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Jim Pitman

# Combinatorial Stochastic Processes

Ecole d'Eté de Probabilités de Saint-Flour XXXII – 2002

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# Foreword

Three series of lectures were given at the 32nd Probability Summer School in Saint-Flour (July 7–24, 2002), by the Professors Pitman, Tsirelson and Werner. The courses of Professors Tsirelson ("Scaling limit, noise, stability") and Werner ("Random planar curves and Schramm-Loewner evolutions") have been published in a previous issue of *Lectures Notes in Mathematics* (volume 1840). This volume contains the course "Combinatorial stochastic processes" of Professor Pitman. We cordially thank the author for his performance in Saint-Flour and for these notes.

76 participants have attended this school. 33 of them have given a short lecture. The lists of participants and of short lectures are enclosed at the end of the volume.

The Saint-Flour Probability Summer School was founded in 1971. Here are the references of Springer volumes which have been published prior to this one. All numbers refer to the *Lecture Notes in Mathematics* series, except S-50 which refers to volume 50 of the *Lecture Notes in Statistics* series.

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|---|---|---|---|
| 1971: vol 307 | 1980: vol 929 | 1990: vol 1527 | 1998: vol 1738 |
| 1973: vol 390 | 1981: vol 976 | 1991: vol 1541 | 1999: vol 1781 |
| 1974: vol 480 | 1982: vol 1097 | 1992: vol 1581 | 2000: vol 1816 |
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| 1979: vol 876 | 1989: vol 1464 | 1997: vol 1717 | |

Further details can be found on the summer school web site
`http://math.univ-bpclermont.fr/stflour/`

Université Blaise Pascal
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*Jean Picard*

# Contents

# Preliminaries

## 0.0. Preface

This is a collection of expository articles about various topics at the interface between enumerative combinatorics and stochastic processes. These articles expand on a course of lectures given at the Ecole d'Eté de Probabilités de St. Flour in July 2002. The articles are also called 'chapters'. Each chapter is fairly self-contained, so readers with adequate background can start reading any chapter, with occasional consultation of earlier chapters as necessary. Following this Chapter 0, there are 10 chapters, each divided into *sections*. Most sections conclude with some *Exercises*. Those for which I don't know solutions are called *Problems*.

**Acknowledgments** Much of the research reviewed here was done jointly with David Aldous. Much credit is due to him, especially for the big picture of continuum approximations to large combinatorial structures. Thanks also to my other collaborators in this work, especially Jean Bertoin, Michael Camarri, Steven Evans, Sasha Gnedin, Ben Hansen, Jacques Neveu, Mihael Perman, Ravi Sheth. Marc Yor and Jim Young. A preliminary version of these notes was developed in Spring 2002 with the help of a dedicated class of ten graduate students in Berkeley: Noam Berger, David Blei, Rahul Jain, Şerban Nacu, Gabor Pete. Lea Popovic, Alan Hammond, Antar Bandyopadhyay, Manjunath Krishnapur and Grégory Miermont. The last four deserve special thanks for their contributions as research assistants. Thanks to the many people who have read versions of these notes and made suggestions and corrections, especially David Aldous, Jean Bertoin, Aubrey Clayton, Shankar Bhamidi, Rui Dong, Steven Evans, Sasha Gnedin, Bénédicte Haas, Jean-François Le Gall, Neil O'Connell. Mihael Perman, Lea Popovic, Jason Schweinsberg. Special thanks to Marc Yor and Matthias Winkel for their great help in preparing the final version of these notes for publication. Thanks also to Jean Picard for his organizational efforts in making arrangements for the St. Flour Summer School. This work was supported in part by NSF Grants DMS-0071448 and DMS-0405779.

## 0.1. Introduction

The main theme of this course is the study of various combinatorial models of random partitions and random trees, and the asymptotics of these models related to continuous parameter stochastic processes. A basic feature of models for random partitions is that the sum of the parts is usually constant. So the sizes of the parts cannot be independent. But the structure of many natural models for random partitions can be reduced by suitable conditioning or scaling to classical probabilistic results involving sums of independent random variables. Limit models for combinatorially defined random partitions are consequently related to the two fundamental limit processes of classical probability theory: Brownian motion and Poisson processes. The theory of Brownian motion and related stochastic processes has been greatly enriched by the recognition that some fundamental properties of these processes are best understood in terms of how various random partitions and random trees are embedded in their paths. This has led to rapid developments, particularly in the theory of continuum random trees, continuous state branching processes, and Markovian superprocesses, which go far beyond the scope of this course. Following is a list of the main topics to be treated:

- models for random combinatorial structures, such as trees, forests, permutations, mappings, and partitions;
- probabilistic interpretations of various combinatorial notions e.g. Bell polynomials, Stirling numbers, polynomials of binomial type, Lagrange inversion;
- Kingman's theory of exchangeable random partitions and random discrete distributions;
- connections between random combinatorial structures and processes with independent increments: Poisson-Dirichlet limits;
- random partitions derived from subordinators;
- asymptotics of random trees, graphs and mappings related to excursions of Brownian motion;
- continuum random trees embedded in Brownian motion;
- Brownian local times and squares of Bessel processes;
- various processes of fragmentation and coagulation, including Kingman's coalescent, the additive and multiplicative coalescents

Next, an incomplete list and topics of current interest, with inadequate references. These topics are close to those just listed, and certainly part of the realm of combinatorial stochastic processes, but not treated here:

- probability on trees and networks, as presented in [292];
- random integer partitions [159, 104], random Young tableaux, growth of Young diagrams, connections with representation theory and symmetric functions [245, 420, 421, 239];
- longest increasing subsequence of a permutation, connections with random matrices [28];

- random partitions related to uniformly chosen invertible matrices over a finite field, as studied by Fulman [160];
- random maps, coalescing saddles, singularity analysis, and Airy phenomena, [81];
- random planar lattices and integrated superbrownian excursion [94].

The reader of these notes is assumed to be familiar with the basic theory of probability and stochastic processes, at the level of Billingsley [64] or Durrett [122], including continuous time stochastic processes, especially Brownian motion and Poisson processes. For background on some more specialized topics (local times, Bessel processes, excursions, SDE's) the reader is referred to Revuz-Yor [384]. The rest of this Chapter 0 reviews some basic facts from this probabilistic background for ease of later reference. This material is organized as follows:

**0.2. Brownian motion and related processes** This section provides some minimal description of the background expected of the reader to follow some of the more advanced sections of the text. This includes the definition and basic properties of Brownian motion $B := (B_t, t \geq 0)$, and of some important processes derived from $B$ by operations of scaling and conditioning. These processes include the Brownian bridge, Brownian meander and Brownian excursion. The basic facts of Itô's excursion theory for Brownian motion are also recorded.

**0.3. Subordinators** This section reviews a few basic facts about increasing Lévy processes in general, and some important facts about gamma and stable processes in particular.

## 0.2. Brownian motion and related processes

Let $S_n := X_1 + \cdots + X_n$ where the $X_i$ are independent random variables with mean 0 and variance 1, and let $S_t$ for real $t$ be defined by linear interpolation between integer values. According to *Donsker's theorem* [64, 65, 122, 384]

$$(S_{nt}/\sqrt{n}, 0 \leq t \leq 1) \xrightarrow{d} (B_t, 0 \leq t \leq 1) \tag{0.1}$$

in the usual sense of convergence in distribution of random elements of $C[0,1]$, where $(B_t, t \geq 0)$ is a *standard Brownian motion* meaning that $B$ is a process with continuous paths and stationary independent Gaussian increments, with $B_t \stackrel{d}{=} \sqrt{t} B_1$ where $B_1$ is standard Gaussian.

**Brownian bridge** Assuming now that the $X_i$ are integer valued, some conditioned forms of Donsker's theorem can be presented as follows. Let $o(\sqrt{n})$ denote any sequence of possible values of $S_n$ with $o(\sqrt{n})/\sqrt{n} \to 0$ as $n \to \infty$. Then [128]

$$(S_{nt}/\sqrt{n}, 0 \leq t \leq 1 \,|\, S_n = o(\sqrt{n})) \xrightarrow{d} (B_t^{\rm br}, 0 \leq t \leq 1) \tag{0.2}$$

where $B^{\rm br}$ is the *standard Brownian bridge*, that is, the centered Gaussian process obtained by conditioning $(B_t, 0 \le t \le 1)$ on $B_1 = 0$. Some well known descriptions of the distribution of $B^{\rm br}$ are [384, Ch. III, Ex (3.10)]

$$(B_t^{\rm br}, 0 \le t \le 1) \stackrel{d}{=} (B_t - tB_1, 0 \le t \le 1) \stackrel{d}{=} ((1-t)B_{t/(1-t)}, 0 \le t \le 1) \quad (0.3)$$

where $\stackrel{d}{=}$ denotes equality of distributions on the path space $C[0,1]$, and the rightmost process is defined to be 0 for $t = 1$.

**Brownian meander and excursion** Let $T_- := \inf\{n : S_n < 0\}$. Then as $n \to \infty$

$$(S_{nt}/\sqrt{n}, 0 \le t \le 1 \,|\, T_- > n) \stackrel{d}{\to} (B_t^{\rm me}, 0 \le t \le 1) \quad (0.4)$$

where $B^{\rm me}$ is the *standard Brownian meander* [205, 71], and as $n \to \infty$ through possible values of $T_-$

$$(S_{nt}/\sqrt{n}, 0 \le t \le 1 \,|\, T_- = n) \stackrel{d}{\to} (B_t^{\rm ex}, 0 \le t \le 1) \quad (0.5)$$

where $B_t^{\rm ex}$ is the *standard Brownian excursion* [225, 102]. Informally,

$$\begin{aligned} B^{\rm me} &\stackrel{d}{=} (B \,|\, B_t > 0 \text{ for all } 0 < t < 1) \\ B^{\rm ex} &\stackrel{d}{=} (B \,|\, B_t > 0 \text{ for all } 0 < t < 1, B_1 = 0) \end{aligned}$$

where $\stackrel{d}{=}$ denotes equality in distribution. These definitions of conditioned Brownian motions have been made rigorous in a number of ways: for instance by the method of Doob $h$-transforms [255, 394, 155], and as weak limits as $\varepsilon \downarrow 0$ of the distribution of $B$ given suitable events $A_\varepsilon$, as in [124, 69], for instance

$$(B \,|\, \underline{B}(0,1) > -\varepsilon) \stackrel{d}{\to} B^{\rm me} \text{ as } \varepsilon \downarrow 0 \quad (0.6)$$

$$(B^{\rm br} \,|\, \underline{B^{\rm br}}(0,1) > -\varepsilon) \stackrel{d}{\to} B^{\rm ex} \text{ as } \varepsilon \downarrow 0 \quad (0.7)$$

where $\underline{X}(s,t)$ denotes the infimum of a process $X$ over the interval $(s,t)$.

**Brownian scaling** For a process $X := (X_t, t \in J)$ parameterized by an interval $J$, and $I = [G_I, D_I]$ a subinterval of $J$ with length $\lambda_I := D_I - G_I > 0$, we denote by $X[I]$ or $X[G_I, D_I]$ the *fragment of* $X$ *on* $I$, that is the process

$$X[I]_u := X_{G_I + u} \qquad (0 \le u \le \lambda_I). \quad (0.8)$$

We denote by $X_*[I]$ or $X_*[G_I, D_I]$ the *standardized fragment of* $X$ *on* $I$, defined by the *Brownian scaling operation*

$$X_*[I]_u := \frac{X_{G_I + u\lambda_I} - X_{G_I}}{\sqrt{\lambda_I}} \qquad (0 \le u \le 1). \quad (0.9)$$

For $T > 0$ let $G_T := \sup\{s : s \le T, B_s = 0\}$ be the last zero of $B$ before time $T$ and $D_T := \inf\{s : s > T, B_s = 0\}$ be the first zero of $B$ after time

$T$. Let $|B| := (|B_t|, t \geq 0)$, called *reflecting Brownian motion*. It is well known [211, 98, 384] that for each fixed $T > 0$, there are the following identities in distribution derived by *Brownian scaling*:

$$B_*[0,T] \stackrel{d}{=} B[0,1]; \qquad B_*[0,G_T] \stackrel{d}{=} B^{\mathrm{br}} \tag{0.10}$$

$$|B|_*[G_T,T] \stackrel{d}{=} B^{\mathrm{me}}; \qquad |B|_*[G_T,D_T] \stackrel{d}{=} B^{\mathrm{ex}}. \tag{0.11}$$

It is also known that $B^{\mathrm{br}}, B^{\mathrm{me}}$ and $B^{\mathrm{ex}}$ can be constructed by various other operations on the paths of $B$, and transformed from one to another by further operations [53].

For $0 < t < \infty$ let $B^{\mathrm{br},t}$ be a *Brownian bridge of length* $t$, which may be regarded as a random element of $C[0,t]$ or of $C[0,\infty]$, as convenient:

$$B^{\mathrm{br},t}(s) := \sqrt{t}B^{\mathrm{br}}((s/t) \wedge 1) \qquad (s \geq 0). \tag{0.12}$$

Let $B^{\mathrm{me},t}$ denote a *Brownian meander of length* $t$, and $B^{\mathrm{ex},t}$ be a *Brownian excursion of length* $t$, defined similarly to (0.12) with $B^{\mathrm{me}}$ or $B^{\mathrm{ex}}$ instead of $B^{\mathrm{br}}$.

**Brownian excursions and the three-dimensional Bessel process** The following theorem summarizes some important relations between Brownian excursions and a particular time-homogeneous diffusion process $R_3$ on $[0,\infty)$, commonly known as the *three-dimensional Bessel process* BES(3), due to the representation

$$(R_3(t), t \geq 0) \stackrel{d}{=} \left(\sqrt{\sum_{i=1}^{3}(B_i(t))^2}, t \geq 0\right) \tag{0.13}$$

where the $B_i$ are three independent standard Brownian motions. It should be understood however that this particular representation of $R_3$ is a relatively unimportant coincidence in distribution. What is more important, and can be understood entirely in terms of the random walk approximations (0.1) and (0.5) of Brownian motion and Brownian excursion, is that there exists a time-homogeneous diffusion process $R_3$ on $[0,\infty)$ with $R_3(0) = 0$, which has the same self-similarity property as $B$, meaning invariance under Brownian scaling, and which can be characterized in various ways, including (0.13), but most importantly as a Doob $h$-transform of Brownian motion.

**Theorem 0.1.** *For each fixed $t > 0$, the Brownian excursion $B^{\mathrm{ex},t}$ of length $t$ is the BES(3) bridge from 0 to 0 over time $t$, meaning that*

$$(B^{\mathrm{ex},t}(s), 0 \leq s \leq t) \stackrel{d}{=} (R_3(s), 0 \leq s \leq t \,|\, R_3(t) = 0).$$

*Moreover, as $t \to \infty$*

$$B^{\mathrm{ex},t} \xrightarrow{d} R_3, \tag{0.14}$$

*and $R_3$ can be characterized in two other ways as follows:*

(i) [303, 436] *The process $R_3$ is a Brownian motion on $[0, \infty)$ started at 0 and conditioned never to return to 0, as defined by the Doob h-transform, for the harmonic function $h(x) = x$ of Brownian motion on $[0, \infty)$, with absorbtion at 0. That is, $R_3$ has continuous paths starting at 0, and for each $0 < a < b$ the stretch of $R_3$ between when it first hits $a$ and first hits $b$ is distributed like $B$ with $B_0 = a$ conditioned to hit $b$ before 0.*
(ii) [345]

$$R_3(t) = B(t) - 2\underline{B}(t) \qquad (t \geq 0) \tag{0.15}$$

*where $B$ is a standard Brownian motion with past minimum process*

$$\underline{B}(t) := \underline{B}[0, t] = -\underline{R_3}[t, \infty).$$

**Lévy's identity** The identity in distribution (0.15) admits numerous variations and conditioned forms [345, 53, 55] by virtue of Lévy's identity of joint distributions of paths [384]

$$(B - \underline{B}, -\underline{B}) \stackrel{d}{=} (|B|, L) \tag{0.16}$$

where $L := (L_t, t \geq 0)$ is the local time process of $B$ at 0, which may be defined almost surely as the occupation density

$$L_t = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t ds 1(|B_s| \leq \varepsilon).$$

For instance,

$$(R_3(t), t \geq 0) \stackrel{d}{=} (|B_t| + L_t, t \geq 0).$$

**Lévy-Itô-Williams theory of Brownian excursions** Due to (0.16), the process of excursions of $|B|$ away from 0 is equivalent in distribution to the process of excursions of $B$ above $\underline{B}$. According to the Lévy-Itô description of this process, if $I_\ell := [T_{\ell-}, T_\ell]$ for $T_\ell := \inf\{t : B(t) < -\ell\}$, the points

$$\{(\ell, \mu(I_\ell), (B - \underline{B})[I_\ell]) : \ell > 0, \mu(I_\ell) > 0\}, \tag{0.17}$$

where $\mu$ is Lebesgue measure, are the points of a Poisson point process on $\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times C[0, \infty)$ with intensity

$$d\ell \frac{dt}{\sqrt{2\pi}\, t^{3/2}} \mathbb{P}(B^{\mathrm{ex},t} \in d\omega). \tag{0.18}$$

On the other hand, according to Williams [437], if $M_\ell := \overline{B}[I_\ell] - \underline{B}[I_\ell]$ is the maximum height of the excursion of $B$ over $\underline{B}$ on the interval $I_\ell$, the points

$$\{(\ell, M_\ell, (B - \underline{B})[I_\ell]) : \ell > 0, \mu(I_\ell) > 0\}, \tag{0.19}$$

are the points of a Poisson point process on $\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times C[0, \infty)$ with intensity

$$d\ell \frac{dm}{m^2} \mathbb{P}(B^{\mathrm{ex}\,|\,m} \in d\omega) \tag{0.20}$$

where $B^{\mathrm{ex}\,|\,m}$ is a *Brownian excursion conditioned to have maximum m*. That is to say $B^{\mathrm{ex}\,|\,m}$ is a process $X$ with $X(0) = 0$ such that for each $m > 0$, and $H_x(X) := \inf\{t : t > 0, X(t) = x\}$, the processes $X[0, H_m(X)]$ and $m - X[H_m(X), H_0(X)]$ are two independent copies of $R_3[0, H_m(R_3)]$, and $X$ is stopped at 0 at time $H_0(X)$. *Itô's law of Brownian excursions* is the $\sigma$-finite measure $\nu$ on $C[0,\infty)$ which can be presented in two different ways according to (0.18) and (0.20) as

$$\nu(\cdot) = \int_0^\infty \frac{dt}{\sqrt{2\pi t^{3/2}}}\,\mathbb{P}(B^{\mathrm{ex},t} \in \cdot) = \int_0^\infty \frac{dm}{m^2}\,\mathbb{P}(B^{\mathrm{ex}\,|\,m} \in \cdot) \tag{0.21}$$

where the first expression is a disintegration according to the lifetime of the excursion, and the second according to its maximum. The identity (0.21) has a number of interesting applications and generalizations [60, 367, 372].

**BES(3) bridges** Starting from three independent standard Brownian bridges $B_i^{\mathrm{br}}, i = 1, 2, 3$, for $x, y \geq 0$ let

$$R_3^{x\to y}(u) := \sqrt{(x + (y - x)u + B_{1,u}^{\mathrm{br}})^2 + (B_{2,u}^{\mathrm{br}})^2 + (B_{3,u}^{\mathrm{br}})^2} \qquad (0 \leq u \leq 1). \tag{0.22}$$

The random element $R_3^{x\to y}$ of $C[0,1]$ is the *BES(3) bridge from x to y*, in terms of which the laws of the standard excursion and meander are represented as

$$B^{\mathrm{ex}} \stackrel{d}{=} R_3^{0\to 0} \text{ and } B^{\mathrm{me}} \stackrel{d}{=} R_3^{0\to\rho} \tag{0.23}$$

where $\rho$ is a random variable with the *Rayleigh density*

$$P(\rho \in dx)/dx = xe^{-\frac{1}{2}x^2} \qquad (x > 0) \tag{0.24}$$

and $\rho$ is independent of the family of Bessel bridges $R_3^{0\to r}, r \geq 0$. Then by construction

$$B_1^{\mathrm{me}} = \rho \stackrel{d}{=} \sqrt{2\Gamma_1} \tag{0.25}$$

where $\Gamma_1$ is a *standard exponential variable*, and

$$(B_1^{\mathrm{me}} \mid B_1^{\mathrm{me}} = r) \stackrel{d}{=} R_3^{0\to r}. \tag{0.26}$$

These descriptions are read from [435, 208]. See also [98, 53, 62, 384] for further background. By (0.22) and Itô's formula, the process $R_3^{x\to y}$ can be characterized for each $x, y \geq 0$ as the solution over $[0,1]$ of the Itô SDE

$$R_0 = x; \qquad dR_s = \left(\frac{1}{R_s} + \frac{(y - R_s)}{(1 - s)}\right) ds + d\gamma_s \tag{0.27}$$

for a Brownian motion $\gamma$.

*Exercises*

**0.2.1.** [384] Show, using stochastic calculus, that the three dimensional Bessel process $R_3$ is characterized by description (i) of Theorem 0.1.

**0.2.2.** Check that $R_3^{x \to y}$ solves (0.27), and discuss the uniqueness issue.

**0.2.3.** [344, 270] Formulate and prove a discrete analog for simple symmetric random walk of the equivalence of the two descriptions of $R_3$ given in Theorem 0.1, along with a discrete analog of the following fact: if $R(t) := B(t) - 2\underline{B}(t)$ for a Brownian motion $B$ then

$$\text{the conditional law of } \underline{B}(t) \text{ given } (R(s), 0 \le s \le t) \text{ is uniform on } [-R(t), 0]. \tag{0.28}$$

Deduce the Brownian results by embedding a simple symmetric random walk in the path of $B$.

**0.2.4. (Williams' time reversal theorem)**[436, 344, 270] Derive the identity in distribution

$$(R_3(t), 0 \le t \le K_x) \stackrel{d}{=} (x - B(H_x - t), 0 \le t \le H_x), \tag{0.29}$$

where $K_x$ is the last hitting time of $x > 0$ by $R_3$, and where $H_x$ the first hitting time of $x > 0$ by $B$, by first establishing a corresponding identity for paths of a suitably conditioned random walk with increments of $\pm 1$, then passing to a Brownian limit.

**0.2.5.** [436, 270] Derive the identity in distribution

$$(R_3(t), 0 \le t \le H_x) \stackrel{d}{=} (x - R_3(H_x - t), 0 \le t \le H_x), \tag{0.30}$$

where $H_x$ is the hitting time of $x > 0$ by $R_3$.

**0.2.6.** Fix $x > 0$ and for $0 < y < x$ let $K_y$ be the last time before $H_x(R_3)$ that $R_3$ hits $y$, let $I_y := [K_{y-}, K_y]$, and let $R_3[I_y] - y$ be the excursion of $R_3$ over the interval $I_y$ pulled down so that it starts and ends at 0. Let $M_y$ be the maximum height of this excursion. Show that the points

$$\{(y, M_y, R_3[I_y] - y) : M_y > 0\}, \tag{0.31}$$

are the points of a Poisson point process on $[0, x] \times \mathbb{R}_{>0} \times C[0, \infty)$ with intensity measure of the form

$$f(y, m)\, dy\, dm\, \mathbb{P}(B^{\text{ex}\,|\,m} \in d\omega)$$

for some $f(y, m)$ to be computed explicitly, where $B^{\text{ex}\,|\,m}$ is a Brownian excursion of maximal height $m$. See [348] for related results.

*Notes and comments*

See [387, 270, 39, 384, 188] for different approaches to the basic path transformation (0.15) from $B$ to $R_3$, its discrete analogs, and various extensions. In terms of $X := -B$ and $M := \overline{X} = -\underline{B}$, the transformation takes $X$ to $2M - X$. For a generalization to exponential functionals, see Matsumoto and Yor [299]. This is also discussed in [331], where an alternative proof is given using reversibility and symmetry arguments, with an application to a certain directed polymer problem. A multidimensional extension is presented in [332], where a representation for Brownian motion conditioned never to exit a (type A) Weyl chamber is obtained using reversibility and symmetry properties of certain queueing networks. See also [331, 262] and the survey paper [330]. This representation theorem is closely connected to random matrices, Young tableaux, the Robinson-Schensted-Knuth correspondence, and symmetric functions theory [329, 328]. A similar representation theorem has been obtained in [75] in a more general symmetric spaces context, using quite different methods. These multidimensional versions of the transformation from $X$ to $2M - X$ are intimately connected with combinatorial representation theory and Littelmann's path model [286].

## 0.3. Subordinators

A *subordinator* $(T_s, s \geq 0)$ is an increasing process with right continuous paths, stationary independent increments, and $T_0 = 0$. It is well known [40] that every such process can be represented as

$$T_t = ct + \sum_{0<s\leq t} \Delta_s \qquad (t \geq 0)$$

for some $c \geq 0$ where $\Delta_s := T_s - T_{s-}$ and $\{(s, \Delta_s) : s > 0, \Delta_s > 0\}$ is the set of points of a Poisson point process on $(0, \infty)^2$ with intensity measure $ds\Lambda(dx)$ for some measure $\Lambda$ on $(0, \infty)$, called the *Lévy measure* of $T_1$ or of $(T_t, t \geq 0)$, such that the *Laplace exponent*

$$\Psi(u) = cu + \int_0^\infty (1 - e^{-ux})\Lambda(dx) \tag{0.32}$$

is finite for some (hence all) $u > 0$. The Laplace transform of the distribution of $T_t$ is then given by the following special case of the *Lévy-Khintchine formula* [40]:

$$\mathbb{E}[e^{-uT_t}] = e^{-t\Psi(u)}. \tag{0.33}$$

**The gamma process** Let $(\Gamma_s, s \geq 0)$ denote a *standard gamma process*, that is the subordinator with marginal densities

$$\mathbb{P}(\Gamma_s \in dx)/dx = \frac{1}{\Gamma(s)} x^{s-1} e^{-x} \qquad (x > 0). \tag{0.34}$$

The Laplace exponent $\Psi(u)$ of the standard gamma process is

$$\Psi(u) = \log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots$$

and the Lévy measure is $\Lambda(dx) = x^{-1}e^{-x}dx$. A special feature of the gamma process is the multiplicative structure exposed by Exercise 0.3.1 and Exercise 0.3.2 . See also [416].

**Stable subordinators** Let $\mathbb{P}_\alpha$ govern a stable subordinator $(T_s, s \geq 0)$ with index $\alpha \in (0,1)$. So under $\mathbb{P}_\alpha$

$$T_s \stackrel{d}{=} s^{1/\alpha} T_1 \tag{0.35}$$

where

$$\mathbb{E}_\alpha[\exp(-\lambda T_1)] = \exp(-\lambda^\alpha) = \int_0^\infty e^{-\lambda x} f_\alpha(x)\,dx \tag{0.36}$$

with $f_\alpha(x)$ the stable($\alpha$) density of $T_1$, that is [377]

$$f_\alpha(t) = \frac{1}{\pi}\sum_{k=0}^\infty \frac{(-1)^{k+1}}{k!}\sin(\pi\alpha k)\frac{\Gamma(\alpha k+1)}{t^{\alpha k+1}}. \tag{0.37}$$

For $\alpha = \frac{1}{2}$ this reduces to the formula of Doetsch [112, pp. 401-402] and Lévy [284]

$$f_{\frac{1}{2}}(t) = \frac{t^{-\frac{3}{2}}}{2\sqrt{\pi}}e^{-\frac{1}{4t}} = \mathbb{P}(\tfrac{1}{2}B_1^{-2} \in dt)/dt \tag{0.38}$$

where $B_1$ is a standard Gaussian variable. For general $\alpha$, the Lévy density of $T_1$ is well known to be

$$\rho_\alpha(x) := \frac{\alpha}{\Gamma(1-\alpha)}\frac{1}{x^{\alpha+1}} \qquad (x > 0) \tag{0.39}$$

Note the useful formula

$$\mathbb{E}_\alpha(T_1^{-\theta}) = \frac{\Gamma(\frac{\theta}{\alpha}+1)}{\Gamma(\theta+1)} \quad (\theta > -\alpha) \tag{0.40}$$

which is read from (0.36) using $T_1^{-\theta} = \Gamma(\theta)^{-1}\int_0^\infty \lambda^{\theta-1}e^{-\lambda T_1}d\lambda$. Let $(S_t, t \geq 0)$ denote the continuous inverse of $(T_s, s \geq 0)$. For instance, $(S_t, t \geq 0)$ may be the local time process at 0 of some self-similar Markov process, such as a Brownian motion ($\alpha = \frac{1}{2}$) or a Bessel process of dimension $2 - 2\alpha \in (0,2)$. See [384, 41]. Easily from (0.35), under $\mathbb{P}_\alpha$ there is the identity in law

$$S_t/t^\alpha \stackrel{d}{=} S_1 \stackrel{d}{=} T_1^{-\alpha} \tag{0.41}$$

Thus the $\mathbb{P}_\alpha$ distribution of $S_1$ is the *Mittag-Leffler distribution* with Mellin transform

$$\mathbb{E}_\alpha(S_1^p) = \mathbb{E}_\alpha((T_1^{-\alpha})^p) = \frac{\Gamma(p+1)}{\Gamma(p\alpha+1)} \quad (p > -1) \tag{0.42}$$