β' and β. The operator $\theta' : V(n) \to V(n+1)$ is the natural trace operator, and is the analogue of the GL(2) level raising operator β'. The operator $\theta : V(n) \to V(n+1)$ is the Atkin–Lehner conjugate of θ', and is thus defined by $\theta = \pi(u_{n+1}) \circ \theta' \circ \pi(u_n)$. This operator is the analogue of β. The third operator, $\eta : V(n) \to V(n+2)$, skips one level and does not have a GL(2) analogue. It is defined by

$$\eta = \pi(\begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}).$$

Theorem 7.5.6 (Generic Oldforms Theorem). *Let* (π, V) *be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Let* N_π *be the paramodular level of* π *and let* W_π *be the newform as in Theorem 7.5.4. Then, for any integer* $n \geq N_\pi$,

$$\dim V(n) = \left\lfloor \frac{(n - N_\pi + 2)^2}{4} \right\rfloor.$$

For $n \geq N_\pi$, *the space* $V(n)$ *is spanned by the linearly independent vectors*

$$\theta'^i \theta^j \eta^k W_\pi, \quad i,j,k \geq 0, \ i+j+2k = n - N_\pi.$$

In particular, all oldforms are obtained by applying level raising operators to the newform and taking linear combinations.

An alternative formulation of this theorem exposes the similarities and differences between oldforms for GL(2) and oldforms in generic representations of $\mathrm{GSp}(4,F)$. Theorem 7.5.6 is equivalent to the statement that for $n \geq N_\pi$

the space $V(n)$ is the direct sum of the subspace spanned by the linearly independent vectors

$$\theta'^i \theta^j W_\pi, \quad i,j \geq 0, i+j = N_\pi - n,$$

and the subspace $\eta V(n-2)$, so that

$$\dim V(n) = n - N_\pi + 1 + \dim V(n-2).$$

Stated this way, we see that oldforms in generic representations of $\mathrm{GSp}(4,F)$ have a structure similar to the structure of oldforms in generic representations of $\mathrm{GL}(2)$, with one difference: in the case of $\mathrm{GSp}(4)$, the space $V(n-2)$ also contributes to $V(n)$ via η. The subspace $\eta V(n-2)$ can be characterized as the subspace of W in $V(n)$ such that $Z(s,W)=0$. We call this characterization the *η Principle*, and discuss it in the next section. Vectors W in $V(n)$ such that $Z(s,W)=0$ are *degenerate*. Degenerate vectors do not exist in the $\mathrm{GL}(2)$ theory, and are a new phenomenon for $\mathrm{GSp}(4)$.

Arbitrary Representations. This work also treats arbitrary paramodular, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character. We prove that the two basic principles of the generic theory hold for arbitrary paramodular representations. These principles are essential for global applications. First of all, there is uniqueness at the minimal paramodular level:

Theorem 7.5.1 (Uniqueness at Minimal Level). *Let (π, V) be an irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Assume that π is paramodular, and let N_π be the minimal paramodular level. Then* $\dim V(N_\pi)=1$.

If π is a paramodular, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, then we call the non-zero elements of $V(N_\pi)$ *newforms*; the theorem asserts that newforms in paramodular representations are essentially unique. The elements of $V(n)$ for $n > N_\pi$ are called *oldforms*. Global applications will require the following theorem. This second basic principle asserts that oldforms are obtained from a newform by applying level raising operators:

Theorem 7.5.7 (Oldforms Principle). *Let (π, V) be an irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Assume that π is paramodular. If v is a non-zero element of the one-dimensional space $V(N_\pi)$ and $n \geq N_\pi$, then the space $V(n)$ is spanned by the (not necessarily linearly independent) vectors*

$$\theta'^i \theta^j \eta^k v, \quad i,j,k \geq 0, \ i+j+2k = n - N_\pi.$$

In other words, all oldforms can be obtained from the newform v by applying level raising operators and taking linear combinations.

In fact, we have determined a basis for $V(n)$ among the spanning set of vectors $\theta'^i\theta^j\eta^k W$ for all paramodular representations and all n. By Theorem 7.5.6, if π is generic, then the spanning vectors $\theta'^i\theta^j\eta^k W$, where $i,j,k\geq 0$ and $i+j+2k=n-N_\pi$, form a basis, and the dimension of $V(n)$ is $[(n-N_\pi+2)^2/4]$. This characterizes generic representations: a representation is generic if and only if the representation is paramodular and $\dim V(n)=[(n-N_\pi+2)^2/4]$ for $n\geq N_\pi$. The bases for $V(n)$ for non-generic, paramodular representations also follow general schemes. There are four patterns for non-generic, paramodular representations. First, it can happen that the vectors $\theta'^i\eta^k W$ where $i,k\geq 0$ and $i+2k=n-N_\pi$ form a basis for $V(n)$, so that $\dim V(n)=[(n-N_\pi+2)/2]$ for $n\geq N_\pi$. This occurs if and only if π is paramodular and of type IIb, IVb, Vb, Vc, VIc, VId or XIb. The second possibility is that the vectors $\theta'^i\theta^j W$ where $i,j\geq 0$ and $i+j=n-N_\pi$ form a basis for $V(n)$, and hence $\dim V(n)=n-N_\pi+1$ for $n\geq N_\pi$. This happens if and only if π is paramodular and of type IIIb or IVc. Third, the vectors $\eta^k W$ where $k\geq 0$ and $2k=n-N_\pi$ form a basis for $V(n)$, so that $\dim V(n)=(1+(-1)^n)/2$ for $n\geq N_\pi$. This occurs if and only if π is paramodular and of type Vd. Finally, it can happen that the vectors $\theta'^i W$ where $i=n-N_\pi$ form a basis for $V(n)$, and thus $\dim V(n)=1$ for $n\geq N_\pi$. This last possibility happens exactly for quadratic unramified twists of the trivial representation, i.e., π is paramodular and of type IVd. See Table A.12 for the dimensions of the spaces $V(n)$ for all irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial character.

Information Carried by a Newform. Finally, our results show that a newform in a paramodular representation carries important canonical information. Let π be a paramodular, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. We saw above that if π is generic, then $\varepsilon(s,\pi)$ and $L(s,\pi)$ can be expressed in terms of the model-independent invariants N_π, ε_π, λ_π and μ_π. Thus, if π is generic, then a newform for π contains all the information present in $\varepsilon(s,\pi)$ and $L(s,\pi)$. Next, assume that π is non-generic. Then the theory of zeta integrals is not available, but based on the generic case it is natural to conjecture the following: if φ_π is the conjectural L-parameter of π, then $\varepsilon(s,\varphi_\pi)$ and $L(s,\varphi_\pi)$ can be expressed in terms of N_π , ε_π, λ_π and μ_π via the same formulas in Corollary 7.5.5 and Theorem 7.5.3. Of course, verifying this conjecture requires knowing φ_π; this appears to be a problem since the Langlands correspondence for $\mathrm{GSp}(4,F)$ is conjectural, so that the L-parameters of general representations are not known. However, it turns out that the desiderata of the conjectural Langlands correspondence, in combination with the classification of induced representations from [ST], do determine the L-parameters of some representations of $\mathrm{GSp}(4,F)$, namely those that are non-supercuspidal. The following theorem implies that any non-generic, paramodular representation is of this type, and is even non-tempered.

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