AN INTRODUCTION TO RELATIVISTIC QUANTUM FIELD THEORY

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Foreword by

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Foreword

It is always astonishing to see one's children grow up, and to find that they can do things which their parents can no longer fully understand. This book is a good example. It was first conceived by Dr. Frederic de Hoffmann and myself as merely a short introduction to the rather simple-minded calculations on π mesons in Volume II of the old book *Mesons and Fields*, published in 1955. In Dr. Schweber's hands Volume I, even then, had developed into a thorough textbook on renormalization in field theory. It has now become a comprehensive treatise on field theory in general.

In the six years since the publication of the two-volume *Mesons and Fields* field theory has made spectacular progress. Some of this progress was stimulated by experiment, e.g., by the discovery that parity is not conserved in weak interactions. Much of it, however, consisted in a deeper search into the foundations of field theory, trying to answer the central question of relativistic quantum theory which Schweber poses himself in Chapter 18 of this book: Do solutions of the renormalized equations of quantum electrodynamics or any meson theories exist? This search has led to the axiomatic approach to quantum field theory which is probably the most promising and solid approach now known, and which is described in Chapter 18.

About half of the present book is devoted to the interaction between fields. This new book contains a thorough discussion of renormalization theory, starting from the general principles and leading to quantitative results in the case of electrodynamics. I do not know of any other treatment of this subject which is equally complete and rigorous. The physicist who is interested in applications of field theory will be happy about the good discussion of the theory of Chew and Low of π-meson scattering, which theory has been so successful in explaining the π-meson phenomena at low energy and which has superseded the methods presented in Volume II *Mesons* of the older book.

The book emphasizes general principles, such as symmetry, invariance, isotopic spin, etc., and develops the theory from these principles. It is never satisfied with superficial explanations. The student who really wants to know and understand field theory, and is willing to work for it, will find great satisfaction in this book.

H. A. BETHE

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Preface

The present book is an outgrowth of an attempted revision of Volume I of *Mesons and Fields* which Professors Bethe, de Hoffmann and the author had written in 1955. The intent at the outset was to revise some of the contents of that book and to incorporate into the new edition some of the changes which have occurred in the field since 1955. Unfortunately, due to the pressure of other duties, Drs. Bethe and de Hoffmann could not assist in the revision. By the time the present author completed his revision, what emerged was essentially a new text. With the gracious consent of Drs. Bethe and de Hoffmann, it is being published under a single authorship.

The motivation of the present book, however, is still the same as for the volume *Fields* on which it is based, in part: to present in a simple and self-contained fashion the modern developments of the quantum theory of fields. It is intended primarily as a textbook for a graduate course. Its aim is to bring the student to the point where he can go to the literature to study the most recent advances and start doing research in quantum field theory. Needless to say, it is also hoped that it will be of interest to other physicists, particularly solid state and nuclear physicists wishing to learn field theoretic techniques.

The desire to make the book reasonably self-contained has resulted in a lengthier manuscript than was originally anticipated. Because it was my intention to present most of the concepts underlying modern field theory, it was, nonetheless, decided to include most of the material in book form. In order to keep the book to manageable length, I have not included the Schwinger formulation of field theory based on the action principle. Similarly, only certain aspects of the rapidly growing field of the theory of dispersion relations are covered. It is with a mention of the Mandelstam representation for the two-particle scattering amplitude that the book concludes. However, some of the topics not covered in the chapters proper are alluded to in the problem section.

Notation

For the reader already accustomed to a variety of different notations, an indication of our own notation might be helpful. We have denoted by an overscore the operation of complex conjugation so that $\bar{a}$ denotes

the complex conjugate of a. Hermitian conjugation is denoted by an asterisk: $(a^*)_{ij} = \overline{a_{ji}}$. Our space-time metric $g_{\mu\nu}$ is such that $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$, and we have differentiated between covariant and contravariant tensors. Our Dirac matrices satisfy the commutation rules $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$. The adjoint of a Dirac spinor u is denoted by $\tilde{u}$, with $\tilde{u} = u^*\gamma^0$.

Acknowledgments

It is my pleasant duty to here record my gratitude to Drs. George Sudarshan, Oscar W. Greenberg and A. Grossman who read some of the early chapters and gave me the benefit of their criticism, and to Professor S. Golden and my other academic colleagues for their encouragement. I am particularly grateful to Professor Kenneth Ford, who read most of the manuscript and made many valuable suggestions for improving it. I am indebted to Drs. Bethe and de Hoffmann for their consent to use some of the material of Volume I of *Mesons and Fields*, to the Office of Naval Research for allowing me to undertake this project in the midst of prior commitments and for providing the encouragement and partial support without which this book could not have been written.

I am also grateful to Mrs. Barbara MacDonald for her excellent typing of the manuscript; to Mr. Paul Hazelrigg for his artful execution of the engravings; and to The Colonial Press Inc. for the masterly setting and printing of a difficult manuscript. I would like to thank particularly the editorial staff of the publisher for efficient and accurate editorial help and for cheerful assistance which made the task of seeing the manuscript through the press a more pleasant one.

Above all, I am deeply grateful to my wife, who offered constant warm encouragement, unbounded patience, kind consideration and understanding during the trying years while this book was being written.

SILVAN S. SCHWEBER

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Part One

THE ONE-PARTICLE EQUATIONS

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Quantum Mechanics and Symmetry Principles

1a. Quantum Mechanical Formalism

Quantum Mechanics, as usually formulated, is based on the postulate that all the physically relevant information about a physical system at a given instant of time is derivable from the knowledge of the state function of the system. This state function is represented by a ray in a complex Hilbert space, a ray being a direction in Hilbert space: If $|\Psi\rangle$ is a vector which corresponds to a physically realizable state, then $|\Psi\rangle$ and a constant multiple of $|\Psi\rangle$ both represent this state. It is therefore customary to choose an arbitrary representative vector of the ray which is normalized to one to describe the state. If $|\Psi\rangle$ is this representative, the normalization condition is expressed as $\langle\Psi \mid \Psi\rangle = 1$, where $\langle\chi \mid \Psi\rangle = \overline{\langle\Psi \mid \chi\rangle}$ denotes the scalar product of the vectors $|\chi\rangle$ and $|\Psi\rangle$.[1] If the states are normalized, only a constant factor of modulus one is left undetermined and two vectors which differ by such a phase factor represent the same state. The system of states is assumed to form a linear manifold and this linear character of the state vectors is called the superposition principle. This is perhaps the fundamental principle of quantum mechanics.

A second postulate of quantum mechanics is that to every measurable (i.e., observable) property, α, of a system corresponds a self-adjoint operator $a = a^*$ with a complete set of orthonormal eigenfunctions $|a'\rangle$ and real eigenvalues a', i.e.,

$$a \mid a'\rangle = a' \mid a'\rangle \tag{1}$$

$$\langle a' \mid a''\rangle = \delta_{a'a''} \tag{2}$$

$$\sum_{a'} |a'\rangle \langle a'| = 1 \tag{3}$$

The symbol $\delta_{a'a''}$ is to be understood as the Kronecker symbol if a' and a'' lie in the discrete spectrum and as the Dirac δ function, $\delta(a' - a'')$, if either or both lie in the continuous spectrum. Similarly, the summation

[1] We shall also use the notation (f, g) to denote the scalar product; $\bar{\lambda}$ denotes the complex conjugate of λ.

sign in the completeness relation Eq. (3) is to be regarded as an integration over the continuous spectrum.

It is further postulated that if a measurement is performed on the system to determine the value of the observable α, the probability of finding the system, described by the state vector $|\Psi\rangle$, to have α with the value a' is given by $|\langle a' \mid \Psi\rangle|^2$. In other words $\langle a' \mid \Psi\rangle$ is the probability amplitude of observing the value a'. A measurement on a system will, in general, perturb the system and, thus, alter the state vector of the system. If as a result of a measurement on a system we find that the observable α has the value a' the (unnormalized) vector describing the system after the measurement is $|a'\rangle \langle a' \mid \Psi\rangle$. An immediate repetition of the measurement will thus again yield the value a' for the observable α. These statements are, strictly speaking, only correct for the case of an observable with a nondegenerate discrete eigenvalue spectrum. These rules, however, can easily be extended to more complex situations.

A measurement of the property α thus channels the system into a state which is an eigenfunction of the operator a. However, only the probability of finding the system in a particular eigenstate is theoretically predictable given the state vector $|\Psi\rangle$ of the system. If this state vector is known, measurements then allow the verification of the predicted probabilities. A measurement of the first kind (i.e., measurements which if repeated immediately give identical results) can also (and perhaps more appropriately) be regarded as the way to prepare a system in a given state.

It is usually the case that several independent measurements must be made on the system to determine its state. It is therefore assumed in quantum mechanics that it is always possible to perform a complete set of compatible independent measurements, i.e., measurements which do not perturb the values of the other observables previously determined. The results of all possible compatible measurements can be used to characterize the state of the system, as they provide the maximum possible information about the system. Necessary and sufficient conditions for two measurements to be compatible or simultaneously performable is that the operators corresponding to the properties being measured commute. A maximal set of observables which all commute with one another defines a "complete set of commuting operators" [Dirac (1958)]. There is only one simultaneous eigenstate belonging to any set of eigenvalues of a complete set of commuting observables.

The act of measurement is thus fundamental to the formulation and interpretation of the quantum mechanical formalism. An analysis of various kinds of physical measurements at the microscopic level reveals that almost every such physical measurement can be described as a collision process. One need only recall that such quantities as the energy of stationary states or the lifetime of excited states can be obtained from scattering cross sections. The realization of the central role of collision proc-

esses in quantum mechanics was of the utmost importance in the recent development of field theory. It also accounts, in part, for the intensive study of the quantum theory of scattering in the past decade.

A collision process consists of a projectile particle impinging upon a target particle, interacting with it, and thereby being scattered. Now initially the projectile particle is far removed from the target. If the force between the particles is of finite range, as is almost always the case, the projectile particle will travel initially as a free particle. Similarly, after it has interacted with the target the scattered particle is once again outside the range of the force field and thus travels as a free particle to the detector. A scattering experiment measures the angular distribution, energy, and other compatible observables of the scattered particles far away from the target, for projectile particles prepared in known states. Thus in making theoretical predictions, the statistical interpretation has only to be invoked for initial and final states of *freely* moving particles or groups of particles in stationary states. Therein lies the importance of collision phenomena from a theoretical standpoint: It is never necessary to give an interpretation of the wave function when the particles are close together and interacting strongly. These remarks also indicate the reason for studying the wave mechanical equations describing freely moving particles which take up Part One of this book.

The postulates introduced thus far allow us to deduce the fact that to every realizable state there corresponds a unique ray in Hilbert space. For if there were several distinct rays which correspond to a single distinct state, then if $|\Psi_1\rangle$, $|\Psi_2\rangle$, etc. are normalized representatives of these rays, by Schwartz's inequality $|(\Psi_1, \Psi_2)|^2 < 1$, i.e., the transition probability from $|\Psi_1\rangle$ to $|\Psi_2\rangle$ is less than one, which cannot be if they represent the same state. Therefore $|\Psi_1\rangle$, $|\Psi_2\rangle$, etc. must be constant multiples of each other. It may, however, be the case that there exist rays in Hilbert space which do not correspond to any physically realizable state. This situation occurs in relativistic field theories or in the second quantized formulation of quantum mechanics. In each of these cases the Hilbert space of rays can be decomposed into orthogonal subspaces $\mathfrak{C}_A$, $\mathfrak{C}_B$, $\mathfrak{C}_C \cdots$ such that the relative phase of the component of a vector in each of the subspaces is arbitrary and not measurable. In other words, if we denote by $|A, l\rangle$ the basis vectors which span the Hilbert space $\mathfrak{C}_A$, and by $|B, j\rangle$ the basis vectors which span $\mathfrak{C}_B$, etc., then no physical measurement can differentiate between the vector

$$\sum_l a_l \,|\, A, l\rangle \oplus \sum_j b_j \,|\, B, j\rangle \oplus \cdots$$

and the vector

$$\sum_l a_l e^{i\alpha} \,|\, A, l\rangle \oplus \sum_j b_j e^{i\beta} \,|\, B, j\rangle \oplus \cdots$$

where $\alpha, \beta, \cdots$ are arbitrary phase factors. The phenomenon responsible for the breakup of the Hilbert space into several incoherent orthogonal subspaces is called a superselection rule [Wick (1952), Wigner (1952a), Bargmann (1953)]. A superselection rule corresponds to the existence of an operator which is not a multiple of the identity and which commutes with all observables. If the Hilbert space of states, $\mathfrak{H}$, decomposes for example into two orthogonal subspaces, $\mathfrak{H}_A$ and $\mathfrak{H}_B$, such that the relative phases of the components of the state vector in the two subspaces is completely arbitrary, then the expectation value of a Hermitian operator that has matrix elements between these two subspaces is likewise arbitrary when taken for a state with nonvanishing components in $\mathfrak{H}_A$ and $\mathfrak{H}_B$. Now for a quantity to be measurable it must surely have a well-defined expectation value in any state. Therefore, a Hermitian operator which connects two such orthogonal subspaces cannot be measurable. An example of this phenomenon is the Hilbert space which consists of the states of 1, 2, 3, $\cdots$, n, $\cdots$ particles each carrying electric charge e. The orthogonal subsets then consist of the subspaces with definite total charge and a Hermitian operator connecting subspaces with different total charge cannot be observable. The superselection rule operating in this case is the charge conservation law, or its equivalent statement: gauge invariance of the first kind (Sec. 7*g*).

An equivalent formulation of the above consists in the statement that all rays within a single subspace are realizable but a ray which has components in two or more subspaces is not. If not all rays are realizable, then clearly no measurement can give rise to these nonrealizable states. They cannot therefore be eigenfunctions of any Hermitian operator which corresponds to an observable property of the system. To be observable a Hermitian operator must therefore satisfy certain conditions (superselection rules). Ordinary elementary quantum mechanics operates in a single coherent subspace, so that it is possible to distinguish between any two rays and all self-adjoint operators are then observable.

Quantum mechanics next postulates that the position and momentum operators of a particle obey the following commutation rules:

$$[q_l, p_j] = i\hbar\delta_{lj} \quad (l, j = 1, 2, 3) \tag{4}$$

For a particle with no internal degrees of freedom, it is a mathematical theorem [Von Neumann (1931)] that these operators are irreducible, meaning that there exists no subspace of the entire Hilbert space which is left invariant under these operators. This property is equivalent to the statements that any operator which commutes with both $\mathbf{p}$ and $\mathbf{q}$ is a multiple of the identity and that every operator is a function of $\mathbf{p}$ and $\mathbf{q}$. The description of the system in terms of the observables $\mathbf{p}$ and $\mathbf{q}$ is complete.

Finally, quantum mechanics postulates that the dynamical behavior of the system is described by the Schrödinger equation

$$i\hbar\partial_t \,|\, ; t\rangle = H \,|\, ; t\rangle \tag{5}$$

where $\partial_t = \partial/\partial t$ and H, the Hamiltonian operator of the system, corresponds to the translation operator for infinitesimal time translations. By this is meant the following: Assume that the time evolution of the state vector can be obtained by the action of an operator $U(t, t_0)$ on the initial state $|\, ; t_0\rangle$ such that

$$|\, t\rangle = U(t, t_0) \,|\, t_0\rangle \tag{6a}$$

$$U(t_0, t_0) = 1 \tag{6b}$$

Conservation of probability requires that the norm of the vector $|\, t\rangle$ be constant in time:

$$\begin{aligned} \langle t \,|\, t\rangle &= \langle t_0 \,|\, t_0\rangle \\ &= \langle t_0 \,|\, U^*(t, t_0)\, U(t, t_0) \,|\, t_0\rangle \end{aligned} \tag{7}$$

and therefore that

$$U^*(t, t_0)\, U(t, t_0) = 1 \tag{8a}$$

This does not yet guarantee that U is unitary. For this to be the case, the following equation must also hold:

$$U(t, t_0)\, U^*(t, t_0) = 1 \tag{8b}$$

This condition will hold if U satisfies the group property:

$$U(t, t_1)\, U(t_1, t_0) = U(t, t_0) \tag{9}$$

If, in Eq. (9), we set $t = t_0$, and assume its validity for $t_0 < t_1$, we then obtain

$$U(t_0, t_1)\, U(t_1, t_0) = 1 \tag{10a}$$

whence

$$U(t_0, t_1) = U^{-1}(t_1, t_0) \tag{10b}$$

and multiplying (10a) on the left by $U^*(t_0, t_1)$ using (8) we obtain

$$U(t_1, t_0) = U^*(t_0, t_1) = U^{-1}(t_1, t_0) \tag{10c}$$

so that U is unitary.

If we let t be infinitesimally close to t_0, with $t - t_0 = \delta t$ then to first order in δt we may write

$$U(t_0 + \delta t, t_0) = 1 - \frac{i}{\hbar} H\delta t \tag{11}$$

In order that U be unitary, H must be Hermitian. The dimension of H is that of an energy. Equation (6a) for the infinitesimal case thus reads

$$|t_0 + \delta t\rangle - |t_0\rangle = -\frac{i}{\hbar} H\delta t \,|\, t_0\rangle \tag{12a}$$

which in the limit as $\delta t \to 0$ becomes Eq. (5) since, by definition,

$$\lim_{\delta t \to 0} (\delta t)^{-1}(|t + \delta t\rangle - |t\rangle) = \partial_t \,|\, t\rangle \tag{12b}$$

1b. Schrödinger and Heisenberg Pictures

In the previous remarks about quantum mechanics, we have defined the state of the system at a given time t by the results of all possible experiments on the system at that time. This information is contained in the state vector $|\, t\rangle_S = |\Psi_S(t)\rangle$. The evolution of the system in time is then described by the time dependence of the state vector which is governed by the Schrödinger equation

$$H_S \,|\, \Psi_S(t)\rangle = i\hbar\partial_t \,|\, \Psi_S(t)\rangle \tag{13}$$

The operators corresponding to physical observables, F_S, are time-independent; they are the same for all time with $\partial_t F_S = 0$. This defines the Schrödinger picture and the subscript S identifies the picture [Dirac (1958)].

Although the operators are time-independent, their expectation value in any given state will in general be time-dependent. Call

$$\langle F_S\rangle = \langle \Psi_S(t) \,|\, F_S \,|\, \Psi_S(t)\rangle \tag{14}$$

then

$$i\hbar \frac{d}{dt}\langle F_S\rangle = \langle \Psi_S(t) \,|\, [F_S, H_S] \,|\, \Psi_S(t)\rangle \tag{15}$$

In the Schrödinger picture we call, by definition, $\dot{F}_S$ that operator for which

$$\langle \dot{F}_S\rangle = \frac{d}{dt}\langle F_S\rangle \tag{16}$$

Let us next perform a time-dependent unitary transformation $V(t)$ on $|\Psi_S(t)\rangle$ which transforms it into the state vector

$$|\Phi(t)\rangle = V(t) \,|\, \Psi_S(t)\rangle \tag{17a}$$

$$V(t)\, V^*(t) = V^*(t)\, V(t) = \mathbf{1} \tag{17b}$$

$$V^*(t) = V^{-1}(t) \tag{17c}$$

Using Eqs. (13) and (17a) we find that $|\Phi(t)\rangle$ obeys the following equation:

$$i\hbar\partial_t \,|\, \Phi(t)\rangle = [i\hbar\partial_t V(t) \cdot V^{-1}(t) + V(t)\, H_S V^{-1}(t)] \,|\, \Phi(t)\rangle \tag{18}$$

If we choose the time-dependent unitary operator, V, to satisfy

$$-i\hbar\partial_t V(t) = V(t)\, H_S V^{-1}(t) \cdot V(t) \tag{19}$$

the transformed state, $|\Phi_H\rangle$, will then be time-independent, i.e., $\partial_t \,|\, \Phi_H\rangle = 0$. The operator $V(t)$ being unitary, the expectation value of the operator F_S in terms of $|\Phi_H\rangle$ is given by